Duke University
Department of Physics

Physics 361 Spring Term 2020

WUN2K FOR LECTURE 16

These are notes summarizing the main concepts you need to understand and be able to apply.

- The **generalized momenta** for a given set of generalized coordinates q_i are defined as $p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$.
- The Lagrange equations can be expressed as $\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i}$.
- In terms of generalized momenta, the Hamiltonian can be defined as $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$. The Lagrangian is a function of q_k, $\dot{q}_k$, and (in general) time t. The Hamiltonian is a function of q_i, p_i, and t. So we can write $\mathcal{H}(q_i, p_i, t) = \sum_i p_i \dot{q}_i - \mathcal{L}(q_k, \dot{q}_k, t)$.
- **Hamilton's equations of motion**, also known as the **canonical equations of motion** are

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, -\dot{p}_i = \frac{\partial \mathcal{H}}{\partial q_i} \tag{1}$$

  These are equivalent to Lagrange's equations (and give the same equations of motion as Newton's Laws). They are $2n$ first-order differential equations, in contrast to Lagrange's n second-order differential equations, where n is number of the coordinates.

- If the coordinate transformation equations $\vec{r}_\alpha = \vec{r}_\alpha(q_i)$ are independent of time (i.e., the coordinates are "natural"), then $\mathcal{H} = T + U = E$, the total energy. If $\partial \mathcal{H}/\partial t = 0$, then $\mathcal{H}$ is conserved. (Note these are different conditions, but they may both apply to a situation.)

- The q_i and p_i are called **canononically conjugate** quantities. If q_i does not appear in the Hamiltonian, then $\dot{p_i} = 0$ and p_i is a constant of the motion (which can be determined from the initial conditions). In this case q_i is called a **cyclic** or **ignorable** coordinate. It then drops out of the Hamiltonian formulation (although $\dot{q_i}$ may still appear in the Lagrangian formulation).

- A how-to for approaching a problem using the Hamiltonian formulation:

  - Select n generalized coordinates q_i.
  - Write T and U in terms of q_i, $\dot{q}_i$.
  - Find p_i for each.
  - Solve for the $\dot{q}_i$ in terms of p_i and q_i.
  - Write $\mathcal{H} = \sum_i p_i q_i - \mathcal{L}$ (or as $T + U$ if you know coordinates are natural.)
  - Write Hamilton's equations.
  - Solve the $2n$ first-order differential equations.