Duke University
Department of Physics

Physics 361 Spring Term 2020

WUN2K FOR LECTURE 13

These are notes summarizing the main concepts you need to understand and be able to apply.

- **Hamilton's Principle** can be considered the basis of classical mechanics. It states: *Of all the possible paths along which a dynamical system may move from one point to another point within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.*
  - This can be stated in terms of calculus of variations as $\delta \int_{t_1}^{t_2}(T - U)dt = 0$. For $\mathcal{L} \equiv T - U = \mathcal{L}(x_i, \dot{x}_i)$, the **Lagrangian**, this can be written $\delta \int_{t_1}^{t_2} \mathcal{L}(x_i, \dot{x}_i)dt = 0$.
  - **Lagrange's equations of motion** are the Euler-Lagrange equations for this integrand, $\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = 0$, $i = 1, 2, 3$.
  - Lagrange's equations, Hamilton's principle, and Newton's second law are all equivalent statements.
  - For N discrete particles, there are $3N$ position coordinates; if there are m equations of constraint relating the coordinates, then there are $n = 3N - m$ **degrees of freedom** describing the motion.
  - We can define **generalized coordinates** $q_1, q_2, ..., q_n$ (the "q_i's"), which are not necessarily Cartesian coordinates but functions of them, $q_i = q_i(\vec{r}_\alpha, t), \dot{q}_i = \dot{q}_i(\vec{r}_\alpha, \dot{\vec{r}}_\alpha, t)$, $\alpha = 1, ..., N$, $i = 1, ..., n$. The derivatives $\dot{q}_i$ are known as **generalized velocities**. The state of a system can be described as a path in the abstract **configuration space** of the generalized coordinates q_i.

- If the number of degrees of freedom is equal to the number of generalized coordinates required to describe the problem, the system is **holonomic**. If the relation between the generalized coordinates and the Cartesian ones does not involve the time, the system is called **natural** or **scleronomic**.
- Hamilton's Principle for the generalized coordinates is $\delta \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i, t)dt = 0$, and Lagrange's equations of motion are $\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$, $i = 1, 2, ...n$. This applies for holonomic systems with conservative non-constraint forces.

- A "how-to" for solving problems with the Lagrangian method:

1. Write down T, U and $\mathcal{L} = T - U$ in a convenient inertial reference frame.
2. Choose some convenient set of n generalized coordinates $q_1, ..., q_n$. Find expressions for the Cartesian coordinates and velocities in terms of the q_i's and $\dot{q}_i$'s.
3. Write $\mathcal{L}$ in terms of the q_i and $\dot{q}_i$.
4. Write the n Lagrange's equations for each q_i.
5. Solve the equations to find the motion, if needed (this might be easy or hard.)