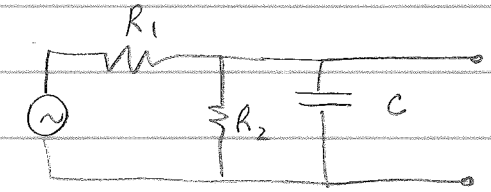
Homework #5 Solutions

1)



$$\hat{H}(j\omega) = \frac{Z_{R_2\|C}}{Z_{R_2\|C} + R_1} = \frac{\dfrac{R_2 \cdot 1/j\omega C}{R_2 + 1/j\omega C}}{\dfrac{R_2 \cdot \frac{1}{j\omega C}}{R_2 + 1/j\omega C} + R_1}$$

(same question as 3 (4), which asks for $|H|$)

$$= \frac{R_2 \cdot 1/j\omega C}{R_2 \cdot 1/j\omega C + R_1(R_2 + 1/j\omega C)}$$

$$= \frac{R_2}{R_2 + R_1 R_2 j\omega C + R_1}$$

$$\boxed{\hat{H}(j\omega) = \frac{R_2}{(R_1 + R_2) + (R_1 R_2 \omega C)j}} = \frac{R_2\left((R_1 + R_2) - j(R_1 R_2 C \omega)\right)}{\left[(R_1 + R_2)^2 + (R_1 R_2 C \omega)^2\right]}$$

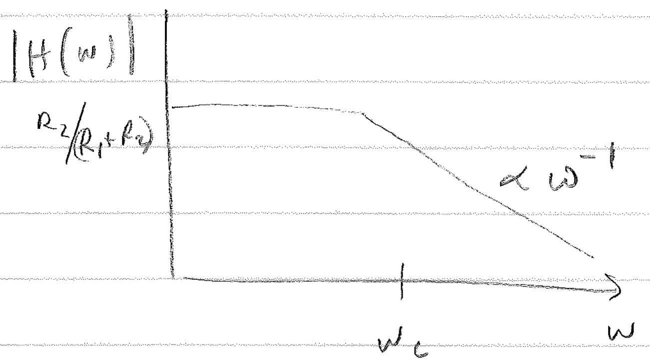
We are not given numerical values, so plot in terms of corner frequency

"Regime change" when $(R_1 + R_2) = R_1 R_2 \omega C$

for $\omega = \omega_c = \dfrac{(R_1 + R_2)}{R_1 R_2 C}$

Sketch: $\omega \gg \omega_c$, $|H| \propto \omega^{-1}$

$\omega \ll \omega_c$, $|H| \sim R_2/(R_1 + R_2)$ | low-pass

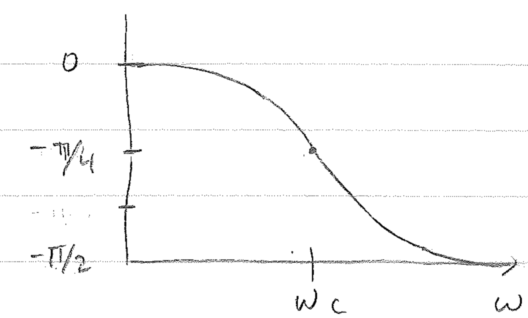


$$|H| = \frac{R_2}{\left((R_1+R_2)^2 + (\omega R_1 R_2 C)^2\right)^{1/2}} = \frac{R_2}{(R_1+R_2)\left[1+\left(\frac{R_1R_2C}{R_1+R_2}\,\omega\right)^2\right]^{1/2}}$$

$$= \frac{R_2}{(R_1+R_2)\left[1+\left(\frac{\omega}{\omega_c}\right)^2\right]^{1/2}}$$

Phase shift $\quad H(j\omega) = \dfrac{R_2\left[(R_1+R_2) - (R_1R_2\omega C)j\right]}{(R_1+R_2)^2 + (R_1R_2\omega C)^2}$

$$\theta = \tan^{-1}\left(\frac{-R_1R_2\omega C}{R_1+R_2}\right) = \tan^{-1}\left(\frac{-\omega}{\omega_c}\right)$$



Low-pass filter type phase shift

2) To show: $P_{avg} = \frac{1}{2} Re(\hat{V}^* \hat{I}) = \frac{1}{2} Re(\hat{V} \hat{I}^*)$

Write these as $\hat{V} = V_0 e^{j(\omega t + \phi_1)}$

$\hat{I} = I_0 e^{j(\omega t + \phi_2)}$

$$P(t) = \underbrace{V(t) I(t)}_{\text{real parts}} = V_0 \cos(\omega t + \phi_1) I_0 \cos(\omega t + \phi_2)$$

$$P_{avg} = \frac{1}{T} \int_0^T V(t) I(t)\, dt \quad \text{over one cycle}$$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$= \frac{V_0 I_0}{T} \int_0^T \cos(\omega t + \phi_1) \cos(\omega t + \phi_2)\, dt$$

$$= \frac{V_0 I_0}{T} \int_0^T \left[\cos \omega t \cos \phi_1 - \sin \omega t \sin \phi_1\right]\left[\cos \omega t \cos \phi_2 - \sin \omega t \sin \phi_2\right] dt$$

odd functions integrated over a cycle (with single $\sin \omega t$) are zero

$$= \frac{V_0 I_0}{T} \int_0^T \left[\cos^2 \omega t \cos \phi_1 \cos \phi_2 + \sin^2 \omega t \sin \phi_1 \sin \phi_2\right] dt$$

$\cos(\phi_1 - \phi_2) = \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2$

$$= \frac{V_0 I_0}{T} \left\{ \underbrace{\left[\int_0^T \left[\cos^2 \omega t\right] dt\right]}_{T/2} \cos \phi_1 \cos \phi_2 + \underbrace{\left[\int_0^T \left[\sin^2 \omega t\right] dt\right]}_{T/2} \sin \phi_1 \sin \phi_2 \right.$$

$$= \frac{V_0 I_0}{2} \left[\underbrace{\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2}_{\cos(\phi_1 - \phi_2)}\right]$$

$$\Rightarrow P_{avg} = \frac{V_0 I_0}{2} \cos(\phi_1 - \phi_2)$$

$$\frac{1}{2}\,\mathrm{Re}\left(\hat{V}^{*} I\right) = \frac{1}{2}\,\mathrm{Re}\left[V_0 e^{-j(wt+\phi_1)} I_0 e^{j(wt+\phi_2)}\right]$$

$$= \frac{V_0 I_0}{2}\,\mathrm{Re}\left[e^{j(-\phi_1+\phi_2)}\right]$$

$$= \frac{V_0 I_0}{2}\cos(-\phi_1+\phi_2)$$

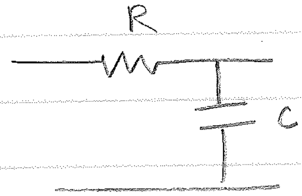
$$= \frac{V_0 I}{2}\cos(\phi_1-\phi_2)$$, same result

$$\frac{1}{2}\,\mathrm{Re}\left(\hat{V}\hat{I}^{*}\right) = \frac{1}{2}\,\mathrm{Re}\left[V_0 e^{j(wt+\phi_1)} I_0 e^{-j(wt+\phi_2)}\right]$$

$$= \frac{V_0 I_0}{2}\,\mathrm{Re}\left[e^{j(\phi_1-\phi_2)}\right]$$

$$= \frac{V_0 I_0}{2}\cos(\phi_1-\phi_2)$$, again same result

2.10) RC low-pass filter with $\frac{|V_{out}|}{|V_{in}|} = 0.5$ @ $f = 5\ kHz$



$$\hat{H} = \frac{1}{1 + j\omega RC} = \frac{1 - j\omega RC}{1 + \omega^2 (RC)^2}$$

$$\frac{|V_{out}|}{|V_{in}|} = |H| = \frac{1}{(1 + \omega^2 (RC)^2)^{1/2}} = \frac{1}{(1 + (\omega/\omega_c)^2)^{1/2}}$$

for $\frac{|V_{out}|}{|V_{in}|} = 0.5 = \frac{1}{(1 + (\omega/\omega_c)^2)^{1/2}}$

$$1 + \frac{\omega^2}{\omega_c^2} = \frac{1}{0.5^2}$$

$$\frac{\omega^2}{\omega_c^2} = \frac{1}{0.5^2} - 1$$

$$\omega_c^2 = \frac{\omega^2}{\frac{1}{(0.5)^2} - 1} \qquad \omega = 2\pi f = 2\pi (5 \times 10^3)$$

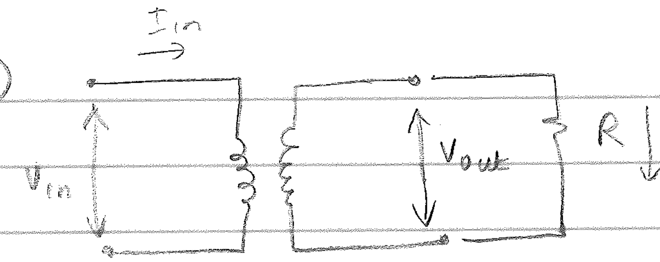
$$\Rightarrow \omega_c = 18.1\ rad/s = 28.5\ kHz$$

$RC = \frac{1}{\omega_c} = 5.5 \times 10^{-5}\ s^{-1}$ } so choose, say $R = 1\ k\Omega$

$C = 1/\omega_c R = 5.5 \times 10^{-8} F$

(any combination with specified product OK)

2.13)



$V_{in} = 120\ V_{rms}$

$V_{out} = 12\ V_{rms}$

$R = 20\ \Omega$

$$V_{out} = \left(\frac{n_2}{n_1}\right) V_{in}$$

$$\text{Turns ratio} = \frac{n_2}{n_1} = \frac{V_{out}}{V_{in}} = \frac{12}{120} = 0.1$$

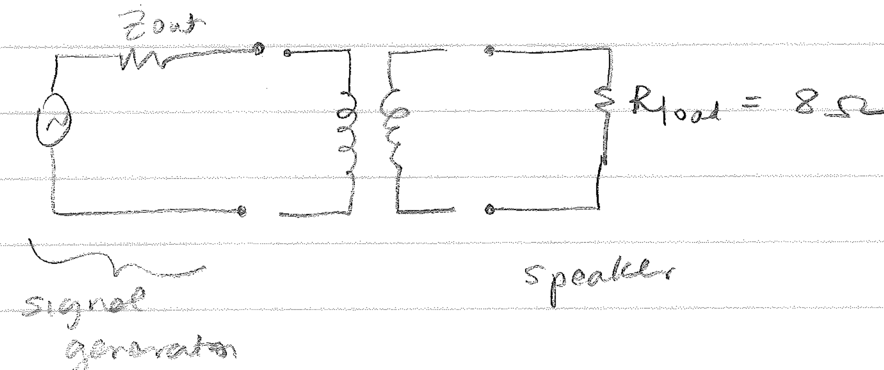
In secondary $$I_{out} = \frac{V_{out}}{R} = \frac{12\ V_{rms}}{20\ \Omega} = 0.6\ A_{rms}$$

In primary $$I_{in} = \left(\frac{n_2}{n_1}\right) I_{out} = (0.1)(0.6\ A_{rms})$$

$$I_{in} = \underline{0.06\ A_{rms}}$$

2.14) Audio signal generator

$Z_{out} = 600\,\Omega$



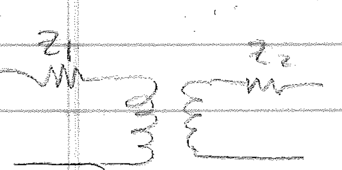
Impedance matching $R_{eff} = \left(\frac{n_1}{n_2}\right)^2 R_L$, want $R_{eff} = 600\,\Omega$

$$\text{Want } \left(\frac{n_1}{n_2}\right) = \sqrt{\frac{R_{eff}}{R_L}} = \sqrt{\frac{600}{8}} = \underline{\underline{8.66}}$$

2.15) Step-down transformer has 1.6 Ω impedance on one side, 40 Ω on the other

$$\left.\begin{aligned} Z_2 &= \frac{V_2}{I_2} = \frac{(n_2/n_1)V_1}{(n_1/n_2)I_1} \\ Z_2 &= \left(\frac{n_2}{n_1}\right)^2 Z_1 \end{aligned}\right\} \Rightarrow \frac{n_1}{n_2} = \sqrt{\frac{Z_1}{Z_2}} = \sqrt{\frac{1.6}{40}} = 0.2$$

It's a step-down transformer



so primary side has larger n

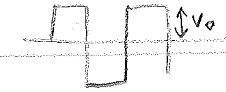
$\Rightarrow$ side 2 is the primary
side 1 is the secondary

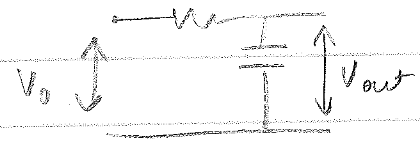
$$V_{out} = \left(\frac{n_2}{n_1}\right) V_{in}$$

$$\Rightarrow V_{out} = 0.2\,(120\ V_{rms}) = \underline{\underline{24\ V_{rms}}}$$

2.16) Low pass filter $\omega_c = 2\pi(100\ Hz)$

Square wave $\quad \omega = 2\pi f = (2\pi) 90\ Hz$
amplitude V_0





$$|H| = \frac{1}{\left(1 + (\omega/\omega_c)^2\right)^{1/2}}$$

$$\frac{\omega}{\omega_c} = \frac{90}{100} = 0.9$$

Attenuates ~~frequencies~~ greater than $\sim \omega_c$

As per example in class, a square wave consists of a sum of sinusoids (see handout)

$\omega, 3\omega, 5\omega, \ldots$; coefficients $V_0\left(\frac{2}{\pi}, -\frac{2}{3\pi}, \frac{2}{5\pi}, \ldots\right)$

$$f(t) = a_0 + \left(\frac{2}{\pi}\right) V_0 \cos \omega_1 t - \left(\frac{2}{3\pi}\right) V_0 \cos \omega_3 t + \ldots$$

The filter attenuates all components with $\omega > 2\pi(100\ Hz)$, so the only remaining component of the square wave will be the fundamental, $\omega = 2\pi(90\ Hz)$

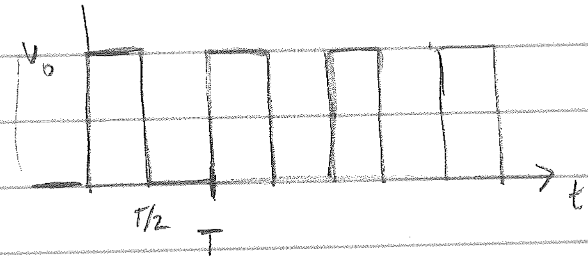
The output will look like a (slightly) distorted 90 Hz sinusoid, amplitude $\frac{2V_0}{\pi}$ (first term of Fourier series from class)

Amplitude out:

$$V_{out} \sim \frac{1}{\left(1 + (\omega/\omega_c)^2\right)^{1/2}} \cdot \frac{2V_0}{\pi}$$

$$= \frac{1}{\left(1 + (0.9)^2\right)^{1/2}} \cdot \frac{2}{\pi} (V_0)$$

$$\sim \underline{\underline{0.47\ V_0}}$$

2.17) 

$f(t) = V_0 \quad 0<t<T/2$
$= 0, \quad T/2 < t < T$

This is the square wave from class with $\frac{V_0}{2}$ offset
(see handout) and a phase shift

$$f(t) = a_0 + 2\sum_{n=1}^{\infty}\left(a_n \cos(n\omega_1 t) - b_n \sin(n\omega_1 t)\right)$$

$\omega_1 = \frac{2\pi}{T}$ In this case, the function is odd, so we can drop the a_n terms, keep the b_n ones

$$n=0 \quad a_0 = \frac{1}{T}\int_{\text{one cycle}} f(t)\cos(0)\,dt = \frac{1}{T}\int_0^{T/2} V_0\,dt = \frac{V_0}{2}$$

$$n=1,2,.. \quad b_n = -\frac{1}{T}\int_0^{T/2} V_0 \sin(n\omega_1 t)\,dt = -\frac{V_0}{T}\left.\frac{(-\cos(n\omega_1 t))}{n\omega_1}\right|_0^{T/2}$$

$\omega_1 = 2\pi/T$

$$= \frac{V_0}{n\cdot 2\pi}\left[\cos\left(\frac{n\omega_1 T}{2}\right) - 1\right] = \frac{V_0}{2\pi n}\left[\cos(n\pi) - 1\right]$$

$n = 1 \rightarrow -2$
$n = 2 \rightarrow 0$
$n = 3 \rightarrow -2$
only odd n's left

$n = 2m-1$

$$b_m = -\frac{2V_0}{(2\pi)(2m-1)}$$

$m = 1, 2, \ldots$

$$f(t) = \frac{V_0}{2} - \frac{2V_0}{2\pi}\left[\underset{n=1}{-2\sin(\omega_1 t)} - \underset{n=3}{\frac{2\sin(3\omega_1 t)}{3}} - \underset{n=5}{\frac{2\sin(5\omega_1 t)}{5}} \ldots\right.$$

$$\boxed{f(t) = \frac{V_0}{2} + \frac{2V_0}{\pi}\left[\sin\omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \frac{1}{5}\sin 5\omega_1 t + \ldots\right]}$$