Homework # 6 Solutions

Problem 1 Fourier Theorem

Periodic function w/ period T

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{C}_n e^{j\omega_n t}, \quad \hat{C}_n = \frac{1}{T}\int_{t'}^{t'+T} f(t) e^{-j\omega_n t}\, dt$$

$$\omega_n = \frac{2\pi n}{T} = n\omega_1$$

$$\omega_1 = \frac{2\pi}{T}$$

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{C}_n \left[\cos\omega_n t + j\sin\omega_n t\right]$$

$$= \hat{C}_0\left[\overset{1}{\cancel{\cos}(0)} + \overset{0}{\cancel{j\sin}(0)}\right] + \sum_{n=-\infty}^{-1} \hat{C}_n\left[\underset{\text{even}}{\cos(n\omega_1 t)} + j\underset{\text{odd for } n\to -n}{\sin(n\omega_1 t)}\right]$$

$$+ \sum_{n=1}^{\infty} \hat{C}_n\left[\cos(n\omega_1 t) + j\sin(n\omega_1 t)\right]$$

$$= \hat{C}_0 + \sum_{n=1}^{\infty} \hat{C}_n \cos(n\omega_1 t) + \sum_{n=-\infty}^{-1} \hat{C}_n \cos(n\omega_1 t)$$

$$+ j\left[\sum_{n=1}^{\infty} \hat{C}_n \sin(n\omega_1 t) + \sum_{n=-\infty}^{-1} \hat{C}_n \sin(n\omega_1 t)\right]$$

We have

$$\hat{C}_n = \frac{1}{T}\int_{t'}^{t'+T} f(t)\left[\cos(n\omega_1 t) - j\sin(n\omega_1 t)\right] dt$$

$$\Rightarrow \hat{C}_n = a_n + j b_n$$

$$\hat{C}_{-n} = \frac{1}{T}\int_{t'}^{t'+T} f(t)\left[\cos(n\omega_1 t) + j\sin(n\omega_1 t)\right] dt$$

$$\Rightarrow \hat{C}_{-n} = a_n - j b_n$$

Now plug in

$$f(t) = \hat{C}_0 + \sum_{n=1}^{\infty} \hat{C}_n \cos(n\omega_1 t) + \sum_{n=1}^{\infty} \hat{C}_{-n} \cos(n\omega_1 t)$$

$$+ j \sum_{n=1}^{\infty} \hat{C}_n \sin(n\omega_1 t) - j \sum_{n=1}^{\infty} \hat{C}_{-n} \sin(n\omega_1 t)$$

$$f(t) = \hat{C}_0 + \sum_{n=1}^{\infty} \Big[(a_n + \cancel{jb_n}) \cos(n\omega_1 t) + (a_n - jb_n) \cos(n\omega_1 t)$$

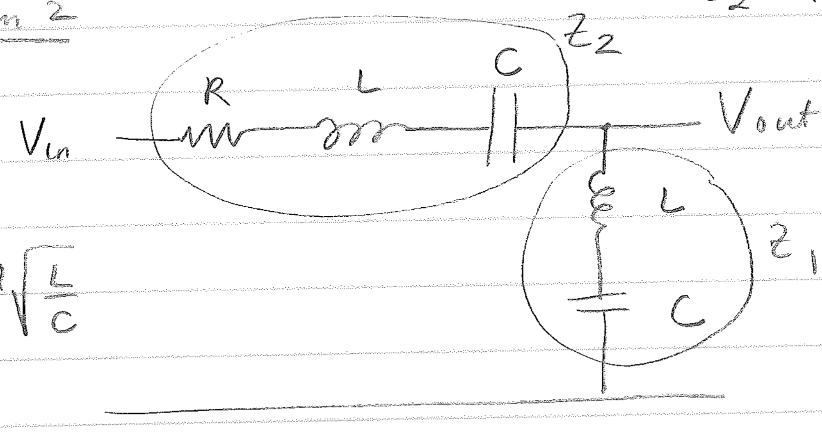
$$+ j(\cancel{a_n} + jb_n) \sin(n\omega_1 t) - j(a_n - jb_n) \sin(n\omega_1 t) \Big]$$

$$\boxed{f(t) = \hat{C}_0 + 2 \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_1 t) - b_n \sin(n\omega_1 t) \right]}$$

which is in the required form

Problem 2

$Z_2 = R + j\omega L + \frac{1}{j\omega C}$



$R = 2\sqrt{\frac{L}{C}}$

$Z_1 = j\omega L + \frac{1}{j\omega C}$

Treat as voltage divider

Transfer function $H(j\omega) = \frac{Z_1}{Z_1 + Z_2}$

$$H(j\omega) = \frac{j\omega L + \frac{1}{j\omega C}}{R + 2(j\omega L) + \frac{2}{j\omega C}} \qquad j\omega \to s$$

$$\hat{H}(\hat{s}) = \frac{\hat{s}L + \frac{1}{\hat{s}C}}{R + 2\hat{s}L + 2/\hat{s}C} = \frac{\hat{s}^2 LC + 1}{\hat{s}RC + 2\hat{s}^2 LC + 2}$$

Plug in $R = 2\sqrt{L/C}$ $\quad \hat{H}(\hat{s}) = \frac{\hat{s}^2 LC + 1}{2\hat{s}\sqrt{LC} + 2\hat{s}^2 LC + 2}$

$$\boxed{H(s) = \frac{\hat{s}^2 LC + 1}{2(1 + \sqrt{LC}\,\hat{s} + LC\hat{s}^2)}}$$

Zeroes: $\hat{s}^2 LC + 1 = 0$

$\hat{s} = \pm j\sqrt{\frac{1}{LC}}$

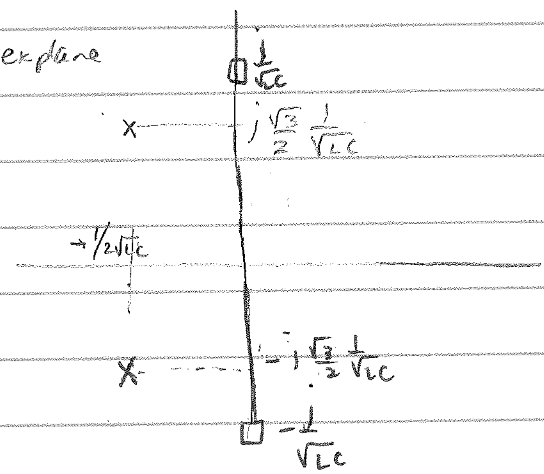
Poles: $1 + \hat{s}\sqrt{LC} + LCs^2 = 0$

$a = LC$
$b = \sqrt{LC}$
$c = 1$

$$\hat{s} = \frac{-\sqrt{LC} \pm \sqrt{LC - 4LC}}{2LC} = -\frac{1}{2\sqrt{LC}} \pm j\frac{\sqrt{3}}{2\sqrt{LC}}$$

Poles: $s_\pm = \dfrac{1}{2\sqrt{LC}}\left(-1 \pm j\sqrt{3}\right)$

On complex plane



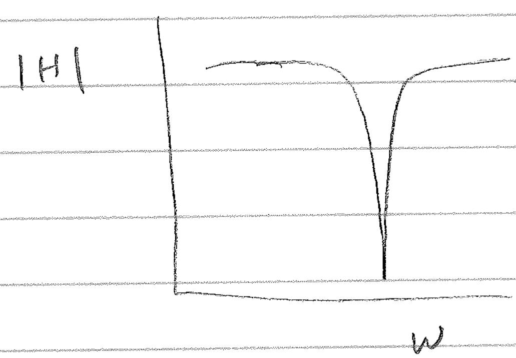
Bode plot $\qquad w_0 = \dfrac{1}{\sqrt{LC}} \qquad s^2 \to -w^2, \quad s \to jw$

Write as $\hat{H}(jw) = \dfrac{-w^2/w_0^2 + 1}{2\left(1 + jw/w_0 - w^2/w_0^2\right)}$

$w \to \infty \qquad |\hat{H}| \propto 1/2$

$w \to 0 \qquad |\hat{H}| \propto -1/2$

zeros for $w = w_0$



A notch filter

In[58]:=
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ClearAll["Global`*"]
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In[59]:=
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H = (1 - omega^2 / omega0^2) / (2 (1 + I omega / omega0 - omega^2 / omega0^2))
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Out[59]=

$$\frac{1-\frac{\text{omega}^2}{\text{omega0}^2}}{2\left(1-\frac{\text{omega}^2}{\text{omega0}^2}+\frac{i\,\text{omega}}{\text{omega0}}\right)}$$

In[64]:=
```
LogLogPlot[Abs[H] /. omega0 → 1, {omega, 0.01, 10000.},
 PlotStyle → Thickness[0.01], PlotRange → {0.001, 0.6}]
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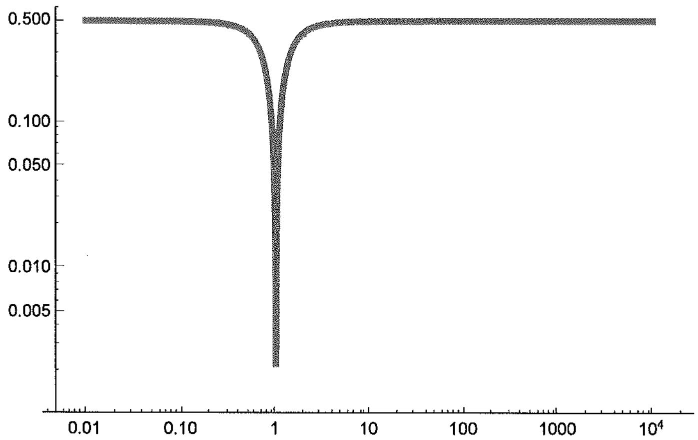
Out[64]=



2.18) Sawtooth signal, $T_{saw} = 1$ ms, V_p at peak

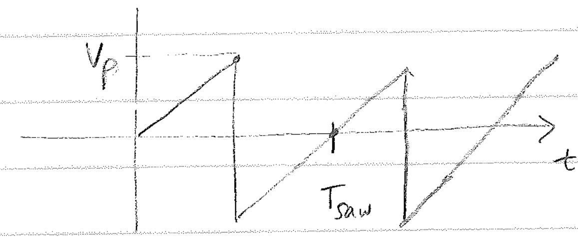
$V_p = 1$ V

Want a circuit that will give a 3 kHz sinusoid as output

Want

$T = \frac{1}{f} = \frac{1}{3 \times 10^3}$

$T = \frac{1}{3}$ ms



$\omega_{saw} = \frac{2\pi}{T_{saw}}$

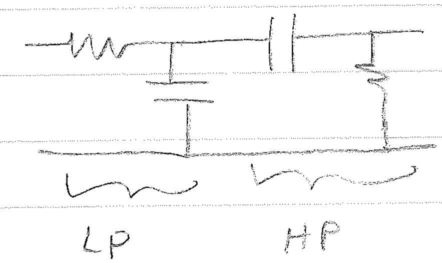
We saw the Fourier components of the sawtooth in Section 2.8

$$f(t) = \frac{2V_p}{\pi}\left[\sin[\omega_{saw} t] - \frac{1}{2}\sin(2\omega_{saw} t) + \frac{1}{3}\sin(3\omega_{saw} t) - \frac{1}{4}\sin(\sin 4\omega_{saw} t) + \dots\right]$$

we want this component only which has 3 x the frequency

$\Rightarrow$ design a bandpass filter that removes frequencies higher and lower than $\omega = 2\pi(3 \times 10^3)$ rad/s

$= 18.8 \times 10^3$ rad/s



2nd stage should have higher impedance (or else buffer)

Choose $\omega_{C_{LP}} = 4\omega_{saw}$

$R = 100\,\Omega$

$C = \frac{1}{4(100)(18.8 \times 10^3)} = 1.3 \times 10^{-7}$ F

Higher impedance

$\omega_{C_{HP}} = 2\omega_{saw}$

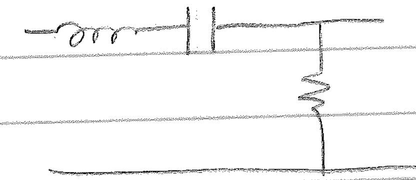
$R = 10\,k\Omega$

$C = \frac{1}{2(10^4)(18.8 \times 10^3)} = 2.6 \times 10^{-9}$ F

$\omega_c = \frac{1}{RC}$

$C = \frac{1}{R\omega_c}$

But a more efficient way is with a bandpass LRC circuit, like the one in class



$$\hat{H} = \frac{j\omega/\omega_1}{\left(1-\omega^2/\omega_1\omega_2 + j\omega/\omega_1\right)}$$

$\omega_1 = \frac{1}{RC}$

$\omega_2 = \frac{R}{L}$

Choose $\sqrt{\omega_1\omega_2} = \omega_0 = \frac{1}{\sqrt{LC}} = 2\pi(3\text{ kHz})$

(resonant frequency)

Choose $LC = (2\pi\, 3\times10^3)^{-2} = 2.8\times10^{-9}$

Choose $C = 10^{-6}\,F$, $L = 2.8\,mH$

We want reasonable attenuation of $2\omega_{saw}$ and $4\omega_{saw}$, so need to choose a relatively small R

$R = 5\,\Omega$ (check with plot - looks fine)

Amplitude of filtered sine wave will be

$$\sim \frac{2V_p}{\pi}\left(\frac{1}{3}\right) = \frac{2(1)}{\pi\, 3} = \underline{\underline{0.212\text{ V}}}$$

```
In[157]:= ClearAll["Global`*"]

In[158]:= Hbp = I (omega / omega1) / (1 - omega^2 / (omega1 omega2)) + I omega / omega1
```

Out[158]= $\dfrac{i\ \text{omega}}{\text{omega1}} + \dfrac{i\ \text{omega}}{\text{omega1}\left(1 - \frac{\text{omega}^2}{\text{omega1 omega2}}\right)}$

```
In[159]:= omega1 = 1 / (R Cap)
```

Out[159]= $\dfrac{1\,000\,000}{R}$

```
In[160]:= omega2 = R / L
```

Out[160]= $\dfrac{R}{L}$

```
In[161]:= R = 5; L = 2.8 × 10^(-3); Cap = 10^(-6);

In[167]:= 1 / Sqrt[L Cap]

Out[167]= 18898.2

In[162]:= Abs[Hbp] /. omega → 2 Pi 3000.

Out[162]= 18.4166

In[163]:= wsaw = 2 Pi 1000.

Out[163]= 6283.19

In[165]:= LogPlot[Abs[Hbp], {omega, wsaw, 4 wsaw},
            PlotStyle → Thickness[0.01], PlotRange -> {{wsaw, 4 wsaw}, {0.1, 100000.}}]
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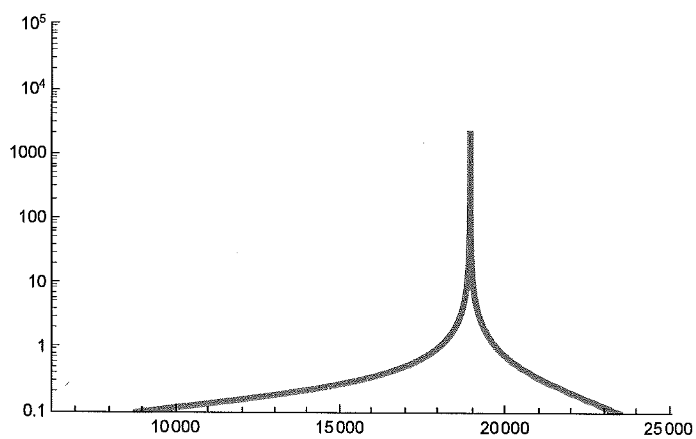
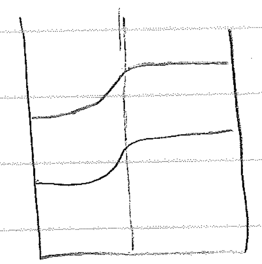
Out[165]=



3.1) These are basically in the text

p-n junction

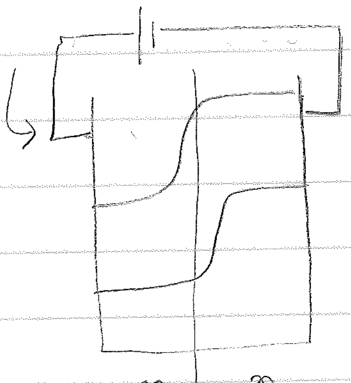
No bias



n (+) p (-)

Net current is zero

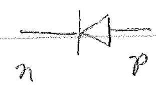
Reverse bias



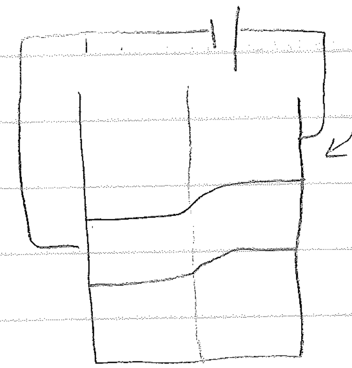
n (+) more p (-) more

⟶

Small current from n to p



Forward bias



n (+) p (-)

⟵

larger current from p to n