Duke University
Department of Physics

Physics 361 Spring Term 2020

WUN2K FOR LECTURE 9

These are notes summarizing the main concepts you need to understand and be able to apply.

- **Linear operators** obey the **principle of superposition**: if $\mathbf{D}x_n(t) = D_n(t)$, then $\mathbf{D}\left(\sum_{n=1}^{N}\alpha_n x_n(t)\right) = \sum_{n=1}^{N}\alpha_n D_n(t)$.
- **Fourier's Theorem** states that *any* arbitrary periodic function, i.e., one for which $f(t+\tau) = f(t)$ where $\tau = 2\pi/\omega$ (subject to conditions generally satisfied by physical functions we will use), can be written $f(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega t + b_n \sin n\omega t)$, where $a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t')\cos n\omega t' dt'$, $b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t')\sin n\omega t' dt'$, $a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t')dt'$.
- These results are very useful for describing motion for arbitrary forces: if you know the solution for sinusoidal forces, then the overall solution for an arbitrary force built out of a superposition of sinusoids can be determined by the superposition of the solutions. For example, the driving force described by an even function can be written $f(t) = \sum_{n=0}^{\infty} f_n \cos\omega t$, and the corresponding solution for the oscillator motion can then be written as $x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n)$, where $A_n = \frac{f_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2\omega^2}}$ and $\delta_n = \tan^{-1}\left(\frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}\right)$.
- **Nonlinear oscillations**
  - Differential equations of the type $m\ddot{x} + f(\dot{x}) + g(x) = h(t)$ are **nonlinear** if $g(x)$ and $f(\dot{x})$ contain powers of $\dot{x}$ and x, respectively, equal or greater than 2. *Superposition no longer applies.* Solutions often need to be calculated numerically, and can exhibit complex behavior. **Deterministic chaos** refers to a behavior of the solutions that shows *extreme sensitivity to initial conditions* (as distinct from simple randomness).

– Linear restoring forces $F = -kx$ have corresponding potential energies $U = \frac{1}{2}kx^2$. Any force can be approximated as linear, and the potential as parabolic, for small enough x. A nonlinear force which has a smaller magnitude than linear in some region is denoted "soft" in that region, and one with a larger magnitude is denoted "hard".

– A phase diagram can be drawn for a nonlinear potential energy using $\dot{x}(x) \propto \sqrt{E - U(x)}$ for a particle of energy E; this will in general be an asymmetric trajectory.