Duke University
Department of Physics

Physics 361 Spring Term 2020

WUN2K FOR LECTURE 7

These are notes summarizing the main concepts you need to understand and be able to apply.

- The energy of the oscillator is $E = T + U$, where $T = \frac{1}{2}m\dot{x}^2$, and $U = \frac{1}{2}kx^2$; $E = \frac{1}{2}kA^2$ is conserved.

- For a 2D oscillator $\vec{F} = -k\vec{r}$, and one obtains sinusoids with frequency ω_0 for $x(t)$ and $y(t)$. By eliminating t, one obtains a relation between y and x, which corresponds physically to an elliptical trajectory of a particle in 2D, for which shape and size are determined by the initial conditions. If ω is not the same in each dimension, one obtains **Lissajous** figures for the superposition of oscillations for $x(t) = A\cos(\omega_x t - \alpha)$ and $y(t) = B\cos(\omega_y t - \beta)$.

- **Phase Diagrams** are an abstract construct helpful for describing motion. One plots (for 1D motion) $x, \dot{x}$ in a 2D space, and the motion of a particle will describe a trajectory $P(x, \dot{x})$ in this space (phase space for n-dimensional space has $2n$ dimensions). For the simple harmonic oscillator, the phase space trajectory is elliptical, corresponding to $\frac{x^2}{2E/k} + \frac{\dot{x}^2}{2E/m} = 1$.

- **Damped Oscillations**

  - The simple harmonic oscillator with no friction or damping force oscillates forever as a **free oscillator**. However in realistic cases, some dissipative forces exist. We will model **damping forces** as being linear retarding forces, and linear functions of the velocity, $\vec{F}_d = -b\vec{v}$.

- The equation of motion for a 1D damped harmonic oscillator with damping force $-b\dot{x}$ is $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, where $\beta \equiv b/2m$ is the **damping parameter** and $\omega_0 = \sqrt{k/m}$ is the angular frequency without damping (also sometimes known as the "natural frequency").
- The general solution is $x(t) = e^{-\beta t}[C_1 e^{\sqrt{\beta^2 - \omega_0^2}t} + C_2 e^{-\sqrt{\beta^2 - \omega_0^2}t}]$.
- The behavior of the solution is determined by the relative values of ω_0 and β:
  * For **underdamping**, $\omega_0^2 > \beta^2$. The solution becomes $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$, which is an oscillation with an exponential decay envelope. The frequency $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ is, for small damping, a little bit smaller than ω_0.
  * For **critical damping**, $\omega_0^2 = \beta^2$, the solution is $x(t) = (A + Bt)e^{-\beta t}$. The system goes to equilibrium monotonically in the minimum time.
  * For **overdamping**, $\omega_0^2 < \beta^2$. The solution becomes $x(t) = e^{-\beta t}[C_1 e^{\omega_2 t} + C_2 e^{-\omega_2 t}]$, where $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$. In this case, the system goes to equilibrium, and its exact behavior depends on the initial conditions.

  In all of these cases, a phase diagram can help understand the motion.