Physics 53

Wave Motion 2

If at first you don't succeed, try, try again. Then quit. No use being a damn fool about it.

— W.C. Fields

Waves in two or three dimensions

Our description so far has been confined to waves in which the energy moves only along one line. For waves in a string this is good enough, but energy in sound, water and light waves generally spreads out in two or three dimensions.

The main new concepts needed in more dimensions are these:

The *direction* of a harmonic wave is specified by giving a *wave vector* **k**. Its direction is that of the energy flow; its magnitude is $k = 2\pi / \lambda$, as in the one-dimensional case. The wavefunction at a position **r** relative to a small source then takes the form

$$\Phi(\mathbf{r}, t) = A\cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi).$$

The energy spreads out in space over larger and larger areas, so the intensity (power per unit area) must decrease with distance from the source. In the simple case where the energy spreads equally in all directions, the intensity at distance r from the source is equal to the power emitted by the source divided by the area of a sphere of radius r:

$$I(r) = \frac{P_{source}}{4\pi r^2}.$$

A wave of this type is a spherical wave. Since intensity is proportional to the square of the amplitude, the amplitude of a spherical wave must fall off with distance as $1/r$.

We are often dealing with the situation where the waves are received by a relatively small detector (ear, eye, or whatever) which is at a large distance from the source. This detector samples only a very small part of the spherical wave, so the curvature of the wavefront is negligible. A good approximation in that case is to treat the waves as one dimensional, moving directly away from the source with constant amplitude. This is the "plane-wave" approximation.

Decibel scale of loudness

Our perception of loudness of a sound is based on the response of our ears to the intensity of the waves entering them. Sound intensities vary over a vast range, but fortunately our ears respond (approximately) to the logarithm of the intensity. For this reason, a logarithmic scale of intensities is commonly used for the loudness of sound.

The standard scale is based on a unit called a **decibel** (db). An arbitrary reference intensity I_0 is chosen (one usually picks $I_0 = 10^{-12}\ \mathrm{W/m^2}$, approximately the faintest audible sound). The received intensity is then converted to the sound **loudness** β, measured in db according to the definition

Loudness (in db)	$\beta = 10 \log_{10}\left(\dfrac{I}{I_0}\right)$

A sound of intensity $1\ \mathrm{W/m^2}$ is painful to most hearers. This is a loudness level $\beta = 120$ db. It is typical for the sound near the stage in a rock concert.

The decibel name comes from "deci", meaning one-tenth, and "bel", a unit named after A.G. Bell, the inventor of the telephone.

The decibel scale is also used to describe amplification or attenuation of signals in electronic equipment. In those cases the relevant variable is power, not intensity.

The Doppler effect

Consider a source emitting harmonic waves along a straight line toward a receiver located on that line. We are interested in the effects of motion along the line of the source, receiver, or both.

If the source is stationary relative to the medium, the waves it emits have equally spaced wave crests. However, if the source is moving the wave crests in front of it are crowded closer together, while those behind it are spaced farther apart. Relative to the medium, the wavelength is *smaller* in front of the source and *larger* behind it. A stationary receiver in front of the moving source will detect more wave crests per second than if the source had been stationary, so the frequency received will be *higher* than that of the source. A receiver placed behind the moving source will detect a *lower* frequency. If the source is stationary but the receiver is moving, similar things happen. A receiver moving toward the source receives more wave crests per second than a receiver at rest. The frequency received is thus higher. Conversely, if the receiver moves away from the source the frequency received is lower.

These phenomena constitute the **Doppler effect**. It is a general property of waves.

For waves in a medium (such as sound) there is a simple formula relating the received frequency f_R to the source frequency f_S:

Medium at rest

Doppler effect (waves in a medium)	$f_R = f_S \dfrac{v - v_R}{v - v_S}$

Here v is the wave speed, v_R is the velocity of the receiver, and v_S is the velocity of the source (all relative to the medium). The drawing shows the case for which both v_R and v_S are positive numbers. Other cases are handled by making one or both of these speeds negative.

Note that as $v_S \to v$ the received frequency becomes infinite. The entire wave collapses into a single pulse, called a shock wave. The "bow wave" created by a boat moving faster than the speed of water waves is a familiar example, as is the "sonic boom" caused by an object traveling faster than the speed of sound.

The formula given is for "mechanical" waves resulting from motion of particles in a medium. Light is different, in that there is no medium supporting the wave, only electric and magnetic fields. The Doppler effect still occurs, but the formula is somewhat different. If the speeds of source and receiver are small compared to the speed of light waves (c), the formula given above can be used as an approximation.

It is through the Doppler effect for light from distant galaxies that we know the universe is expanding in a way consistent with the "big bang" model.

Interference of harmonic waves

Now we look at the situation where two one-dimensional harmonic waves exist simultaneously in the same medium. At first we assume they travel in the same direction, and to simplify things further we will assume they have the same amplitude. They are described by the wavefunctions

$$y_1(x,t) = A\cos(k_1 x - \omega_1 t)$$
$$y_2(x,t) = A\cos(k_2 x - \omega_2 t + \phi)$$

To find the *total* wavefunction, given by $y = y_1 + y_2$, we use a trigonometric identity:

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right).$$

For our case this gives

$$y(x,t) = 2A\cdot\cos\tfrac{1}{2}\left(\Delta k\cdot x - \Delta\omega\cdot t - \phi\right)\cdot\cos\left(k_{av}x - \omega_{av}t + \tfrac{1}{2}\phi\right)$$

where we have introduced the notations

$$\Delta k = k_2 - k_1,\ \Delta\omega = \omega_2 - \omega_1$$

$$k_{av} = \tfrac{1}{2}(k_1 + k_2),\ \omega_{av} = \tfrac{1}{2}(\omega_1 + \omega_2)$$

This somewhat complicated formula for $y(x,t)$ gives the general solution to our problem. We will examine some special cases.

Simple interference

Let the two waves have the same values of k and ω (same wavelength and frequency) and travel in the same direction. Then we find from the general formula

$$y(x,t) = 2A\cos(\phi/2)\cdot\cos(kx - \omega t + \phi/2).$$

This has the form of the wavefunction of a *single* harmonic wave with amplitude

$$A_{\text{tot}} = 2A\cos(\phi/2).$$

Let us compare the intensity of this resultant wave with the intensity of one of the original waves alone. The individual waves both had the same amplitude, and therefore the same intensity, which we call I_1. Then $I_1 = KA^2$, where the constant K depends on what kind of medium the wave is in. Since the resultant wave is in the same medium, it has intensity $I = KA_{\text{tot}}{}^2$. Squaring both sides of the above equation and multiplying by K, we find

Simple interference	$I = 4I_{1)}\cos^2(\phi/2) = 2I_1(1+\cos\phi)$

We have used the identity for the half-angle to get the second form, which is often more useful.

The resultant intensity varies between 0 and $4I_1$, depending on the value of ϕ. The two extremes occur when ϕ is an *even* multiple of π ($I = 4I_1$), and when ϕ is an odd multiple of π ($I = 0$).

These cases have special names:

Constructive interference	$\phi = 0, \pm 2\pi, \pm 4\pi, \ldots$
Destructive interference	$\phi = \pm \pi, \pm 3\pi, \ldots$

What might cause two waves to be out of phase with each other and thus interfere?

- They may have been emitted by two otherwise identical sources with different initial phases. Different sources which emit waves having a definite phase relation between them are called **coherent**. The two speakers in a stereo system are coherent sources, and the stereo illusion arises from interference between the waves they emit.
- The waves may have started together in phase but traveled different paths to the receiver, such that the total distance traveled from the source is not the same for both. If wave 1 travels distance x_1 and wave 2 travels distance x_2, then the *phase difference* between them as they arrive at the receiver is given by

  $$\phi = k(x_2 - x_1).$$

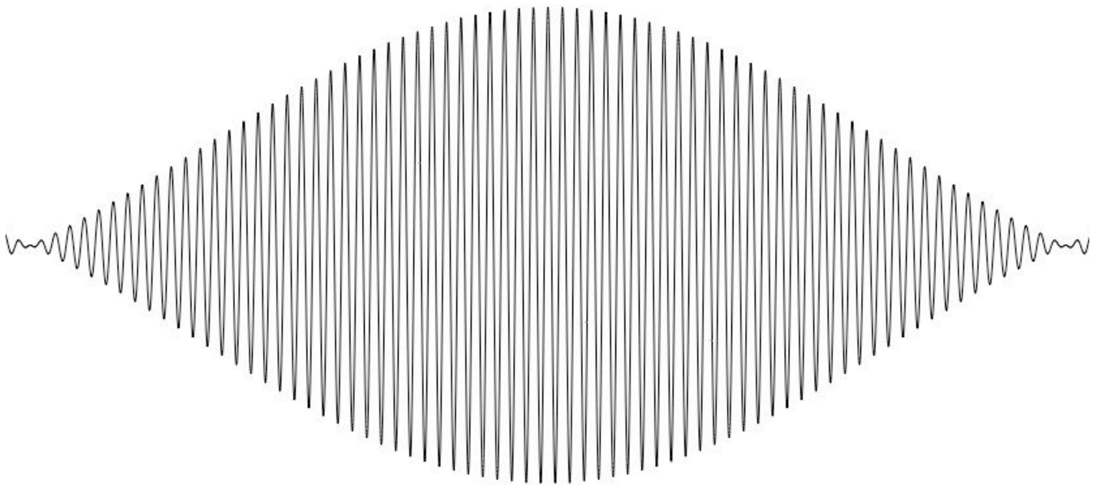
  This is a very useful formula in practice.
- One or both of the waves may have been reflected at an interface between two media. If the reflection is from a medium in which the waves move more slowly, the reflected wave is inverted, which is equivalent to adding π to the wave's phase.

The symbol ϕ in the intensity formula represents the *total* phase difference due to all these effects.

Wave groups

Real waves begin and end; they cannot be single harmonic waves, but must be a superposition of waves with different wavelengths and frequencies. A real wave thus does not have a precisely defined wavelength or frequency, but rather involves distributions of these, with an average value and a "spread" around the average. Of course, any wave also lacks a precise “position” at which it can be said to be located.

Shown is a wave "group" centered at $x = 0$.



Call the spread in position Δx. To make such a wave requires superposition of harmonic waves with wave numbers having a spread of size Δk around the average value of k.

To make a highly "localized" wave requires superposition of a large number of different wave numbers, so the more precisely we know *where* the wave group is, the less precisely we know its wave number (or wavelength). Conversely, to have a relatively well defined wave number, the wave must spread over many wavelengths. There is a mathematical theorem relating these "uncertainties" in position and wave number:

$$\Delta x \cdot \Delta k \geq \tfrac{1}{2}.$$

In quantum theory the momentum of an object is proportional to k, and in that context this inequality becomes the famous Heisenberg Uncertainty Principle.

The motion of such a wave group can be complicated. The individual peaks of the waves move with the **wave speed**:

$$\text{Wave speed: } v_w = \frac{\omega_{av}}{k_{av}}.$$

The pattern as a whole moves with the **group speed**:

$$\text{Group speed: } v_g = \frac{d\omega}{dk}.$$

These speeds are not in general the same. The energy density is substantial only where the wave amplitude is large, so **energy travels at the group speed**.

It is possible for the wave speed to be either smaller or larger than the group speed. Information transmitted by a wave is carried by its energy, and thus travels at the group speed. No information can travel faster than the speed of light in vacuo, called c. There is no such restriction on the wave speed.

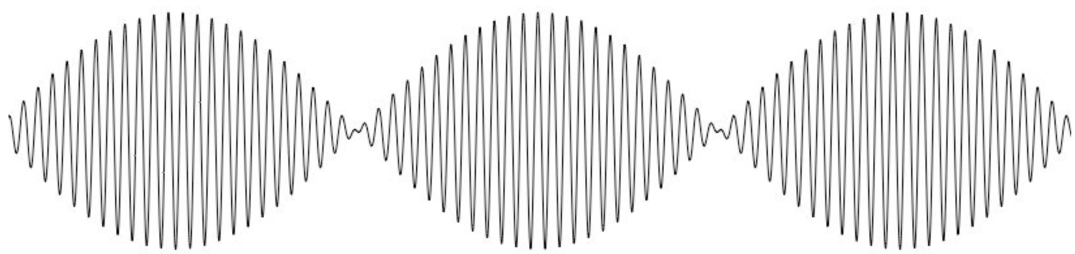
Beats

A relatively simple case of wave groups occurs with superposition of only two harmonic waves, of slightly different wavelengths and frequencies.

This combination will result in a wave which at $t = 0$ has the form:

$$y(x,0) = 2A \cdot \cos\left(\tfrac{1}{2}\Delta k \cdot x\right) \cdot \cos(k_{av}x).$$

The figure below shows this function for $k_{av} = 50$, $\Delta k = 2$.



The short wavelength oscillations are from the $\cos(k_{av}x)$ term, while the slowly varying term $\cos\left(\tfrac{1}{2}\Delta k \cdot x\right)$ "modulates" the amplitude, giving the overall envelope.

As time goes on, the waves move to the right, changing the pattern. The individual peaks travel with the wave speed; the groups in the envelope travel with the group speed. These speeds might not be the same.

Now consider the number of large groups (called **beats**) that pass a given point per second. The length L of one group is given by $\tfrac{1}{2}\Delta k \cdot L = \pi$, so $L = 2\pi / \Delta k$. Since the group moves at the speed $v_g = \Delta\omega / \Delta k$, the time it takes the group to pass a particular point is $T = L / v_g = 2\pi / \Delta\omega$. The number of groups passing per second, called the **beat frequency**, is $1/T$, so we have:

Beat frequency	$f_{beat} = \dfrac{\lvert\omega\rvert}{2\pi} = \lvert\Delta f\rvert$

The beat frequency is the *magnitude* of the difference between the two original frequencies. This simple formula has many applications, from tuning pianos to radar speed detectors.