Problem 1 [3 pts]
In its rest frame, quasar Q2203+29 produces a hydrogen emission line of wavelength 121.6 nm. Astronomers on Earth measure a wavelength of 656.8 nm for this line. Determine the redshift parameter ($z$) and the speed of recession for this quasar.

From class we derived:

$$z = \frac{\Delta \lambda}{\lambda_0} = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} - 1} \Rightarrow \frac{v}{c} = \frac{(z + 1)^2 - 1}{(z + 1)^2 + 1}$$

Here, $\lambda_0 = 121.6$ nm, and $\Delta \lambda = (656.8 - 121.6)$ nm = 535.2 nm.
Therefore, $z = 4.40$ (note: positive, so the Quasar is in fact receding).
This gives $\frac{v}{c} = \frac{28.2}{50.2} = 0.93 \Rightarrow v = 0.93c$.

Problem 2 [2 pts]
Quasar 3C 446 is extremely variable: its luminosity at optical wavelengths has been observed to change by a factor of 40 in as little as 10 days. Using the redshift parameter $z = 1.404$ measured for 3C 446, determine the time for the luminosity variation as measured in the quasar’s rest frame.

Also in class we derived:

$$z + 1 = \frac{\Delta t_{\text{obs}}}{\Delta t_0}$$

Here;

$\Delta t_{\text{obs}} = 10$ days, $z = 1.404 \Rightarrow \Delta t_0 = 4.2$ days

This is the time for the same observed (on Earth) variation in the Quasars rest frame. This is, therefore, the time we must take into account when considering models for what is happening with Quasars (or anything else!).
Problem 3 [5 pts]
Using the aid of space-time diagrams:

(a) Show that if two events are space-like separated, there is a Lorentz frame in which they are simultaneous. Does a Lorentz frame exist in which they are simultaneous if the two events are time-like separated? Why or why not?

(b) Show that if two events are time-like separated, there is a Lorentz frame in which they occur at the same point (same spacial coordinate values). Is this also true if the two events are space-like separated? Why or why not?

(c) If in a Lorentz frame two events $A$ and $B$ occur at times $t_A$ and $t_B$ respectively where $t_B > t_A$ (that is, $B$ occurs after $A$), does a Lorentz frame exist in which $A$ occurs after $B$? If this is possible, what must the relationship between $A$ and $B$ in the first Lorentz frame be (space-like, time-like, or null, separated)?
Problem 4 [4 pts]
Consider the collision of a photon with wavelength $\lambda_i$ with an electron of mass $m_e$ at rest. After the collision the scattered photon makes an angle $\theta$ with respect to the initial photon direction, and has wavelength $\lambda_f$. Write down the conservation of relativistic energy and momentum equations and using these derive the Compton scattering equation:

$$\Delta \lambda = \lambda_f - \lambda_i = \frac{h}{m_ec}(1 - \cos \theta)$$

Calculate the total relativistic energy and momentum components (defining positive $x$ direction as shown).

Initially: $E_i = \frac{hc}{\lambda_i} + m_e c^2, \quad p_{ix} = \frac{h}{\lambda_i}, \quad p_{iy} = 0$

Finally: $E_f = \frac{hc}{\lambda_f} + \gamma m_e c^2, \quad p_{fx} = \frac{h}{\lambda_f} \cos \theta + \gamma m_e v_e \cos \phi, \quad p_{fy} = \frac{h}{\lambda_f} \sin \theta - \gamma m_e v_e \sin \phi$

Conservation of relativistic energy, and momenta in the $\hat{x}$ and $\hat{y}$ directions gives the equations:

$$\frac{hc}{\lambda_i} + m_e c^2 = \frac{hc}{\lambda_f} + \gamma m_e c^2 \quad (1)$$

$$\frac{h}{\lambda_i} = \frac{h}{\lambda_f} \cos \theta + \gamma m_e v_e \cos \phi \quad (2)$$

$$0 = \frac{h}{\lambda_f} \sin \theta - \gamma m_e v_e \sin \phi \quad (3)$$

We need to eliminate $\phi$ and $v_e$ (which also appears in $\gamma = (1 - (v_e/c)^2)^{-1/2}$).

$$(2)^2 \Rightarrow \left( \frac{h}{\lambda_i} - \frac{h}{\lambda_f} \cos \theta \right)^2 = \gamma^2 m_e^2 v_e^2 \cos^2 \phi$$

$$(3)^2 \Rightarrow \left( \frac{h}{\lambda_f} \right)^2 \sin^2 \theta = \gamma^2 m_e^2 v_e^2 \sin^2 \phi$$

Adding these equations, and using $\sin^2 \phi + \cos^2 \phi = 1$ gives:

$$\left( \frac{h}{\lambda_i} \right)^2 + \left( \frac{h}{\lambda_f} \right)^2 - 2 \frac{h^2}{\lambda_i \lambda_f} \cos \theta = \gamma^2 m_e^2 v_e^2 \quad (4)$$

OK, so we’ve gotten rid of $\phi$, now let’s eliminate $v_e$ (and $\gamma$). Squaring (1) gives:

$$\left( \frac{h}{\lambda_i} \right)^2 - 2 \frac{h^2}{\lambda_i \lambda_f} + \left( \frac{h}{\lambda_f} \right)^2 = m_e^2 c^2 (\gamma - 1)^2$$

Subtracting this from (4) gives:

$$\frac{2h^2}{\lambda_i \lambda_f} (1 - \cos \theta) = \gamma^2 m_e^2 v_e^2 - m_e^2 c^2 (\gamma - 1)^2 = 2m_e^2 c^2 (\gamma - 1)$$

$$\Rightarrow \frac{h^2}{\lambda_i \lambda_f} (1 - \cos \theta) = m_e c^2 (\gamma - 1) = m_e c h \left( \frac{1}{\lambda_i} - \frac{1}{\lambda_f} \right) \quad \text{(by substituting in (1))}$$

Multiplying both sides by $\lambda_i \lambda_f$ finally gives:

$$\lambda_i - \lambda_f = \frac{h}{m_e c} (1 - \cos \theta)$$
Problem 5 [2 pts]
Calculate the energies (in eV) and wavelengths (in nm) of the shortest wavelength photons emitted by a downward transitioning electron in each of the Lyman, Balmer, and Paschen series of the hydrogen atom. In what regions of the electromagnetic spectrum are these?

Since \( E = \frac{hc}{\lambda} \) the shortest wavelength photons will correspond to those of highest energy for a given series of lines. Recall the energy levels for the hydrogen atom are given by; \( E_n = \frac{-13.6 \text{eV}}{n^2} \), with the Lyman series corresponding to transitions to/from the \( n = 1 \) (ground) state, the Balmer series corresponding to transitions to/from the \( n = 2 \) state, and the Paschen series corresponding to transitions to/from the \( n = 3 \) state. In each case the highest energy photons will correspond to transitions from \( n \rightarrow \infty \) (Note: in the previous version of the solutions I erroneously did this for the lowest energy transitions!). Also note that \( hc = 1.242 \times 10^{-6} \text{eV.m} \) which is useful for calculating wavelengths with energies in eV.

Lyman series lines: photons of greatest energy correspond to transitions from \( n \rightarrow \infty \) to \( n = 1 \):

\[
E_\gamma = 13.6 \text{eV} - 0 \text{eV} = 13.6 \text{eV} \Rightarrow \lambda = \frac{hc}{E_\gamma} = 91.3 \text{nm} \quad \text{(Ultraviolet region)}
\]

Balmer series lines: photons of greatest energy correspond to transitions from \( n \rightarrow \infty \) to \( n = 2 \):

\[
E_\gamma = 3.40 \text{eV} - 0 \text{eV} = 3.40 \text{eV} \Rightarrow \lambda = 365 \text{nm} \quad \text{(Ultraviolet/Visible region)}
\]

Paschen series lines: photons of greatest energy correspond to transitions from \( n \rightarrow \infty \) to \( n = 3 \):

\[
E_\gamma = 1.51 \text{eV} - 0 \text{eV} = 1.51 \text{eV} \Rightarrow \lambda = \frac{hc}{E_\gamma} = 821 \text{nm} \quad \text{(Infrared region)}
\]

Problem 6 [2 pts]
Show that, at room temperature, the thermal energy \( kT \approx 1/40 \text{eV} \). At what temperature is \( kT \) equal to 1 eV? to 13.6 eV? (the latter is the temperature at which the thermal energy is sufficient to ionize hydrogen.)

At room temperature, \( T \sim 20^\circ \text{C} = 293 \text{K} \). With \( k = 1.38 \times 10^{-23} \text{J/K} = 8.617 \times 10^{-5} \text{eV/K} \), we have for the thermal energy of photons at this temperature:

\[
E = kT = 0.0252 \text{eV} \sim \frac{1}{40} \text{eV}
\]

For \( kT = 1 \text{eV} \Rightarrow T = \frac{1 \text{eV}}{8.617 \times 10^{-5} \text{eV/K}} = 11,600 \text{K} \).

For \( kT = 13.6 \text{eV} \Rightarrow T = (11600 \text{K})(13.6) = 1.58 \times 10^5 \text{K} \).

To put this in some perspective, we’ll see later in the course that for conditions in the early universe the thermal energies of particles are of order GeV for times about \( 10^{-8} \text{s} \) after the Big Bang.
Problem 7 [2 pts]
The cosmos is pervaded by a 3 K radiation field, which is regarded as the “echo” of the Big Bang. This radiation field is called the Cosmic Microwave Background. Calculate the energy and wavelength of this radiation. In what region of the electromagnetic spectrum is this?

For $T = 3 \text{ K}$, $E = kT = 2.6 \times 10^{-4} \text{eV}$, $\lambda = \frac{hc}{E} \approx 5 \text{ mm}$. So this EM radiation that we detect is in the microwave region, hence its name.

Problem 8 [5 pts]
Consider throwing 0.5 kg balls through a doorway (of width 1 m) at 30 km/hour at a wall 5 m from the doorway.

(a) Assuming the wave nature of all particles, what is the de Broglie wavelength of these thrown balls?

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{Js}}{(0.5 \text{ kg})(8.33 \text{m/s})} = 1.6 \times 10^{-34} \text{m}$$

(b) Given that these ball’s are acting like waves with a de Broglie wavelength, $\lambda$, estimate the distance to the first minimum of the resulting diffraction pattern on the wall.

The angle $\theta$ that represents the first minimum is given by: $d \sin \theta = \lambda$. For very small angles (which this is because $\lambda/d$ is very small); $\sin \theta \approx \theta \approx y/L$.

$$\Rightarrow y = L\frac{\lambda}{d} = 8 \times 10^{-34} \text{m}$$

This is about $10^{20}$ times smaller than the size of the proton, so the wavelike nature of macroscopic objects in the real world is completely unnoticeable.

(c) What is the same distance (to the first minimum for electrons of energy $10\text{MeV}$ passing through a single slit of width 500 nm, on a screen 2 m from the slit.

For $E_e = 10\text{MeV}$ electrons (rest mass, $m_e = 0.511\text{MeV}$), we can calculate the momentum from $E_e^2 = p_e^2 c^2 + m_e^2 c^4$ and see that the rest mass term is negligible so that:

$$p_e \approx E_e/c = 10 \text{ MeV}/c = 1.6 \times 10^{-12} \text{J}/\text{c} = 5.33 \times 10^{-21} \text{kgm/s} \Rightarrow \lambda = \frac{h}{p} = 1.24 \times 10^{-13} \text{m}$$

Therefore, following the same calculation from (b) we get:

$$y = L\frac{\lambda}{d} = (2 \text{ m}) \frac{1.24 \times 10^{-13}}{5 \times 10^{-7} \text{m}} = 4.96 \times 10^{-7} \text{m} \approx 500 \text{nm}$$

which is about 0.005 mm, observable with a microscope.
Problem 9 [5 pts]
Consider a gas of neutral hydrogen atoms.

(a) At what temperature will equal numbers of atoms have electrons in the ground state and in the second excited state \((n = 3)\) ?

From the Boltzmann equation:

\[
\frac{N_3}{N_1} = \frac{g_3}{g_1} e^{(E_3 - E_1)/kT}
\]

where, \(E_1 = -13.6 \text{ eV}\), \(E_3 = -13.6 \text{ eV}/3^2 = -1.51 \text{ eV}\), \(g_1 = 2\), \(g_3 = 2(3)^2 = 18\), \(k = 8.617 \times 10^{-5} \text{ eV}/K\). For \(N_3 = N_1\) we have:

\[
9 e^{-(12.09)/(8.617 \times 10^{-5} T)} = 1 \Rightarrow -1.403 \times 10^5 K = \ln(1/9) = -2.20 \Rightarrow T = 6.38 \times 10^4 K
\]

(b) At a temperature of 85,400 K, when an equal number \((N)\) of atoms are in the ground state and in the first excited state, how many atoms are in the second state \((n = 3)\) ? Express your answer in terms of \(N\).

When \(T = 85400 K\);

\[
\frac{N_3}{N_1} = 9 e^{-(12.09)/(8.617 \times 10^{-5})(8.54 \times 10^4)} = 9 e^{-1.64} = 1.74 \Rightarrow N_3 = 1.74 N
\]

(c) As the temperature \(T \to \infty\), how will the electrons in the hydrogen atom be distributed, according to the Boltzmann equation? That is, what will be the relative numbers of electrons in the \(n = 1, 2, 3, \ldots\) orbitals?

As \(T \to \infty\), \(e^{-\Delta E/kT} \to 1\). Therefore, for any orbital \(n\), the number of atoms/electrons in this state compared to the ground will be:

\[
\frac{N_n}{N_1} = \frac{2n^2}{2} = n^2
\]

For any two orbitals states \(n\) and \(m\), the relative numbers will be:

\[
\frac{N_n}{N_m} = \frac{n^2}{m^2}
\]

Note: for \(T >> 10^4 K\) almost all the \(H\) is ionized, but it is true that for the rare \(H\) atom that is not ionized, these ratios will represent the correct relative probabilities of being in a given state.