Duke University
Department of Physics

Physics 361 Spring Term 2020

WUN2K FOR LECTURE 10

These are notes summarizing the main concepts you need to understand and be able to apply.

- A specific example of a nonlinear equation is the *van der Pol* equation, $\ddot{x}+\mu(x^2-a^2)\dot{x}+\omega_0^2 x=0$, where μ is a small positive constant, and a is another constant. This system exhibits *negative damping* if $|x|<|a_c|$, where a_c, some critical value: this is a phenomenon in which amplitude increases with time. If $|x|>|a_c|$, the usual diminishing amplitude and "spiraling in" on the phase diagram results. In this system, in both cases, the motion moves towards a stable oscillation, which in phase space is known as a **limit cycle**.

- The **plane pendulum** — a mass m oscillating on a rigid rod — is another nonlinear system, with equation of motion (no forcing) $\ddot{\phi}+\omega_0^2\sin\phi=0$. For small oscillations, this is approximately the simple linear pendulum. There are analytic techniques for next-approximation solutions, but we will use a computer. The phase diagram is helpful for understanding properties of bounded and unbounded motion.

- A driven, damped plane pendulum of mass m and length L has the EOM $\Gamma = I\ddot{\phi} = -b\dot{\phi} - mgL\sin\phi + F_0 L\cos\omega t$, where I is the moment of inertia, b is the damping coefficient, and F_0 is tangential drivign force of angular frequency ω. We can rewrite this EOM in terms of $\beta=\frac{b}{2m}$, $\omega_0^2=g/L$ and the dimensionless **drive strength** $\gamma=\frac{F_0}{mL\omega_0^2}=\frac{F}{mg}$. The new EOM is $\ddot{\phi}+2\beta\dot{\phi}+\omega_0^2\sin\phi=\gamma\omega_0^2\cos\omega t$. The solutions to this equation exhibit quasiperiodic behavior and chaos, depending on the parameters. The γ for the nonlinear pendulum is known as a "control parameter" for the system. For small driving $\gamma << 1$, the oscillator is nearly linear, and one gets a near-sinusoidal solution. As γ increases,

the sinusoids get distorted by the presence of harmonics. Eventually one sees *period-doubling* of the oscillations, as γ continues to increase. A *bifurcation point* is the γ value at which the period doubles. A "bifurcation diagram" for the nonlinear pendulum plots ϕ at a periodic time versus γ. An approximate behavior, found in many nonlinear systems is that the bifurcation points γ_n get progressively more finely spaced, such that $(\gamma_{n+1} - \gamma_n) \sim \frac{1}{\delta}(\gamma_n - \gamma_{n-1})$, where $\gamma = 4.6692016$ is the **Feigenbaum number**. Eventually, when the bifurcation points get extremely close together as a function of γ, the system is chaotic and exhibits erratic oscillations.

- A **Poincaré section** is a visualization tool that helps understand how the phase diagram describing a system evolves: it's a projection of successive snapshots in time of the $\dot{\phi}$ vs ϕ plane. When a system is chaotic, the Poincareé section often exhibits rich structure, and *strange attractors*.