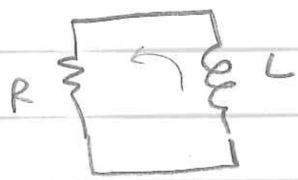
Homework # 4 Solutions

RL circuit



Kirchoff's Loop Rule

$$-IR - L\frac{dI}{dt} = 0$$

$$\frac{dI}{dt} = -\frac{IR}{L}$$

$$\int_{I_0}^{I} \frac{dI}{I} = -\int_0^t \frac{R}{L}\, dt$$

$$\ln \frac{I}{I_0} = -\frac{R}{L} t$$

$$I = I_0 e^{-R/L\, t} = I_0 e^{-t/\tau}$$

where $\tau = L/R$

$$V(t) = I(t)R = I_0 R e^{-t/\tau}$$

$Q \to x$

$I \to v$

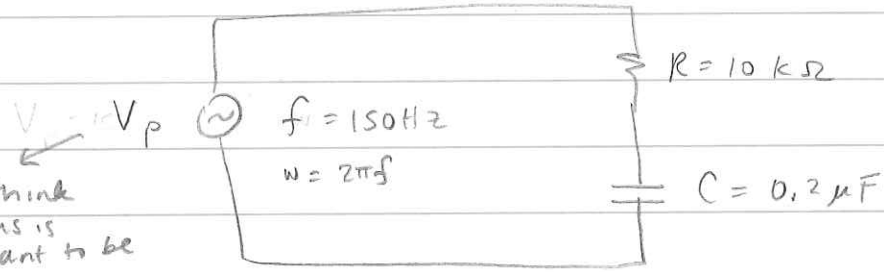
Mechanical analogy:

$\rightarrow \quad -vR - L\frac{dv}{dt} = 0$ $\rightarrow$ acceleration

like $\rightarrow \quad ma = -bv$ in mechanics

Like a constant drag force $\propto$ velocity

2.6)



I think this is meant to be Vp = 1V

$$\hat{Z}_{tot} = R + \frac{1}{jwC} = R - j(1/wC)$$

Magnitude $|\hat{z}| = \sqrt{R^2 + 1/(wC)^2} = \sqrt{(10^3)^2 + 1/((2\pi(150))(0.2\times10^{-6}))^2}$

$= \underline{\underline{11.3\ k\Omega}}$

Phase $\theta = \tan^{-1}\left(\frac{-1}{RwC}\right)$

$= \tan^{-1}\left(\frac{-1}{(10\times10^3)(2\pi)(150)(0.2\times10^{-6})}\right)$

$= -0.488\ rad$

$= \underline{\underline{-27.95^\circ}}$

2.7) $w = 10^3$ rad/s

$$\hat{V}_R = \hat{I}\,\hat{Z}_R$$

$$\hat{V}_C = \hat{I}\,\hat{Z}_c$$

$$\hat{I} = \frac{\hat{V}_p}{\hat{Z}_{tot}} = \frac{V_p e^{jwt}}{R - j/wC}$$

Across resistor:

$$\Rightarrow \hat{V}_R = \frac{V_p e^{jwt}\, R}{R - j/wC} = \frac{V_p R\, e^{jwt}(R + j/wC)}{R^2 + 1/(wC)^2}$$

Peak amplitude is the magnitude

$$|\hat{V}_R| = \frac{V_p R}{R^2 + 1/w^2C^2}\left(R^2 + 1/(wC)^2\right)^{1/2} = \frac{V_p R}{\sqrt{R^2 + 1/w^2C^2}}$$

$$|\hat{V}_R| = \frac{(1\,V)(10\times 10^3)}{\left((10\times 10^3)^2 + 1/\left((10^3)^2(0.2\times 10^{-6})^2\right)\right)^{1/2}}$$

$$|\hat{V}_R| = \underline{\underline{0.89\,V}}$$

Across capacitor: $$\hat{V}_C = \frac{V_p e^{jwt}}{R - j/wC}\left(1/jwC\right) = \frac{V_p\left(\frac{-j}{wC}\right)(R + j/wC)}{(R^2 + 1/(wC)^2)}\, e^{jwt}$$

$$= V_p\left(\frac{1/w^2C^2 - jR/wC}{(R^2 + (1/w^2C^2))}\right)e^{jwt} = V_p\left(\frac{1 - jRwC}{1 + R^2w^2C^2}\right)e^{jwt}$$

$$|\hat{V}_C| = |V_p|\frac{(1 + R^2w^2C^2)^{1/2}}{(1 + R^2w^2C^2)} = |V_p|\frac{1}{(1 + R^2w^2C^2)^{1/2}}$$

$$= \frac{(1\,V)}{(1 + (10\cdot 10^3)^2(10^3)^2(0.2\times 10^{-6})^2)^{1/2}}$$

$$|\hat{V}_C| = \underline{\underline{0.45\,V}}$$

2.8) 

$L = 1\ H$
$R = 1\ k\Omega$
$C = 2\times10^{-6}\ F$

Resonant frequency $w_0 = \dfrac{1}{\sqrt{LC}}$

$$w_0 = \frac{1}{\sqrt{(2\times10^{-6})(1)}} = \underline{\underline{707\ rad/s}}$$

2.9) $\hat{Z} = R + jwL - \frac{1}{jwC}$, $f = 2500\ Hz$

Magnitude $|\hat{Z}| = \left(R^2 + \left(wL - \frac{1}{wC}\right)^2\right)^{1/2}$

$$= \left(1000^2 + \left(2\pi(2500)(1) - \frac{1}{((2\pi)(2500)(2\times10^{-6}))}\right)^2\right)^{1/2}$$

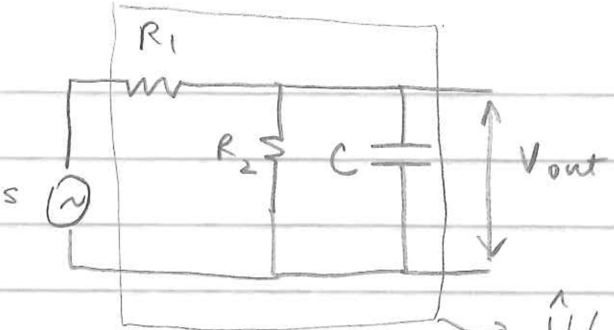
$$|\hat{Z}| = \underline{\underline{15.7\ k\Omega}}$$

Phase $\theta = \tan^{-1}\left(\dfrac{wL - \frac{1}{wC}}{R}\right) = \tan^{-1}\left(\dfrac{2\pi(2500)(1) - \frac{1}{((2\pi)(2500)(2\times10^{-6}))}}{(1000}\right)$

$\cdot 180/\pi$

$$\theta = \underline{\underline{86.35}}\ ^{\circ}$$

2.11)



$\hat{H}(jw)$ describes network response

$$|V_{out}| / |V_{in}| = |\hat{H}(jw)|$$

Treat as voltage divider

$$\hat{H}(jw) = \frac{R_2 \parallel \hat{Z}_C}{R_2 \parallel \hat{Z}_C + R_1}$$

$$= \frac{\frac{R_2(-j/wC)}{R_2 - j/wC}}{\frac{R_2(-j/wC)}{R_2 - j/wC} + R_1} = \frac{-(R_2/wC)\,j}{R_2(-j/wC) + R_1(R_2 - j/wC)}$$

$$= \frac{-R_2 j}{-jR_2 + R_1R_2wC - jR_1} = \frac{-R_2 j}{(R_1R_2wC) - j(R_2+R_1)}$$

$$= \frac{-R_2 j\left(R_1R_2wC + j(R_1+R_2)\right)}{(R_1R_2wC)^2 + (R_1+R_2)^2} = \frac{R_2(R_1+R_2) - R_2 j R_1R_2wC}{(R_1R_2wC)^2 + (R_1+R_2)^2}$$

$$|\hat{H}(jw)| = \frac{R_2}{(R_1R_2wC)^2 + (R_1+R_2)^2}\left[(R_1+R_2)^2 + (R_1R_2wC)^2\right]^{1/2}$$

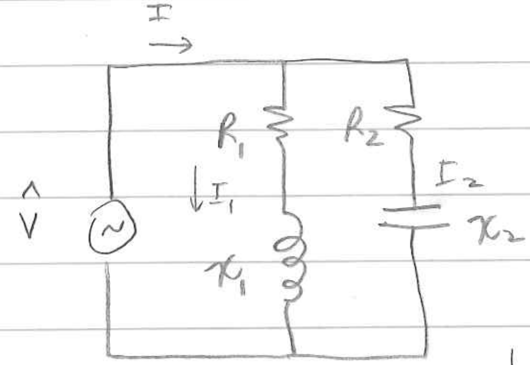
$$\boxed{\frac{|V_{out}|}{|V_{in}|} = |\hat{H}(jw)| = \frac{R_2}{\sqrt{(R_1+R_2)^2 + (wCR_1R_2)^2}}}$$

as required

$\hat{z}_1 = jx_1 = j\omega L$

$\hat{z}_2 = jx_2 = -53.1\,\Omega$
$= -j/\omega C$
$-1/\omega C = x_2 = -53.1\,\Omega$
$C = -1/(x_2\omega)$

2.12)



$R_1 = 20\,\Omega$

$x_1 = 37.7\,\Omega$ ← this means

$x_2 = -53.1\,\Omega$

$R_2 = 10\,\Omega$

$\hat{z}_1 = jx_1$

$\hat{z}_2 = jx_2$

$\hat{z}_1 = j\omega L, \quad \hat{z}_2 = -j/\omega C$

$\hat{V}_{rms} = 230\,V$

$f = 60\,Hz, \quad \omega = 2\pi(60)\,\frac{rad}{s}$

$V_{peak} = |\hat{V}| = \sqrt{2}(230)\,V$

$$\hat{V} = \hat{Z}_{tot}\,\hat{I}$$

$$\hat{I} = \frac{\hat{V}}{\hat{Z}_{tot}} \qquad \hat{Z}_{tot} = (R_1 + \hat{z}_1)\,||\,(R_2 + \hat{z}_2)$$

$$\hat{Z}_{tot} = \frac{(R_1 + jx_1)(R_2 + jx_2)}{R_1 + jx_1 + R_2 + jx_2}$$

Do this with Mathematica – see attached

```
In[219]:= ClearAll["Global`*"]

In[220]:= Z = (R1 + I chi1) * (R2 + I chi2) / (R1 + I chi1 + R2 + I chi2)
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$$\text{Out[220]= } \frac{(i\,\text{chi1} + \text{R1})\ (i\,\text{chi2} + \text{R2})}{i\,\text{chi1} + i\,\text{chi2} + \text{R1} + \text{R2}}$$

```
In[221]:= Current = V / Z
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$$\text{Out[221]= } \frac{(i\,\text{chi1} + i\,\text{chi2} + \text{R1} + \text{R2})\ \text{V}}{(i\,\text{chi1} + \text{R1})\ (i\,\text{chi2} + \text{R2})}$$

```
In[222]:= omega = 2 Pi f
```

Out[222]= $2\ f\ \pi$

```
In[223]:= Abs[Current]
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$$\text{Out[223]= Abs}\left[\frac{(i\,\text{chi1} + i\,\text{chi2} + \text{R1} + \text{R2})\ \text{V}}{(i\,\text{chi1} + \text{R1})\ (i\,\text{chi2} + \text{R2})}\right]$$

```
In[224]:= omega = 2 Pi f
```

Out[224]= $2\ f\ \pi$

```
In[225]:= chi1 = 37.7

Out[225]= 37.7

In[226]:= chi2 = -53.1

Out[226]= -53.1

In[227]:= V = 230
```

(rms)

```
Out[227]= 230

In[228]:= Abs[Current] /. R1 → 20 /. R2 → 10 /. f → 60.

Out[228]= 3.36346
```

3.36 A rms → (note this is rms current)

```
In[229]:= (Arg[Current] /. R1 → 20 /. R2 → 10 /. f → 60.) * 180. / Pi

Out[229]= -9.89206
```

-9.89° phase shift