Duke University
Department of Physics

Physics 361 Spring Term 2020

# WUN2K FOR LECTURE 17

These are notes summarizing the main concepts you need to understand and be able to apply.

- For the two-body central force problem, with no external forces, we have $\vec{r}_1$ and $\vec{r}_2$ with respect to some origin, and $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ is the difference vector between them. For this case, $U = U(r)$, i.e., the potential energy depends only on the distance between the particles. We have $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$, where $M = m_1 + m_2$, is the center of mass (CM) vector. We define the **reduced mass** $\mu = \frac{m_1 m_2}{m_1 + m_2}$.
  - The Langrangian can be written as the sum of a center-of-mass term and a "relative" term, $\mathcal{L} = \mathcal{L}_{\rm CM} + \mathcal{L}_{\rm rel} = \frac{1}{2}M\dot{\vec{R}}^2 + (\frac{1}{2}\mu\dot{\vec{r}}^2 - U(r))$. The relative contribution "acts like" the motion of a single particle of reduced mass $\mu$.
  - Angular momentum is conserved and the motion is in a 2D plane.
  - In polar coordinates, we have $\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$. The coordinate $\phi$ is cyclic, and Lagrange's equations for $\phi$ give $\ell = \mu r^2\dot{\phi} =$ constant.
  - In polar coordinates, the radial equation of motion is $\mu r\dot{\phi}^2 - \frac{dU}{dr} = \mu\ddot{r}$.
  - One can rewrite the radial equation as $\mu\ddot{r} = -\frac{d}{dr}U_{\rm eff}(r)$, and treat the motion as under the influence of an **effective potential energy** $U_{\rm eff}(r)$, $U_{\rm eff}(r) \equiv U(r) + \frac{\ell^2}{2\mu r^2}$, where $\ell$ is the conserved angular momentum, $\mu$ is the reduced mass, and $U(r)$ is the true potential energy. The quantity $U_{\rm cf} \equiv \frac{\ell^2}{2\mu r^2}$, with dimensions of energy, is known as the "centrifugal potential energy", and is associated with the *fictitious* **centrifugal force** $F_{\rm cf} = \mu r\dot{\phi}^2$. Conventionally, one chooses $U(r) = 0$ for $r \to \infty$.

– If one plots $U_{\text{eff}}(r)$ vs $r$, as $r$ increases, one goes from an infinite positive effective potential at $r = 0$, to a negative well, to zero from below as $r \to \infty$. The motion depends on $E$: for $E > 0$, the motion is *unbounded*, with a minimum $r$ value corresponding to the intersection of $U_{\text{eff}}(r)$ with $E$. For $E < 0$, there will be a *bounded* motion, with $r_{\min}$ and $r_{max}$ corresponding to the intersection of $E$ with the walls of the "well": these are the *turning points*.