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A pseudo-matched filter for chaos

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A matched filter maximizes the signal-to-noise ratio of a signal. In the recent work of Corron et al. [Chaos 20, 023123 (2010)], a matched filter is derived for the chaotic waveforms produced by a piecewise-linear system. This system produces a readily available binary symbolic dynamics that can be used to perform correlations in the presence of large amounts of noise using the matched filter. Motivated by these results, we describe a pseudo-matched filter, which operates similarly to the original matched filter. It consists of a notch filter followed by a first-order, low-pass filter. We compare quantitatively the matched filter’s performance to that of our pseudo-matched filter using correlation functions. On average, the pseudo-matched filter performs with a correlation signal-to-noise ratio that is 2.0 dB below that of the matched filter. Our pseudo-matched filter, though somewhat inferior in comparison to the matched filter, is easily realizable at high speed (>1 GHz) for potential radar applications. © 2012 American Institute of Physics. [http://dx.doi.org/10.1063/1.4754437]

Recently, Corron et al. developed an analog matched filter that uses the digital symbolic dynamics of a piecewise-linear chaotic system. This matched filter for chaos is derived from the analytic solution of their uniquely designed chaotic system. However, the majority of chaotic systems do not have analytically solvable dynamics. In order to extend the ideas presented by Corron et al. to other systems with symbolic dynamics, this problem must be approached empirically, rather than analytically. To do so, we derive a pseudo-matched filter for this particular piecewise-linear system. The matched filter, which is optimal, serves as a baseline to benchmark the pseudo-matched filter’s performance. Quantitative comparisons are established using the two filters in a chaos radar application. Our hope is that, based on our analysis of the chaos in the Fourier and time domains, filtering techniques can be developed for applications that use the symbolic dynamics of higher dimensional chaotic systems in the presence of noise.

I. INTRODUCTION

A conventional radar system measures the distances of targets in the field of view using a signal source, a transmitter, and a receiver. In Fig. 1, a radar transmitter broadcasts a signal $u(t)$ from the source toward an intended target, and the receiver detects a version of the transmitted signal that is reflected off of the target. Prior to transmission, a copy of the radar signal’s information is digitally sampled and stored as $s_n$. The received signal $v(t)$, which picks up environmental noise, is filtered and correlated with $s_n$. Typical radar signals are non-repeating in order to avoid multiple points of correlation and, therefore, the correlation will peak only when the transmitted and received signals are aligned. Using the time of the output peak in the correlation, the measured range from the transmitter to the target is determined.

The performance of a radar is determined by its ability to identify the correlation time between the transmitted and received signal. In the correlation function, the width of the peak scales inversely with the bandwidth of the transmitted signal and sets the spatial resolution of the radar. In addition, the height of the correlation peak above the noise floor, also known as signal-to-noise ratio (SNR), is determined by the length of the transmitted and sampled waveforms as well as the noise from the environment. Thus, the digital storage capacity of the radar sets the maximum SNR in the correlation measurement. State-of-the-art radar systems use high-frequency, broadband signals, where the digital sampling and storage of the signals can be costly. These radars must balance the bandwidth and cost of the systems design while maintaining its performance.

A simple example of an inexpensive, non-repeating, broadband signal source is amplified electrical noise. In the past, electrical noise has been used by radar systems to perform ranging measurements.\(^1\)\(^2\) The high bandwidth of these noise generators yields high-resolution ranging information but requires fast sampling and large data storage capacities. Some recent techniques have been proposed to minimize the necessary sampling capacity of noise radars using analog delay lines for signal storage.\(^3\) But these methods limit the ranging capabilities of the radar.

Various deterministic signal sources have been studied in efforts to minimize the necessary data storage rate and capacity of a radar. As one example, pseudo-random binary sequences (PRBSs) are often used as a radar signal sources. To be implemented as a radar signal source, a PRBS is up-converted to a suitable frequency band before transmitting and then down-converted at the receiver before correlation.\(^4\)\(^5\) The main advantage of a PRBS is the ability to use one-bit digital samplings of the binary sequences, thereby requiring low amounts of data-storage capacity. This allows for longer sequences of the transmitted waveform to be stored, thus enhancing the radar’s SNR without increasing the cost of the
ties that make them ideally suited as signal sources for radar form generators, which are believed to have several proper-
serves as the signal source (see Fig. 1) and is transmitted,
chaotic dynamics. In the proposed radar, the chaotic signal
state (symbolic dynamics) that completely characterizes its
simultaneously an analog chaotic signal and a binary switching
noise-like properties.

system. The main disadvantage of a PRBS is that it requires
computational power to generate and its sequence eventually
repeats, which ultimately limits its performance. Many other
radar concepts like this one exist, each with advantages and
disadvantages, and today the radar community continues to
develop broadband signal sources.

One novel approach to a radar is to use chaotic wave-
form generators, which are believed to have several proper-
ties that make them ideally suited as signal sources for radar applications.8 One defining feature of a chaotic system is
that it can generate a signal that does not repeat in time. Cha-
otic signals are also often inherently broadband. High-speed
chaos has been observed in optical and electronic systems
with frequency bandwidths that stretch across several giga-
hertz.9–11 Such broadband chaos has been studied for its
applications in high-resolution ranging and in imaging.12–15
These applications use the non-repeating aspects of the high-
speed chaos.

However, conventional chaos radars do not take full
advantage of the chaotic signal source. In addition to produc-
ning broadband, non-repeating signals, chaotic systems are
deterministic and extremely sensitive to small perturbations.
By not using these properties, chaos radars add no benefit
over noise radars, requiring high-sampling to perform corre-
lations. In addition, many modern proposals for chaos-based
applications implement chaos using digital synthesis,16
which requires digital processors and up/down-conversion
for radar. Thus, to take full advantage of chaotic systems in a
radar application, a chaos radar needs to benefit from the
determinism or sensitivity of analog chaos in addition to its
noise-like properties.

Recently, Corron et al. proposed a novel chaos radar
concept that uses the analog dynamics produced by a
piecewise-linear harmonic oscillator.17–19 It produces simul-
taneously an analog chaotic signal and a binary switching
state (symbolic dynamics) that completely characterizes its
chaotic dynamics. In the proposed radar, the chaotic signal
serves as the signal source (see Fig. 1) and is transmitted,
while a copy of a switching state is stored using a one-bit
digital sampling. For a radar receiver, Corron et al. derived
the analytical form of a filter that is matched to a basis func-
tion, which is inherently encoded in the chaos produced by
the system. A matched filter is a linear operation that opti-
mizes the SNR of a signal in the presence of additive white
Gaussian noise (AWGN).20 Their matched filter, when
applied to the received signal, can be used in conjunction
with the stored symbolic dynamics in the correlation operation of
Fig. 1. This technique uses deterministic aspects of the chaos,
making it an improvement over a noise radar. With
reduced data storage and an optimal SNR, their architecture
could effectively reduce the cost of a radar system.

Corron et al. implement their piecewise-linear design
using an inductance-resistance-capacitance (LRC) oscillator
that operates in the kHz frequency range.18 It is difficult to
realize a high-frequency version (>1 GHz) of this system
because of parasitic capacitances and inductances associated
with high-speed electronics.21 In addition, at high-speeds,
there are inherent time delays in the propagation of signals
in LRC circuits.22 As it stands, there is no high-frequency
realization of the piecewise-linear system from Ref. 18 or
the associated matched filter. In order to have resolutions
that are comparable to state-of-the-art radar systems, the
waveforms and switching states produced by this chaotic
system must be scaled to higher-frequencies and to broader
bandwidths. Thus, techniques for simplifying the design by
Corron et al. and increasing its speed are of interest.

To begin simplifying their approach, we present a set of
standard filters (first order low-pass filter and notch filter).
Cascading these standard filters allows us to realize a pseudo-
matched filter for the chaos produced by the piecewise-linear
harmonic oscillator. We define a pseudo-matched filter as a
sub-optimal linear operation (when compared to the matched
filter) that performs comparably to the matched filter for the
system. As we will show, the pseudo-matched filter is empiri-
cally derived from the observed spectral properties of the
chaos and is suited to perform correlations with the system’s
symbolic dynamics for applications like chaos radar.

In addition, as a first step toward a high-frequency
implementation of Corron et al.’s system, we present a sim-
ple, high-speed design for the pseudo-matched filter. The
design includes filters that operate at high-frequencies, which
are inexpensive, well characterized, and readily available.
Motivated by the chaos-based radar system proposed by Ref.
19, we are interested in high-speed architectures that also
take advantage of a chaotic system’s symbolic dynamics.
Our pseudo-matched filter for chaos shows that the approach
by Corron et al. does not need an analytically derived filter
and can benefit from integrating readily available filters for
radar applications.

II. MATCHED FILTER REVIEW

To motivate our analysis, we briefly review the charac-
teristics of the chaotic system and matched filter presented in
Ref. 18 within the context of a radar application. Consider a
harmonic oscillator with negative damping −β and with a
piecewise-constant driving term s(t) whose behavior is gov-
erned by the differential equation

\[ \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + \omega_0^2 x = s(t) \]
\[ \ddot{u}(t) - 2 \beta \dot{u}(t) + (\omega_0^2 + \beta^2)(u(t) - s(t)) = 0, \] (1)

together with a guard condition on the output variable \( u(t) \) that switches the sign of \( s(t) \)

\[ s(t) = \begin{cases} 1, & \text{if } u(t') \geq 0 \\ -1, & \text{if } u(t') < 0 \end{cases}, \] (2)
at times \( t' \) where \( \dot{u}(t') = 0 \).

Figure 2 shows a time series of the variables \( u(t) \) and \( s(t) \) with the parameters \( \omega_0 = 2\pi \) and \( \beta = \ln(2) \). We integrate Eqs. (1) and (2) using MATLAB’s ODE45, where the switching condition is monitored using the integrator’s event-detection algorithm. The attractor for this system is low dimensional and is plotted in Fig. 2(b). The dynamics can also be viewed as a chaotic shift map,\(^{18} \) as seen in Fig. 2(c). For this system, \( u(t) \) oscillates with a growing amplitude and fixed oscillation frequency \( f_o = \omega_0/2\pi \) about a piece-wise constant line \( s(t) \). The switching times of \( s(t) \) depend on the local maxima and minima of \( u(t) \), and the times between the local maxima and minima of \( u(t) \) are fixed by the fundamental frequency \( f_o \). Thus, the maximum rate of switching in \( s(t) \) is limited to \( 1/f_o \). Using a one-bit digital sampling of \( s(t) \) at a sampling frequency that is greater than or equal to \( f_o \), we are able to store a record of all switching state values \( s_n \). Similar to the case for a PRBS information about the transmitted waveform can be stored with minimal sampling and memory, enhancing the potential SNR of a radar correlation measurement.

In addition to its data storage capabilities, chaos from this system can be further exploited using a matched filter. Corran et al. demonstrate that the switching information \( s(t) \) can be used with a matched filter at the radar receiver to perform a correlation over a given bit-sequence.\(^{19} \) The matched filter is described by

\[ \dot{y}(t) = v(t - 2\pi/\omega_o) - v(t), \] (3)

\[ \dot{\xi}_m(t) + 2\beta \dot{\xi}_m(t) + (\omega_0^2 + \beta^2)\xi_m(t) = (\omega_0^2 + \beta^2)y(t), \] (4)

where \( v(t) \) is the input signal and \( \xi_m(t) \) is the analog output of the matched filter. In Fig. 3, we examine the output of the matched filter when it is driven by \( v(t) = u(t) + \text{AWGN} \). The original signals \( u(t) \) and \( s(t) \) are plotted in Figs. 3(a) and 3(b). The switching state \( s(t) \) is plotted with a one-bit digital sampling \( s_n \) at a sampling frequency \( f_o \). In Fig. 3(c), the time series \( v(t) \) has a SNR of \(-5.9 \text{dB} \), where \( \text{SNR} = 10\log_{10}(\text{SNR}_{\text{input}}) \), \( \text{SNR}_{\text{input}} = \sigma_u^2/\sigma_{\text{AWGN}}^2 \), and where \( \sigma_u^2 \) and \( \sigma_{\text{AWGN}}^2 \) are the input signal \( u(t) \) and additive noise powers, respectively (see the Appendix for details on the additive noise).

The matched filter output, \( \dot{\xi}_m(t) \), when driven by \( v(t) \), shows a maximum SNR at specific times along the waveform. We note that, in Fig. 3(d), \( \dot{\xi}_m(t) \) is a nearly noise-free waveform that transitions approximately between two states, one positive and negative (defined by a dotted black line at \( \xi_m(t) = 0 \)), where the sign of the waveform is dictated by the system’s symbolic dynamics. To demonstrate this, we compare \( s_n \) to \( \dot{\xi}_m(t) \). Using a correlation between \( s(t) \) and \( \dot{\xi}_m(t) \), we determine the time delay through the matched filter \( \tau_m \) to be approximately \( 1/(2f_o) \). We compensate for the

![FIG. 2. Chaos from a piecewise-linear harmonic oscillator with negative damping. (a) Time series of the analog variable \( u(t) \) (green) and the nonlinear switching state \( s(t) \) (blue dashed line). (b) Chaotic attractor in phase space. (c) Chaotic shift map created by sampling \( u(t) \) using the times \( t' \) from Eq. (2), where \( s_n = u(t') \) if \( |u(t')| - 1 < 0 \).](Image 56x104 to 292x327)

![FIG. 3. Temporal evolution of (a) \( u(t) \), (b) \( s(t) \), (c) \( \xi_m(t) \) with SNR = -5.9 dB, and (d) \( \xi_m(t) \). The horizontal axes in (b) and (c) are shifted by \( \tau_m \) and \( 2\tau_m + \tau_m \), respectively, where \( \tau_m \) is the propagation distance to an intended target and \( \tau_m \) is the time delay through the matched filter. Above the signals \( \xi_m(t) \) and \( s(t) \) (blue), a single-bit discrete sampling of the waveforms (red dots) is shown.](Image 320x113 to 556x411)
delay, and sample $\xi_m(t)$ at $f_o$, assigning binary values using the relation: $-1$ if $\xi_m(t_n) \leq 0$ and $+1$ if $\xi_m(t_n) > 0$, where $t_n$ is the $n$th sampling time. These binary values, also shown in Fig. 3(d), match $s_n$, demonstrating that the matched filter’s output follows the switching information from $s(t)$.

In a simulated radar application, Corron et al. use a tapped delay line to perform a time-domain correlation operation between the digitally stored $s_n$ and the analog output $\xi_m(t)$. The tapped delay line is described by

$$Z_m(t) = \sum_{n=1}^{N} s_n \xi_m(t-t_n),$$  \hspace{1cm} (5)

where $N$ is the length of the stored sequence and $Z_m(t)$ is the output of the correlator (we have used the linearity of Eqs. (1)–(4) to rearrange the operations in Ref. 19). In this case, the output of the matched filter follows the switching information from $s(t)$. The output sum $Z_m(t)$ peaks at time $t = 2\tau_{pd} + \tau_m$.

To better understand the operations performed by a tapped delay line, we use a pictorial representation of Eq. (5). In Fig. 4, the output of the matched filter $\xi_m(t)$ splits into $N$ copies, each of which are successively delayed by times $t_{df} = n$, where $n$ is an integer. The resulting copies are multiplied by the corresponding stored $s_n$ and summed continuously in time. The output $Z_m(t)$ peaks at $2\tau_{pd} + \tau_m$, where $\tau_{pd}$ and $\tau_m$ are the propagation delays of the signal to the target and through the matched filter, respectively. In the context of Fig. 1, the correlation operation of this chaos radar can be performed between a transmitted and received signal using the tapped delay line in Eq. (5). Thus, using the chaotic waveform generator from Ref. 18, combined with the matched filter and tapped delay line, we arrive at a chaos radar.29

This particular chaos radar benefits in two ways from the deterministic characteristics of the system’s chaos. The first benefit is the link between the non-repeating waveform $u(t)$ and the switching state $s(t)$: A binary sampling of $s(t)$ can completely characterize the dynamics in $u(t)$. A second benefit is the ability to derive a matched filter that optimizes the output SNR of the receiver. The matched filter for chaos, combined with a tapped delay line, provides an architecture for relatively quick and inexpensive correlations between a binary sequence and a recovered analog signal, both generated from a single chaotic system. These benefits present a dramatically simplified platform for an inexpensive, analog, high-performance radar.

III. MATCHED FILTER ANALYSIS

With this background and motivation, we now examine Corron et al.’s matched filter for chaos in the frequency domain. Using the transfer functions of Eqs. (1), (3), and (4), we examine the spectral properties of the matched filter. For the purposes of our analysis, we mainly focus on the magnitudes of transfer functions. We will show that the phase of the matched filter is approximately linear with frequency $f$ when $f < f_o$ and, thus, preserves the transition information in $s_n$. We will then derive empirically a pseudo-matched filter using a combination of standard filters that also preserves the transition information in $s_n$. Lastly, we compare the pseudo-matched filter performance to the true matched filter for chaos in a simulated radar application.

First, we analyze the spectral properties of the chaotic dynamics from $u(t)$ and $s(t)$ as well as the driving signal $v(t)$. In Fig. 5, we plot the spectral amplitudes for $u(t)$, $s(t)$, and $v(t)$, where $v(t) = u(t) + AWGN (SNR = -5.9 \text{ dB})$. One should take note that, in Fig. 5(a), there is no maxima in the frequency spectrum at $f_o$, the fundamental frequency of oscillation, because the phase of $u(t)$ switches by $\pi$ each time $s(t)$ switches states. This demonstrates that, if the spectrum of $u(t)$ is scaled-up in high-frequency system ($> 1 \text{ GHz}$), the bandwidth in $u(t)$ would stretch over several gigahertz. In addition to its radar properties, a high-frequency broadband spectrum in $u(t)$ would provide a useful carrier signal for low-profile or ultra-wideband technologies.23,24 In Fig. 5(c), the frequency spectrum of the AWGN in $v(t)$ covers information about $u(t)$. The matched filter is engineered to correlate with the underlying waveform in $v(t)$.

Next, we examine the transfer function of the matched filter. We take the Fourier transform of Eqs. (3) and (4) and...
obtain the transfer functions $H_{\text{input}}$ and $H_{o}$. We combine these transfer functions to obtain the transfer function for the matched filter $H_{m}$. They read

$$H_{\text{input}}(\nu) = \frac{\hat{y}}{\hat{v}} = e^{2\pi i \nu} - 1,$$

$$H_{o}(\nu) = \frac{\hat{z}_{m}}{\hat{y}} = \frac{4\pi^{2} + \beta^{2}}{4\pi^{2}(1 - \nu^{2}) + \beta(4\pi
i\nu + \beta)},$$

$$H_{m}(\nu) = \frac{\hat{z}_{m}}{\hat{v}} = H_{\text{input}}H_{o},$$

where $\nu = f/f_{o}$ and $\hat{z}_{m}$, $\hat{y}$, and $\hat{v}$ are the Fourier transforms of $z_{m}(t)$, $y(t)$, and $v(t)$, respectively. We plot the magnitudes of $H_{\text{input}}$, $H_{o}$, and $H_{m}$ as a function of frequency $\nu$ in Fig. 6. In Fig. 6(d), we also plot the phase of $H_{m}$, where the phase is approximately linear with frequency for $\nu < 1$ and thus preserves timing information in $s_{o}$.

We analyze $H_{\text{input}}$ and $H_{o}$ individually to better understand the matched filter’s transfer function. We factorize $H_{\text{input}}$ into two linear operations ($H_{\text{input}} = H_{\text{notch}}H_{\text{integrator}}$), a notch filter and an integrator

$$H_{\text{notch}}(\nu) = \frac{\hat{y}}{\hat{q}} = e^{2\pi i \nu} - 1,$$

$$H_{\text{integrator}}(\nu) = \frac{\hat{q}}{\hat{v}} = \frac{1}{2\pi i \nu},$$

where $\hat{q}$ is the Fourier transform of an intermediate input-output variable and $H_{\text{notch}}$ and $H_{\text{integrator}}$ are the transfer functions of a notch filter and integrator, respectively. We plot the magnitudes of $H_{\text{notch}}$ and $H_{\text{integrator}}$ in Fig. 7. In Fig. 7(a), the magnitude of the notch filter’s transfer function goes to minus infinity (on a log scale) at integer multiples of the fundamental frequency $f_{o}$. Similar types of filters have been used previously in chaotic systems to stabilize periodic orbits using continuous-time control methods.25,26 The transfer function in Eq. (10) is a standard operation for integration. In Fig. 7(b), the function $|H_{\text{integrator}}|$ diverges to infinity at $\nu = 0$ and falls off at a rate of $1/\nu$ with increasing $\nu$. When cascaded, $H_{\text{notch}}$ and $H_{\text{integrator}}$ complement one another to form a filter that preserves low frequencies, eliminates integer multiples of $f_{o}$, and cuts out high-frequencies (see Fig. 6(a)).

Upon inspection of $H_{o}$ in Eq. (7), we see that it takes on a functional form that is similar to the Fourier transform of the dynamical system in Eq. (1), given by

$$H_{\text{system}}(\nu) = \frac{\hat{u}}{\hat{s}} = \frac{4\pi^{2} + \beta^{2}}{4\pi^{2}(1 - \nu^{2}) + \beta(4\pi i\nu + \beta)},$$

where $\hat{u}$ and $\hat{s}$ are the Fourier transforms of $u(t)$ and $s(t)$, respectively. The only difference between $H_{o}$ and $H_{\text{system}}$ is the sign of the $4\pi i\nu$ term in the denominator, which does not affect the magnitude of either transfer function, and thus Eq. (7) contains specific spectral information about the system’s dynamics. This difference does, however, introduce a constant phase shift. When the transfer function $H_{o}$ is applied with $H_{\text{input}}$, it reshapes the spectrum of the transfer function for the matched filter $H_{m}$ near $\nu = 1$ and creates an approximately linear phase for $f < f_{o}$, as seen in Figs. 6(c) and 6(d). Thus, the operations of the matched filter for chaos can be separated into four criteria: (i) a notch filter that eliminates the fundamental oscillation frequency $f_{o}$, (ii) an integrator that preserves the low frequencies and (iii) cuts off high frequencies falling off as $1/\nu$, and (iv) a dynamical filter that reshapes the transfer function near the fundamental oscillation frequency $f_{o}$. These four criteria are the foundation for our derivation of the pseudo-matched filter for chaos.
IV. PSEUDO-MATCHED FILTER

Our strategy for designing a pseudo-matched filter for chaos is to simplify the four criteria of the matched filter using transfer functions from components that are readily available at high-speed. In Sec. V, we satisfy criteria (i) and (ii) using a single transfer function. We also show that criterion (iii) can be accomplished without an integrator. Lastly, we demonstrate that criterion (iv) is not necessary for our applications.

To begin constructing our pseudo-matched filter, we select a different notch filter that is shifted in frequency but still blocks the fundamental frequency \( f_0 \). Most notch filters block integer multiples \( (n = 0, 1, 2, 3, \ldots) \) of a single frequency. Instead, we choose a notch filter that is shifted to block odd integer multiples \( (2n + 1 = 1, 3, 5, \ldots) \) of a single frequency. Since the matched filter attenuates frequencies above \( \nu = 1 \), we conjecture that the only important spectral notch is at \( f_0 \), and all high-order even notches are not included in our pseudo-matched filter. The transfer function of our shifted-notch filter is

\[
H_{\text{shifted-notch}}(\nu) = \frac{\hat{v}_{\text{out}}}{\hat{v}_{\text{in}}} = \frac{1}{2} \left( 1 + e^{i\nu} \right), \tag{12}
\]

where \( \hat{v}_{\text{in}} \) and \( \hat{v}_{\text{out}} \) are the Fourier transforms of the input signal \( v_{\text{in}} \) and output signal \( v_{\text{out}} \), respectively. We plot the magnitude of \( H_{\text{shifted-notch}} \) in Fig. 8(a) (compare to \( H_{\text{notch}} \) from Fig. 7(a)). In both plots, the fundamental frequency \( f_0 \) is blocked. However, in Fig. 8(a), the lower frequencies \( (\nu < 0.5) \) are not cut. Thus, the shifted-notch filter performs two of the four operations from the matched filter; (i) it eliminates the fundamental oscillation frequency \( f_0 \), and (ii) preserves low frequencies.

In the time domain, the shifted-notch filter of Eq. (12) is expressed by

\[
v_{\text{out}}(t) = \frac{1}{2} \left( v_{\text{in}}(t) + v_{\text{in}}(t - \pi/\omega_0) \right). \tag{13}
\]

We compare Eq. (13) to Eq. (3) and note that the output is no longer related to the input through a derivative. Also, the time-shift on the input signal is halved \( (\pi/\omega_0) \) instead of \( 2\pi/\omega_0 \) and the shifted input \( v_{\text{in}}(t - \pi/\omega_0) \) is summed with the present state \( v_{\text{in}}(t) \). In an experimental setting using high-speed electronics, where \( v_{\text{in}} \) and \( v_{\text{out}} \) are voltages, this shifted-notch filter can be realized using a voltage divider, time delays (realized, for example, by coaxial cables), and an isolating hybrid junction, as illustrated in Fig. 9. The lengths of the two cables used in this realization of the filter are chosen such that the difference in propagation times for electromagnetic waves to propagate through them is \( \tau_B - \tau_A = \pi/\omega_0 \). The isolating hybrid junction sums the outputs \( v(t - \tau_A) + v(t - \tau_B) \). We shift time \( t \rightarrow t + \tau_A \) to arrive at the output signal \( v_{\text{out}} \) in Eq. (14). This realization of the shifted-notch filter can scale to high-speed voltages (>1 GHz).

Continuing the construction of the pseudo-matched filter, we use a first-order low-pass filter to attenuate high frequencies, rather than an integrator. We avoid the need for an integrator because the shifted-notch does not cut off low frequencies. The transfer function of the low-pass (L-P) filter is

\[
H_{\text{L-P}}(\nu) = \frac{\hat{x}_{\text{out}}}{\hat{x}_{\text{in}}} = \frac{1}{1 + 2\pi i \nu / \nu_L}, \tag{14}
\]

where \( x_{\text{in}} \) and \( x_{\text{out}} \) are the Fourier transforms of the input and output signals, respectively, and the low-pass cutoff frequency is \( \nu_L = f_L/f_0 \). We plot the magnitude of \( H_{\text{L-P}} \) in Fig. 8(b). In the figure, \( H_{\text{L-P}} \) leaves the spectral amplitude of frequencies below \( \nu_L \) unchanged, while suppressing frequencies above \( \nu_L \). Beyond \( \nu = \nu_L \), the rate of the spectral roll-off of \( |H_{\text{L-P}}| \) is not \( \sim \nu^{-1} \), but rather \( \sim (1 + \nu)^{-1} \). A first-order low-pass filter is a standard electronic component for filtering an

FIG. 8. Magnitudes of the transfer functions (a) \( H_{\text{shifted-notch}} \), (b) \( H_{\text{L-P}} \), (c) \( H_p \). In (d), the phase of \( H_p \) is given as a function frequency.

FIG. 9. Pictorial realization for a high-speed (>1 GHz) shifted-notch filter for voltages \( v_{\text{in}} \) and \( v_{\text{out}} \). An example of a broadband, high-frequency power splitter is the Mini-Circuits ZFRSC-42-S, and an example of a broadband, high-frequency hybrid-junction is the M/A-COM H-9.
input voltage $x_{in}$ to obtain an output voltage $x_{out}$ and satisfies approximately the third component of the matched filter criteria (iii). We note that higher-order low-pass filters (Butterworth, Chebyshev, etc.) are also available at high-speed.

When constructing our pseudo-matched filter, we neglect the dynamical filter that reshapes the spectrum (iv). We show that using just the shifted-notch and low-pass filters allows us to achieve comparable performance to the true matched filter in a simulated radar application. Thus, we cascade the shifted-notch and low-pass filters ($H_{\text{shifted-notch}}H_{\text{low-pass}}$) to arrive at the transfer function of our pseudo-matched filter

$$H_p(v) = \frac{\tilde{v}_{\text{out}}}{\tilde{v}_{\text{in}}} = \frac{1 + e^{i\pi v}}{2 + 4\pi v/v_L}, \quad (15)$$

where $\tilde{v}_{\text{in}}$ and $\tilde{v}_{\text{out}}$ are the Fourier transforms of the input and output signals of the filter, respectively. We plot the magnitude and phase of $H_p$ in Fig. 8(c). In the figure, the phase of $H_p$ is approximately linear and thus preserves timing information from $v(t)$. For comparison to the matched filter, see Fig. 6(c). Qualitatively, the two filters follow similar trends in both magnitude and phase. The phase in the pseudo-matched filter has a lower slope in its frequency dependence; a lower slope just constitutes a shorter time delay through the filter. However, it is clear by comparison of the magnitudes that the pseudo-matched filter is not performing the same operations as the matched filter.

We now apply the pseudo-matched filter to the chaotic waveform generated by Eqs. (1) and (2) and examine its output signals of the filter, respectively. We plot the magnitude and phase of $v_p$ in Fig. 8(c). In the figure, the phase of $H_p$ is approximately linear and thus preserves timing information from $v(t)$. For comparison to the matched filter, see Fig. 6(c). Qualitatively, the two filters follow similar trends in both magnitude and phase. The phase in the pseudomatched filter has a lower slope in its frequency dependence; a lower slope just constitutes a shorter time delay through the filter. However, it is clear by comparison of the magnitudes that the pseudo-matched filter is not performing the same operations as the matched filter.

We compensate for the delay, and sample $x(t)$ to obtain an output voltage $x_{out}$ with a lower slope just constitutes a shorter time delay through the filter. However, it is clear by comparison of the magnitudes that the pseudo-matched filter is not performing the same operations as the matched filter.

We now apply the pseudo-matched filter to the chaotic waveform generated by Eqs. (1) and (2) and examine its output in the time-domain. We drive the pseudo-matched filter with $v(t) = u(t) + AWGN$, where $v(t)$ has a SNR of $-5.9$ dB (see Fig. 3(c)). In Fig. 10(a), we plot the pseudo-matched filter’s output $\xi_p(t)$. From the figure, we see that pseudo-matched filter has effectively removed the main oscillation frequency $f_0$, and what remains is a digital-like signal $\xi_p(t)$.

We note that a considerable amount of noise is still present in comparison to Fig. 3(d). Using a correlation between the original $s(t)$ and $\xi_p(t)$, we determine the time delay through the pseudo-matched filter $t_p$ to be approximately $0.14/f_0$. We compensate for the delay, and sample $\xi_p(t)$ at $f_p$, assigning binary values using the relation: $-1$ if $\xi_p(t_n) \leq 0$ and $+1$ if $\xi_p(t_n) > 0$, where $t_n$ is the nth sampling time. In Fig. 10(a), we see that, with this particular SNR, the discrete sampling of $\xi_p(t)$ is equivalent to $s_p$ from Fig. 3(b).

Next, we parallel the construction of the time delay tap from Eq. (5) for the output of the pseudo-matched filter. In this case, the time delay tap is

$$\chi_p(t) = \sum_{n=1}^{N} s_n \xi_p(t - t_n), \quad (16)$$

where $\chi_p(t)$ is the output of the time-domain correlation. We plot an example of $\chi_p(t)$ in Fig. 10(b) using the same chaos and $s_n$ that were used to calculate $\chi_m(t)$ in Fig. 4. By visually comparing $\chi_p(t)$ to $\chi_m(t)$, we see that the output correlation peaks are qualitatively similar, but $\chi_p(t)$ has more noise. In the remaining section, we establish criteria for quantitatively comparing these correlation waveforms and use these criteria to weight each filter’s performance.

V. MATCHED VS. PSEUDO-MATCHED

In order to quantitatively compare the performances of the matched and pseudo-matched filters, we examine each filter’s ability to recover sequences of the system’s symbolic dynamics within the context of a chaos radar. In all radar applications, the ability to correctly identify the location of the correlation peak in the correlation operation is the useful measure. Therefore, in order to compare the two filters, we examine their performances based on the peak width and output SNR of $\chi_{m,p}(t)$. We also present an approximate analytical form for each correlation’s output SNR.

The peak widths of the output-correlation functions give the resolutions of each radar system. We measure $\Delta_m$ and $\Delta_p$, the full-width at half maximum (FWHM) time of the correlation output peaks using the matched and pseudo-matched filters, respectively. For the most ideal measure of each filter’s correlation peak width, we measure $\Delta_{m,p}$ in cases where no noise is present in the received waveform $v(t) = u(t)$. We note that these widths are independent of $N$, the number of stored data points in the correlation calculation of Eqs. (5) and (16). Using a Gaussian fit to the peak of $\chi_{m,p}(t)$, we obtain peak widths $\Delta_n f_0 = 0.55$ and $\Delta_p f_0 = 0.73$ (see Appendix). Using these values of $\Delta_{m,p}$ and scaling $f_0$ to 1 GHz, we calculate the theoretical resolutions of the matched and pseudo-matched filters to be 0.17 m and 0.22 m, respectively. In this example, the ranging resolutions differ by 5 cm. Thus, this is not a critical difference for radar applications that localize targets like planes or cars, and the pseudo-matched filter has an acceptable ranging resolution in comparison to the matched filter.

Next, we measure the output correlation SNR’s of the matched and pseudo-matched filters using the correlation

![FIG. 10. Output of pseudo-matched filter. (a) Time series of the output of the matched filter $\xi_p(t)$ (blue) while driven by $v(t) = u(t) + AWGN$ (SNR = $-5.9$ dB). The signal $\xi_p(t)$ is sampled with uniform spacing (red dots) at a clock frequency $f_c = c_{\text{clock}}/2\pi$. Above the waveform, a single-bit discrete sampling of the waveform is shown. From the figure, we see that all of the relevant information from $s_p$ is encoded in $\xi_p(t)$. (b) The switching state $s(t)$ is sampled and $s_n$ is stored for $N = 100$. The output of the pseudo-matched filter $\xi_p(t)$ drives Eq. (16) and the output $x_p(t)$ peaks at time $t = 2t_{pd} + t_p$.](image-url)
peak heights $a_{m,p}$ and the surrounding correlation noise floors. The output SNR in $\chi_{m,p}(t)$ is

$$SNR_{m,p} = \frac{a_{m,p}^2}{\sigma_{N|m,p}^2},$$ (17)

where $a_{m,p}$ is the peak height of $\xi_{m,p}(t)$ from a Gaussian fit (see Appendix) and $\sigma_{N|m,p}^2$ is the output variance of the correlation noise floor (note that the mean of the noise floor $\approx 0$) for the matched and pseudo-matched filters, respectively. We present a summary of these quantities in the block diagram shown in Fig. 11(a). In the diagram, we also review the waveforms and processes used for generating $\chi_{m,p}(t)$ and $\xi_{m,p}(t)$ and highlight the two relevant quantities, $SNR_{input}$ and $SNR_{m,p}$. We calculate $SNR_{m,p}$ as a function of the input $SNR_{input}$. The results of these calculations are given in Fig. 11(b). In addition, we use the distributions from $\xi_{m,p}(t)$ and $\xi_{p}(t)$ from the two different cases $\nu(t) = AWGN$ and $\nu(t) = u(t)$ to derive an analytical prediction for $SNR_{m,p}$ (see Appendix for derivations). These theoretical predictions are plotted with dotted lines in Fig. 11(b). These plots represent the performances of the matched and pseudo-matched filters in a simulated radar.

From Fig. 11(b), it is clear that the matched filter outperforms the pseudo-matched filter in the output $SNR$ of a radar correlation. Without noise in the system, the matched and pseudo-matched filters perform with output correlation $SNR$'s of $2.6 + 10\log_{10}(N)$ dB and $1.3 + 10\log_{10}(N)$ dB, respectively. For $SNR_{input} = 1/100$, the output $SNR$'s decrease to $-2.0 + 10\log_{10}(N)$ dB and $-44 + 10\log_{10}(N)$ dB, respectively. In Fig. 11(b), the average difference between $SNR_{m}$ and $SNR_{p}$ is 2.0 dB. We note that this difference is independent of $N$ and therefore fully characterizes the filter performances. Thus, where small loss is acceptable in the performance of the radar, the pseudo-matched filter is a simpler alternative to the system’s analytically matched filter for chaos.

As a final example, we use the theoretical $SNR_{m,p}$ to predict when the matched and pseudo-matched filters will fail in a radar application. Failure occurs when $SNR_{m,p}$ falls below a certain threshold. For a radar system that is capable of storing $N = 50$ data points and has a desired output correlation $SNR$ of $33$ dB, the matched and pseudo-matched filters will fail at a $1/SNR_{input}$ of approximately 25 and 70, respectively. If, in this application, the input $SNR$ is such that $1/SNR_{input} < 10$, then a radar with either the matched or pseudo-matched filter will be able to range, on average, without failure. The choice between the matched and pseudo-matched filter is therefore an application-dependent problem, and, as the bandwidth of this system scales higher, one must also begin to weigh each filter’s high-frequency capabilities as well as its baseline performance.

VI. CONCLUSIONS

In conclusion, for the chaotic system presented in Ref. 18, we derive empirically a sub-optimal filter for increasing the $SNR$ of a chaotic waveform. This sub-optimal filter performs approximately three out of four of the linear operations from the matched filter for chaos; (i) eliminates the fundamental oscillation frequency $f_o$, (ii) preserves the low frequencies, and (iii) cuts off high frequencies. Our filter, deemed a pseudo-matched filter, is composed of a shifted-notch filter and a first-order low-pass filter. In the context of a radar concept that uses a time delay tap as a correlation measure, we have shown that the pseudo-matched filter may be an acceptable and simplified substitute for the matched filter. In addition, we acknowledge that the pseudo-matched filter can be further improved using higher order low-pass filters and additional shifted-notch filters. We present this current version of the pseudo-matched filter to illustrate our method and emphasize its simplicity.

We note that our analysis highlights the flexibility and robustness of Corron et al.’s findings. The chaos from the dynamical system in Eqs. (1) and (2), even in the presence of large amounts of noise, can be processed by a linear filter to perform correlations with its symbolic dynamics. We capitalize on this system’s elegance to create a simpler, so-called pseudo-matched filter. Although sub-optimal, the pseudo-matched filter shows that the advantages of this chaotic system can be adapted for applied settings that use commercially available, high-speed filters. In the future, in order to make use of this specific high-speed, pseudo-matched filter, we acknowledge that a high-frequency version of this dynamical system must also be constructed. But, it is our hope that, in

FIG. 11. (a) Block diagram for testing the matched and pseudo-matched filters in a simulated radar application. (b) Output-correlation $SNR$’s of the matched (blue [ ]) and pseudo-matched (red [ ]) filters scaled by $N$ on a logarithmic scale as a function of $1/\text{SNR}_{input}$. For each value of $\text{SNR}_{input}$, 100 calculations of $\text{SNR}_{m,p}$ were performed using sequence $s_n$ for $n = n_0$ to $n = n_0 + N$, where $n_0$ is a random positive integer and $N = 50$. The mean value of the calculated $\text{SNR}_{m,p}$ is plotted with the respective standard deviations. The blue and red dotted lines give the theoretical predictions of the $\text{SNR}_{m,p}$ as references for the matched and pseudo-matched filters, respectively. Cases for larger $N$ were verified to have quantitatively similar results.
the meantime, this work inspires others to investigate deeper into the symbolic dynamics of all chaotic systems and find applications that benefit from an empirically derived pseudo-matched filter for chaos (LADAR, communications, etc.).

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APPENDIX A: ADDITIVE NOISE

Because the output from MATLAB’s ODE45 uses a variable timestep, we resample $u(t)$ using a linear interpolation with time steps $\delta t$, where $\delta t_0 = 10^{-2}$. To simulate environmental noise, we add noise to the waveform $u(t)$ using random numbers spaced by time units $\delta t$. The random numbers are calculated from a Gaussian distribution with zero mean. For different points along the $1/SNR_{input}$ axis of Fig. 11(b), the variance of the AWGN is varied accordingly.

APPENDIX B: GAUSSIAN FITS

We measure the correlation peak width and height using a Gaussian fit

$$f(t) = a_{m,p} e^{-(t-(2r_{p1}+r_{m,p}))^2/2c_{m,p}^2}, \quad (B1)$$

where $a_{m,p}$ and $c_{m,p}$ are free parameters that are fit to the correlation peak heights and widths. Using $f(t)$ to fit $x_m,p(t)$, we obtain a FWHM peak width $\Delta m,p = 2\sqrt{2\ln(2)c_{m,p}^2}$ and peak height $a_{m,p}$.

APPENDIX C: ANALYTICAL SNR’S

We derive analytical forms for the output-correlation SNR of the matched and pseudo-matched filters. To do so, we approximate Eq. (17) as

$$SNR_{m,p} = \frac{\sigma_{m,p}^2}{\sigma_{N,m,p}^2} \sim \frac{(A_{m,p} N)^2}{\sigma_{1|m,p}^2 N + \sigma_{2|m,p}^2 N}, \quad (C1)$$

where $A_{m,p}$ is a constant that characterizes the growth rate of the correlation peak height with $N$, $\sigma_{1|m,p}^2$ is a constant determined in the noise-free case where $v(t) = u(t)$, and $\sigma_{2|m,p}^2$ is a function of $SNR_{input}$ in the case where $v(t) = AWGN$. Recall that the numerators and denominators of Eq. (C1) represent the power of peak heights of the correlation and the surrounding noise floor, respectively. We derive each of the three terms $A_{m,p}$, $\sigma_{1|m,p}^2$, and $\sigma_{2|m,p}^2$ in the following sections.

The correlation peak heights for the matched and pseudo-matched filters grow at different rates. In the correlation operations of Eqs. (5) and (16), the peaks occur at times $t_{m,p}^* = 2r_{pd} + 2r_{m,p}$ for the matched and pseudo-matched filters, respectively. At time $t_{m,p}$, $x_m,p(t) = 0$ and the output correlation is

$$x_{m,p}(t_{m,p}) \sim \sum_{n=1}^{N} |\xi_{m,p}(t_{m,p} - t_n)| \sim A_{m,p} N. \quad (C2)$$

We approximate $A_{m,p}$ from the local maxima of $|\xi_{m,p}(t)|$. To do so, we examine the noise-free case where $v(t) = u(t)$ and collect a subset of points $|\xi_{m,p}(t')|$, where $t'$ are the times of local maxima in $|\xi_{m,p}(t)|$. We average $|\xi_{m,p}(t')|$ to obtain $A_m = 1.34$ and $A_p = 1.02$ using a time-length $t_{f0} \sim 10^3$.

To approximate the value of $\sigma_{1|m,p}^2$, we also examine $x_{m,p}(t)$ in the noise-free case where $v(t) = u(t)$. The deterministic noise floor in a correlation measurement is also known as its side-lobes; the side-lobes result from non-zero contributing terms in the correlation $x_{m,p}(t)$ when $t \neq t_{m,p}$. Using the central limit theorem, we approximate the variance of these nonzero terms as $\sigma_{1|m,p}^2 N$, where $\sigma_{1|m,p}$ is the variance of the signal $x_{m,p}(t)$. In this approximation, we find that $\sigma_{1|m}^2 = 1.00$ and $\sigma_{1|p}^2 = 0.78$.

It remains to calculate the contributions to the noise floor of the correlation from additive noise. To do so, we examine $x_{m,p}(t)$ in the case where $v(t) = AWGN$. Similar to the case for the side-lobes, we use the central limit theorem to approximate the contribution of the AWGN to the noise floor of the correlation as $\sigma_{2|m,p}^2 N$, where $\sigma_{2|m,p}$ is the variance of the signal $x_{m,p}(t)$. However, $\sigma_{2|m,p}^2$ depends on the variance of the input AWGN

$$\sigma_{2|m,p}^2 = a_{m,p}^2 \sigma_{AWGN}^2, \quad (C3)$$

where $a_{m,p}$ is the noise attenuation factor of the matched and pseudo-matched filters, respectively. We measure the values $a_m = 1/75$ and $a_p = 1/64$. Lastly, we use that $\sigma_{AWGN}^2 = \sigma_u^2/(SNR_{input})$ to rewrite Eq. (C1) as

$$SNR_{m,p} \sim \frac{N A_{m,p}^2}{\sigma_{1|m,p}^2 + \sigma_{m,p}^2 SNR_{input}}, \quad (C4)$$

where $\sigma_u^2 = 1.34$ is the power of the chaotic signal $u(t)$. We plot Eq. (C4) as a function of $1/SNR_{input}$ for the matched and pseudo-matched filters in Fig. 11(b).


