Mechanical Waves

This set of notes contains a review of wave motion in mechanics, emphasizing the mathematical formulation that will be used in our discussion of electromagnetic waves.

Mechanical Waves

A mechanical system can transfer energy (and momentum) from one place to another in two ways. The simplest is to send an object — for example, a bullet — carrying mass, momentum and energy from the “source” to the “receiver”. A more complex method is called wave motion. In the case discussed here the energy-momentum transfer results from interaction of microscopic particles in the space between source and receiver. The system of these particles (which may be gas, liquid, or solid) is called the medium. Examples are sound waves, water waves, and waves in an elastic medium.

In the present course our main interest will be in electromagnetic waves, for which there is no material medium; but the mathematical description of all waves is basically the same.

Mechanical waves are created by the interaction between neighboring particles in the medium. Energy and momentum are transferred from one particle to the next by this interaction, and the net effect is to pass these quantities along from the source to the receiver. One characterizes this transfer by three quantities: its direction, its speed \( v \), and its rate of energy transfer, measured by its intensity \( I \).

General description of wave motion. The mathematical formula describing a wave is a function of position and time, called a wave function. For a mechanical wave this formula gives the status of a given point in the medium at a given time. In the case of one-dimensional wave moving in the \( x \)-direction, this function has the general form

\[
y(x,t) = f(x - vt).
\]

Here \( y \) represents a physical property of the medium. For sound waves, it is the variation in pressure relative to the normal pressure. For waves in a string it is the transverse displacement of the particles of the string from their equilibrium positions.

For electromagnetic waves it is usually taken to be the electric field strength. The fact that \( x \) and \( t \) appear only in the combination \( x - vt \) is what makes this a case of a traveling wave. The “disturbance” of the medium (e.g., displacement of the particles from their equilibrium positions) represented by \( f \) moves at speed \( v \) in the \( +x \)-direction.

If the disturbance moved in the \( -x \)-direction, the combination would be \( x + vt \).

Detailed analysis of the microscopic behavior of the particles of the medium leads to a relation between the behavior in space and that in time. It takes the general form

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}.
\]

The derivatives denoted by the symbol \( \partial \) are called partial derivatives. In calculating them all variables other than the one being considered are treated as constants.
This is called the wave equation. Any function that satisfies it represents a possible wave in the medium. It is easy to show that the function $y(x,t) = f(x - vt)$ satisfies the wave equation, as long as $f$ has first and second derivatives.

**Wave speed.** The analysis that leads to the wave equation in a particular case also determines $v$ in terms of properties of the medium. For mechanical waves the formula for $v$ has a generic form:

$$v = \sqrt{\text{Stiffness}/\text{Inertia}}.$$

For example, in a stretched string the wave speed is given by $v = \sqrt{T/\mu}$, where $T$ is the string tension and $\mu$ is the mass per unit length of the string.

**Superposition of waves.** Suppose we have two different functions, $y_1(x,t)$ and $y_2(x,t)$, both of which satisfy the wave equation and thus describe possible waves in the medium. It is easy to verify that the function $y(x,t) = y_1 + y_2$ also satisfies the wave equation and is thus describes another possible wave. But physically this wave is a combination of the waves described by $y_1$ and $y_2$. What this means is the following:

If two waves exist simultaneously in the same medium, the net effect is a wave for which the wave function is the sum of the wave functions of the two individual waves. This important aspect of wave motion is called the Principle of Superposition. It gives rise to the phenomena of interference and diffraction, characteristic of waves.

**Energy transport; intensity.** We are interested in wave motion because it is a method of transporting energy from one place to another. Usually we have in mind a source of this energy, which transfers it to the nearby particles of the medium with which it interacts. The energy then moves out through the medium, and a portion of it impinges on a receiver of some kind.

An example is a person talking to you: the source is the vocal apparatus of the person talking; the energy of the sound waves spreads out into the air, and some of it impinges on your eardrums (the receiver), setting them into vibration and sending signals through nerves to your brain.

To quantify this transfer of energy we use the intensity. This is the amount of wave energy that passes in unit time through unit area perpendicular to the wave direction. The energy per unit time (power) impinging on a receiver which presents area $A$ perpendicular to energy flow of intensity $I$ is given by $P = IA$.

An important general property of wave energy is that the intensity is proportional to the square of the wave strength; that is $I = C \cdot y^2$, where $C$ is a constant which depends on the specific properties of the medium.

**Interference.** As discussed above, two waves can come together in the medium to form a resultant wave, with the wave functions related by $y(x,t) = y_1 + y_2$. The intensity of the resultant will be $I = C \cdot y^2 = C \cdot (y_1 + y_2)^2$. The intensities of each of the waves alone
would be $I_1 = C \cdot y_1^2$ and $I_2 = C \cdot y_2^2$, so we find $I = I_1 + I_2 + C \cdot y_1 y_2$. The resultant intensity is not simply the sum of the individual intensities. This is the phenomenon called interference. Because the “interference term” $C \cdot y_1 y_2$ can be positive, negative, or zero, the resultant intensity can be greater that the sum of the individual intensities, or less, or the same. It is in this respect that transport of energy by wave motion differs most markedly from transport by moving objects with mass.

For example, the total energy transported by two bullets is the sum of their individual energies.

**Harmonic waves.** The function $f$ we have been using to represent wave motion can be almost any smooth function, and there are cases of many different kinds. But the most important case is where $f$ is a sinusoidal function, meaning the particles of the medium oscillate about their equilibrium positions in simple harmonic motion. In this case we have a harmonic wave, and the wave function takes the form

$$y(x,t) = A \cos(kx - \omega t + \phi).$$

Here $A$ is called the amplitude. The frequency of the oscillation is $f = \omega / 2\pi$. At a given time the distance between successive points where $y = A$, called the wavelength, is given by $\lambda = 2\pi / k$. The speed of the wave is $v = f \lambda = \omega / k$. The phase of the wave is the argument of the cosine, and the number $\phi$, called the initial phase, is the value of the phase at $x = 0$ and $t = 0$.

The reason for using $\omega$ and $k$ instead of $f$ and $\lambda$ is simply to avoid writing numerous factors of $2\pi$.

The given wave function describes a harmonic wave of a single frequency and wavelength. No actual wave like this exists (because it would have no beginning or end in space) but it represents a good approximation for many cases. In addition, any realistic wave motion can be described as a superposition of harmonic waves.

**Interference of harmonic waves.** Consider two waves of equal frequency and wavelength, moving in the same direction, but with a difference in their phases. Let the two waves be represented by

$$y_1 = A \cos(kx - \omega t) \quad \text{and} \quad y_2 = A \cos(kx - \omega t + \delta).$$

The phase difference between these waves is the number $\delta$. A calculation using a trigonometric identity shows that the resultant wave is given by

$$y = y_1 + y_2 = 2A \cos(\delta / 2) \cdot \cos(kx - \omega t + \delta / 2).$$

Our interest is in the intensity — specifically, the average intensity over a cycle of the oscillation, which is what one measures. We use the fact that averaged over a cycle $(\sin^2 \omega t)_{av} = (\cos^2 \omega t)_{av} = 1 / 2$. Then we find for the two waves and the resultant wave:

$$I_{1av} = I_{2av} = \frac{1}{2} C \cdot A^2, \quad I_{av} = \frac{1}{2} C \cdot [2A \cos(\delta / 2)]^2.$$
This gives us the formula for the intensity of the resultant wave in terms of that of one wave alone:

\[ I_{av} = 4I_1^2 \cos^2(\delta / 2) = 2I_1^2 (1 + \cos \delta). \]

If \( \delta \) is an integer multiple of \( 2\pi \), we have \( I_{av} = 4I_1^2 \); this is constructive interference. If \( \delta \) is an odd multiple of \( \pi \) we have \( I_{av} = 0 \); this is destructive interference.

If the amplitudes of the two interfering waves are not the same these results are a bit more complicated, but the conditions for constructive and destructive interference are the same.

What gives rise to the phase difference \( \delta \) in practical situations? There are several possibilities. The simplest is that we have waves from two sources that emit waves of the same type but not exactly synchronized in time; the waves from the two speakers of a stereo sound system are a common case. Another common case is that the two waves start out together but spilt apart and one travels a greater distance than the other before they are brought together again. A third important case arises when waves are reflected from a boundary between two media with different wave speeds.

In the present course the last two causes will be the most important to us.

**Standing waves in a string fixed at both ends; harmonics.** When a wave impinges on the boundary between two different media, in general two things happen to the energy: part of it continues into the second medium as the transmitted wave, at a different speed; the other part reverses direction and travels back through the original medium as the reflected wave.

A general property of wave reflection concerns phase changes in the reflected wave. Let the wave speed in the original medium be \( v_1 \) and that in the other medium \( v_2 \). Then there is a general rule applying to all kinds of waves:

- If \( v_1 > v_2 \) the reflected wave undergoes a phase change of \( \pi \).
- If \( v_1 < v_2 \) the reflected wave undergoes no phase change.

Consider the important case of a stretched string attached to a wall at \( x = 0 \), with a wave moving in the \(-x\)-direction toward the wall. This wave will reflect from the wall, creating a wave moving away from the wall in the \(+x\)-direction. The transmitted wave’s speed in the massive wall is quite small, so \( v_1 > v_2 \); this means the reflected wave undergoes a sudden phase change of \( \pi \).

Since \( \cos(\theta + \pi) = - \cos \theta \), this phase change has the effect of multiplying the wave function by \(-1\).

The total wave is the superposition of the incident wave and the reflected wave. The latter travels in the \(+x\)-direction, so we have \( y(x,t) = A\cos(kx + \omega t) - A\cos(kx - \omega t) \).

Negligible energy is transferred to the massive wall, so the transmitted wave is negligible and the reflected wave has essentially the same amplitude as the incident wave.
Another trigonometric identity allows us to write the resultant wave function as 
\( y(x,t) = 2A \sin kx \cdot \sin \omega t \). We see that the space and time variations are now in separate factors. This kind of wave does not travel or transport energy over a distance. It is called a standing wave.

In practical situations the string is also fixed at the other end, with a reflection there also. If the distance between the fixed ends is \( L \) we have the restriction that \( y = 0 \) for \( x = L \) (at all times). This can only be true if \( \sin kL = 0 \), or \( kL = \pi, 2\pi, 3\pi, ... \) Using \( k = 2\pi / \lambda \), we see that this restricts the allowed wavelengths: \( \lambda = 2L / n \), where \( n = 1, 2, 3, ... \)

Correspondingly, the frequencies allowed for these standing waves are restricted:
\[
 f_n = nf_1, \text{ where } f_1 = v / 2L \text{ and } n = 1, 2, 3, ...
\]

These frequencies constitute the harmonics of the vibrating string. The lowest, \( f_1 \), is usually called the fundamental.

Musical instruments based on vibrating strings make use of this set of harmonics. A similar set applies to standing sound waves in wind instruments.

Other wave phenomena. We have treated here only combinations of waves with the same frequency and wavelength. A combination of two waves with slightly different frequencies results in the phenomenon of beats, pulses of energy that travel in the same direction as the individual waves but usually with a different speed.

Another phenomenon is the Doppler effect, a shift in the received frequency because of motion of the source and/or receiver.

The present course will not cover material that makes use of these phenomena.

Sample problems and solutions.

1. The bugle produces its musical pitches using a set of harmonics like those for the vibrating string. That is, one can set up in the bugle’s air column standing waves with frequencies \( f_n = nf_1 \), where \( n = 1, 2, 3, ... \) and \( f_1 \) is the fundamental frequency. Traditional bugle calls are based on only four musical notes from these harmonics. Their frequencies (for a certain bugle) are 330, 440, 550 and 660 Hz.

   a. To what values of \( n \) do these frequencies belong?

   b. What is the fundamental frequency of this bugle?

a. Assuming these are successive harmonics, clearly they belong to \( n = 3, 4, 5, 6 \).

b. They are all multiples of 110 Hz, which is the fundamental frequency.

   It is very difficult in practice for a bugler to produce a frequency this low.
Two speakers driven by the same source are arranged as shown from above. A listener is at the point shown. The speakers emit sound of frequency 170 Hz. The speed of sound is 340 m/s.

a. What is the phase difference $\delta$ between the waves as they arrive at the listener?

b. If each speaker separately would result in sound of intensity $I_0$ at the listener’s location, what intensity does the listener receive with both speakers active?

c. What is the next higher frequency for which the listener would receive zero intensity?

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Because the speakers are driven by the same source, waves emitted by them start out in phase, i.e., a peak in the wave leaves each speaker at the same time. But when they arrive at the listener they have traveled different distances. Describe each wave by the wave function $y(x,t) = A \cos(kx - \omega t)$, where $y$ represents the pressure variation in the sound wave and $x$ is the distance the wave has traveled. (As usual, $k = 2\pi / \lambda$, $\omega = 2\pi f$, and $v = f\lambda$.) The wave from the top speaker travels distance 3 m to reach the listener; that from the bottom speaker travels 5 m. So at the location of the listener their wave functions are: $y_{\text{top}} = A \cos(k \cdot 3 - \omega t)$ and $y_{\text{bottom}} = A \cos(k \cdot 5 - \omega t)$. The phase difference (the difference between the arguments of the cosines) is $\delta = k \cdot (5 - 3) = k \cdot 2$.

This illustrates a general rule: The phase difference between two waves arising from a path difference $\Delta x$ is given by $\delta = k \cdot \Delta x$.

In the present case, the wavelength is $\lambda = v / f = 340 / 170 = 2$ m. So we find $\delta = (2\pi / 2) \cdot 2 = 2\pi$.

b. From the general formula on page 4 above: $I = 2I_0(1 + \cos\delta) = 2I_0(1 + 1) = 4I_0$. This is a case of constructive interference.

c. Write the phase difference in terms of the frequency: $\delta = (2\pi / \lambda) \cdot \Delta x = (2\pi f / v) \cdot \Delta x$. This shows that $\delta$ increases as the frequency is raised. Zero intensity results from destructive interference, which occurs when $\cos\delta = -1$, i.e., when $\delta$ is an odd multiple of $\pi$. Starting from the case when $f = 170$ Hz, for which $\delta = 2\pi$, we see that the next higher odd multiple of $\pi$ is $3\pi$. So we set $\delta = 3\pi$ and solve for the frequency: $3\pi = (2\pi f / v) \cdot \Delta x = (2\pi f / 340) \cdot 2$. This gives $f = 255$ Hz.