A gauge-invariant formulation of the Nizhnik-Veselov-Novikov equation

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1 introduction
   solitons
   Korteweg-de Vries equation
   Inverse Scattering Transform

2 two-dimensional generalizations of the KdV-equation
   Kadomtsev-Petviashvili equation
   Nizhnik-Veselov-Novikov equation

3 gauge-invariant NVN-equation
   derivation
   gauge transformations
   modified NVN-equation

4 conclusions
A **solitary wave** solution of a partial differential equation \( \Delta(x, t, u) = 0 \) is a travelling wave solution of the form

\[
    u(x, t) = w(x - \gamma t) = w(z),
\]

with \( w(z \to -\infty) = 0 \) and \( w(z \to \infty) = \text{const.} \).

A **soliton** is a solitary wave which asymptotically preserves its shape and velocity upon nonlinear interaction with solitary waves, or more generally, with another (arbitrary) localized disturbance.
KdV equation describes waves propagating in a shallow channel of water.
Korteweg-de Vries equation 2

- waves are long in comparison with the total depth
- amplitudes are small
- viscous effects may be neglected

nondimensional form

\[ u_t + 6uu_x + u_{xxx} = 0 \]
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How to find solutions?
Lax pair

Time-independent Schrödinger equation:

\[ L_1 \psi = \psi_{xx} + u(x, t) \psi = \lambda \psi \]

The time dependence of its eigenfunctions is given by:

\[ L_2 \psi = \psi_t - (\gamma + u_x) \psi + (4\lambda + 2u) \psi_x = 0 \]

The compatibility condition

\[ [L_1, L_2] = 0 \]

yields the KdV equation.
Solve the Schrödinger equation for a given initial value \( u(x, 0) \).

- **discrete eigenvalues**, \( \lambda = \kappa_n^2 > 0, \ n = 1, 2, ..., N \) with the eigenfunctions: \( \psi_n (x \to \infty, t = 0) \sim c_n (0) \exp (-\kappa_n x) \)

- **continuous eigenvalues** for \( \lambda = -k^2 < 0 \) with the eigenfunctions:
  \[
  \psi (x \to \infty, 0) \sim e^{-ikx} + r (k, 0) e^{ikx} \\
  \psi (x \to -\infty, 0) \sim a (k, 0) e^{-ikx}
  \]

\[\implies \text{scattering data at } t = 0:\]
\[S (\lambda, 0) = (\{\kappa_n, c_n (0)\}_{n=1}^N, r (k, 0), a (k, 0))\]
Determine the time evolution of the scattering data via $L_2$

\[
\kappa_n = \text{const.}, \quad n = 1, 2, \ldots, N
\]
\[
c_n(t) = c_n(0) \exp(4\kappa_n^3 t), \quad n = 1, 2, \ldots, N
\]
\[
a(k, t) = a(k, 0)
\]
\[
r(k, t) = r(k, 0) \exp(8ik^3 t)
\]

\[\Rightarrow\text{scattering data at time } t:\]
\[
S(\lambda, t) = (\{\kappa_n, c_n(t)\}_{n=1}^{N}, r(k, t), a(k, t))
\]
Problem: reconstruct the potential $u(x, t)$ from the scattering data $S(\lambda, t)$. Define

$$F(x; t) = \sum_{n=1}^{N} c_n^2(t) \exp(-\kappa_n x) + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k, t) e^{ikx} dk$$

Solve the linear integral equation (Gel’fand-Levitan-Marchenko equation):

$$K(x, y; t) + F(x + y; t) + \int_{z}^{\infty} K(x, z; t) F(z + y; t) dz = 0$$

The potential can finally be reconstructed via

$$u(x, t) = 2 \frac{\partial}{\partial x} K[(x, x; t)]$$
reflectionless potentials

\[ r (k, 0) = r (k, t) = 0 \Rightarrow \text{soliton solutions} \]

\[ F (x; t) = \sum_{n=1}^{N} c_n^2 (t) \exp (-\kappa_n x) \]

The kernel and the inhomogeneous term in the GLM equation are reduced to finite sums!

For \( N = 1 \):

\[ u (x, t) = 2\kappa_1^2 \text{sech}^2 \{ \kappa_1 (x - 4\kappa_1^2 t + x_1) \} \]

where \( x_1 = \ln c_1^2 (0) \) is a constant.
Inverse Scattering Transform

\( u(x, 0) \)
Inverse Scattering Transform

\[ u(x, 0) \xrightarrow{\text{direct problem}} S(\lambda, 0) \]
Inverse Scattering Transform

\[ u(x, 0) \xrightarrow{\text{direct problem}} S(\lambda, 0) \]

\[ S(\lambda, 0) \xrightarrow{\text{time evolution}} S(\lambda, t) \]
Inverse Scattering Transform

$u(x, 0)$ \hspace{2cm} direct problem \hspace{2cm} $S(\lambda, 0)$

$u(x, t)$ \hspace{2cm} inverse problem \hspace{2cm} $S(\lambda, t)$

time evolution
interaction of solitons
interaction of solitons
interaction of solitons
interaction of solitons
interaction of solitons
Kadomtsev-Petviashvili equation

The KP equation is weakly two-dimensional and describes shallow surface waves. It’s the physically motivated generalization.

\[
(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0
\]

where \(\sigma^2 = \pm 1\) (Kadomtsev and Petviashvili [1970]).
The Nizhnik-Veselov-Novikov equation is seen as the natural two-dimensional generalization. It has no known application in physics.

\[ U_t + \kappa_1 U_{\xi\xi} + \kappa_2 U_{\eta\eta} + 3\kappa_1 (U \partial_{\eta}^{-1} U_{\xi})_{\xi} + 3\kappa_2 (U \partial_{\xi}^{-1} U_{\eta})_{\eta} = 0 \]

(Nizhnik [1980]; Veselov, Novikov [1984])
Instead of $[L_1, L_2] = 0$ we require $[L_1, L_2] = B(U) L_1$. 

$L_1 \psi = \psi_{\xi\eta} + U \psi = 0$

$L_2 \psi = \psi_t + \kappa_1 \psi_{\xi\xi\xi} + \kappa_2 \psi_{\eta\eta\eta} + 3\kappa_1 (\partial^{-1}_{\xi} U_{\xi}) \psi_{\xi} + 3\kappa_2 (\partial^{-1}_{\xi} U_{\eta}) \psi_{\eta} = 0$

with $B$ given by the expression

$$B = 3\kappa_1 \partial^{-1}_{\eta} U_{\xi\xi} + 3\kappa_2 \partial^{-1}_{\xi} U_{\eta\eta}.$$
derivation of the gauge-invariant NVN-equation

Starting from the more general linear representation

\[ L_1 \psi = \psi_{\xi \eta} + u_1 \psi_\xi + v_1 \psi_\eta + U \psi = 0; \]
\[ L_2 \psi = \psi_t + u_3 \psi_{\xi \xi \xi} + v_3 \psi_{\eta \eta \eta} + u_2 \psi_{\xi \xi} + v_2 \psi_{\eta \eta} + \tilde{u}_1 \psi_\xi + \tilde{v}_1 \psi_\eta + \tilde{U} \psi = 0. \]

with arbitrary functions \( u_1, v_1, U, u_3, v_3, u_2, v_2, \tilde{u}_1, \tilde{v}_1 \) and \( \tilde{U} \) of \( \eta, \xi \) and the time \( t \). Calculate

\[ [L_1, L_2] = BL_1. \]

The presence of the field \( \tilde{U} \) allows for gauge-invariance!
gauge-invariant NVN-equation

With

\[ B(W) = 3\kappa_1 \partial_\eta^{-1} W_{\xi\xi} + 3\kappa_2 \partial_\xi^{-1} W_{\eta\eta} \neq 0 \]

we find a gauge invariant form of the NVN-equation

\[ W_t + \kappa_1 W_{\xi\xi\xi} + \kappa_2 W_{\eta\eta\eta} + 3\kappa_1 (W \partial_\eta^{-1} W_\xi)_\xi + 3\kappa_2 (W \partial_\xi^{-1} W_\eta)_\eta = 0 \]

with \( W := U - u_1\xi - u_1\nu_1 \).
Consider the gauge transformation

$$\psi \rightarrow \psi' = g^{-1}\psi$$

with an arbitrary function $g(\xi, \eta, t)$. The field variables transform as

$$U'_1 = U_1 + \frac{g_\eta}{g}, \quad V'_1 = V_1 + \frac{g_\xi}{g}$$

$$U' = U + \frac{g_{\xi\eta}}{g} + U_1\frac{g_\xi}{g} + V_1\frac{g_\eta}{g}.$$
Invariants of the gauge transformation:

\[ \mathcal{W}_1 := u_1'\xi - v_1'\eta = u_1\xi - v_1\eta = 0 \]
\[ \mathcal{W} := U' - u_1\xi - u_1'v_1 = U - u_1\xi - u_1v_1 \]

The linear problems read:

\[
L_1^{(\mathcal{W})}\psi = \psi_{\xi\eta} + \mathcal{W}\psi = 0;
\]
\[
L_2^{(\mathcal{W})}\psi = \psi_t + \kappa_1\psi_{\xi\xi\xi} + \kappa_2\psi_{\eta\eta\eta} + 3\kappa_1(\partial_\eta^{-1}\mathcal{W}_\xi)\psi_\xi + 3\kappa_2(\partial_\xi^{-1}\mathcal{W}_\eta)\psi_\eta = 0
\]

Triad representation:

\[ [L_1^{(\mathcal{W})}, L_2^{(\mathcal{W})}] = B(\mathcal{W})L_1^{(\mathcal{W})} \]
$u_1\xi = v_1\eta \Rightarrow$ express the field variables $u_1$ and $v_1$ through a potential $\phi(\xi, \eta, t)$:

$$(v_1, u_1) := (\phi_\xi, \phi_\eta).$$

The invariant $W$ of the gauge transformation is of the form

$$W(\phi, U) = U - u_1\xi - u_1v_1 = U - \phi_\xi\eta - \phi_\xi\phi_\eta.$$
The modified NVN-equation

The gauge with \( \phi = 0, \: U \neq 0 \) gives the original NVN-equation

\[
U_t + \kappa_1 U_{\xi\xi\xi} + \kappa_2 U_{\eta\eta\eta} + 3\kappa_1 (U \partial^{-1}_\eta U_\xi)_\xi + 3\kappa_2 (U \partial^{-1}_\xi U_\eta)_\eta = 0
\]

The gauge with \( \phi \neq 0, \: U = 0 \) describes a new modified NVN-equation

\[
\phi_t + \kappa_1 \phi_{\xi\xi\xi} + \kappa_2 \phi_{\eta\eta\eta} - \frac{\kappa_1}{2} \phi_\xi^3 - \frac{\kappa_3}{2} \phi_\eta^3 \\
-3\kappa_1 \phi_\xi \partial^{-1}_\eta (\phi_\eta \phi_\xi_\xi) - 3\kappa_2 \phi_\eta \partial^{-1}_\xi (\phi_\xi \phi_\eta_\eta) = 0
\]

Both are gauge invariant to each other.
Miura transformation

Miura-type transformation:

\[ U = -\phi_{\xi\eta} - \phi_{\xi}\phi_{\eta} \]

It is possible to construct solutions of the NVN-equations from solutions of the mNVN-equation.
• several nonlinear equations can be solved by IST methods
conclusions

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- approach relies on genius guesswork
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2 dimensional generalization in physics of KdV: KP-equation
Mathematical 2 dimensional generalization of KdV: NVN-equation
NVN-equation can be derived within a gauge-invariant formulation
conclusions

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• NVN-equation can be derived within a gauge-invariant formulation
Thanks for your attention!