

1 Isospectral Hamiltonians - Precursor to Supersymmetry

A good source for further information regarding this topic is G. Junker's text *Supersymmetry in Quantum and Statistical Physics*.

1.1 Introduction

Imagine we are given a Hamiltonian $H^{(1)}$ that can be factored into the form $H^{(1)} = A^\dagger A$ for some operator A . We can define another Hamiltonian $H^{(2)} = AA^\dagger$. Our second Hamiltonian is automatically Hermitian. Now consider an eigenstate $|\phi\rangle$ of the first Hamiltonian that has energy E :

$$\begin{aligned} H^{(1)}|\phi\rangle &= E|\phi\rangle \\ A^\dagger A|\phi\rangle &= E|\phi\rangle \end{aligned}$$

Operating on both sides with A yields

$$\begin{aligned} A(A^\dagger A)|\phi\rangle &= A(E|\phi\rangle) \\ (AA^\dagger)(A|\phi\rangle) &= E(A|\phi\rangle) \\ H^{(2)}(A|\phi\rangle) &= E(A|\phi\rangle) \\ H^{(2)}|\psi\rangle &= E|\psi\rangle \end{aligned}$$

Hence, there is an eigenstate $|\psi\rangle = A|\phi\rangle$ of $H^{(2)}$ with this same energy eigenvalue. We could equally well have begun with the statement that $H^{(2)}|\psi\rangle = E|\psi\rangle$ and then found a state of $H^{(1)}$ given by $|\phi\rangle = A^\dagger|\psi\rangle$ with the same energy. Note that a Hamiltonian of the two given forms will have only non-negative eigenvalues: $\langle\psi|A^\dagger A|\psi\rangle = ||A\psi\rangle|^2 \geq 0$. Equality holds only if the state is annihilated by A . Analogous statements hold for A^\dagger . In order to have both $|\phi\rangle$ and $|\psi\rangle$ properly normalized, we need

$$\begin{aligned} |\phi\rangle &= E^{-1/2}A^\dagger|\psi\rangle \\ |\psi\rangle &= E^{-1/2}A|\phi\rangle \end{aligned}$$

This reveals an issue with the procedure: what about the case when $E = 0$? Suppose that $|\phi\rangle$ has vanishing energy. The eigenvalue equation then reads

$$H^{(1)}|\phi\rangle = A^\dagger A|\phi\rangle = 0$$

or $A^\dagger|\psi\rangle = 0$ so that $|\phi\rangle$ was originally null. Hence in the case where there's a state with vanishing energy $E = 0$, either $|\psi\rangle$ or $|\phi\rangle$ must be a null state. At most one of these two Hamiltonians may have a state with this vanishing eigenvalue. Additionally, there is only full, unbroken supersymmetry (SUSY) if one of the two groundstates indeed *does* have a vanishing energy eigenvalue. We'll see later that failure to satisfy this condition is equivalent to a ground state which is not invariant under SUSY transformations: a hallmark of a broken symmetry (at best).

An important class of operators A with which this analysis is often carried out take the form

$$\boxed{\begin{aligned} A &= \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \\ A^\dagger &= -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \end{aligned}}$$

Here, $W(x)$ is known as the *superpotential*. These yield paired Hamiltonians

$$\begin{aligned} H^{(-)} := H^{(1)} = A^\dagger A &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + W(x)^2 - \frac{\hbar}{\sqrt{2m}} W'(x) \\ H^{(+)} := H^{(2)} = AA^\dagger &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + W(x)^2 + \frac{\hbar}{\sqrt{2m}} W'(x) \end{aligned}$$

The above is often stated with units $\hbar = 2m = 1$. From here on, I will adopt this convention when convenient.

The condition that $A|\phi_0\rangle = 0$ for the ground state $|\phi_0\rangle$ of $H^{(-)}$ becomes

$$\left(\frac{d}{dx} + W(x)\right)\phi_0(x) = 0$$

which is known as the *Riccati equation*.

1.2 Simple Application: Simple Harmonic Oscillator

To begin, we quickly note that picking $W(x) = (m\omega^2/2)^{1/2}x$ will generate two paired Hamiltonians both of which are simple harmonic oscillators, one with potential $\hbar\omega$ higher than the other.

$$V_{1,2} = \frac{1}{2}m\omega^2x^2 \pm \frac{1}{2}\hbar\omega$$

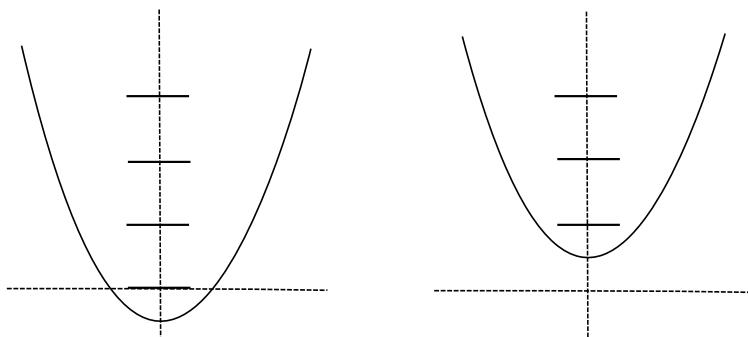


Figure 1: The energy levels of the simple harmonic oscillator are evenly spaced.

We see that one of these two Hamiltonians has a ground state energy of $E = 0$ and so, SUSY is an unbroken symmetry relating these two potentials. Now we repeatedly use the fact that supersymmetry is relating one system to a shifted version of itself. Because the potential is the same up to a constant shift, this means the ground state of the second Hamiltonian has energy $E = \hbar\omega$. Hence, there is a state of the first Hamiltonian with the same energy, its first excited state. The existence of this state in the first potential means the second Hamiltonian has an equivalent first excited state, just shifted. This, then, yields by supersymmetry a second excited state of the first Hamiltonian. This process can be continued *ad infinitum* and explains why the energy levels of a simple harmonic oscillator are evenly spaced.

In reality, these results are nothing new; we couldn't obtain using the familiar a and a^\dagger operators from the typical algebraic treatment of the harmonic oscillator. In going through this analysis, one sees that the operators A and A^\dagger are performing the roles of the operators a and a^\dagger . Indeed, $W(x) = (m\omega^2/2)^{1/2}x$ implies

$$A = \sqrt{\hbar\omega} a$$

$$A^\dagger = \sqrt{\hbar\omega} a^\dagger$$

Regardless, it's pleasing that the forms of the ladder operators are suggested by this more general supersymmetric technique. We'll see the same in other contexts.

1.3 Two Quick Examples

Taking our superpotential to be $W(x) = \tanh x$, we get two paired potentials

$$V_1(x) = 1 \quad V_2(x) = 1 - 2 \operatorname{sech}^2 x$$

The second of these is the *Rosen-Morse potential*, which is relevant to the study of Fermi liquids and to the Korteweg-de Vries equation. I have no idea about either of these topics. But it does have two nice and readily observed properties. First, because the constant potential has no bound states, the Rosen-Morse potential can have at most one, corresponding to a zero-mode. Indeed, we find such a zero energy solution does exist and takes the form

$$\phi_0(x) = C \operatorname{sech} x$$

The second observation is that the famous reflectionless behavior of the Rosen-Morse potential follows immediately from the fact that it is paired to a constant potential.

Another example, with no interesting corollaries that I know of, is the superpotential $W(x) = \Theta(x) - \Theta(-x)$. The two paired potentials are then $V(x) = 1 \pm 2\delta(x)$, an attractive and repulsive δ -function potential.

1.4 Treatment of the Pöschl-Teller Potential

We can use the method of isospectral Hamiltonians to evaluate the spectrum of a rather difficult potential by relating it to a very simple system. Take $W(x) = \tan x$. This choice gives

$$A = \frac{d}{dx} + \tan x$$

$$A^\dagger = -\frac{d}{dx} + \tan x$$

We get the two paired Hamiltonians:

$$H^{(-)} = A^\dagger A = -\frac{d^2}{dx^2} - 1$$

$$H^{(+)} = AA^\dagger = -\frac{d^2}{dx^2} + 2 \sec^2 x - 1$$

The second of these has been previously studied under the name of the Pöschl-Teller potential.

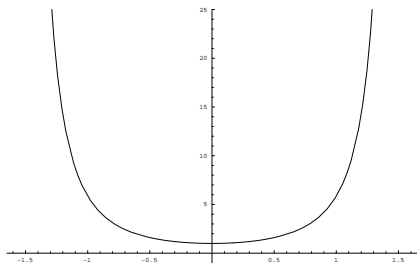


Figure 2: The Pöschl-Teller potential $V_2(x) = 2 \sec^2 x - 1$, which is paired to the free particle in a box.

Since the second potential is naturally limited in spatial extent by the divergence of $\sec^2 x$ at $x = \pm \frac{\pi}{2}$, we impose the same restriction on the first Hamiltonian as well. The system described by $H^{(-)}$ is therefore revealed to be a free particle in a box, with eigenstates

$$\phi_n(x) = \begin{cases} \sqrt{\frac{2}{\pi}} \sin nx & \text{for } n = 2, 4, 6, \dots \\ \sqrt{\frac{2}{\pi}} \cos nx & \text{for } n = 1, 3, 5, \dots \end{cases}$$

and corresponding eigenenergies $E_n = n^2 - 1$. In particular, $E_1 = 0$, so there will not be an equivalent eigenstate of $H^{(+)}$. Furthermore, solving the Riccati equation would yield the ground state of vanishing energy:

$$\left(\frac{d}{dx} + \tan x\right)\phi_0(x) = 0 \longrightarrow \phi_0 = C \cos x$$

Take the eigenspectrum of $H^{(-)}$ over to $H^{(+)}$ to find the energy eigenstates.

$$\psi_n(x) = \begin{cases} E_n^{-1/2} \left(\frac{d}{dx} + \tan x\right) \sqrt{\frac{2}{\pi}} \sin nx & \text{for } n = 2, 4, 6, \dots \\ E_n^{-1/2} \left(\frac{d}{dx} + \tan x\right) \sqrt{\frac{2}{\pi}} \cos nx & \text{for } n = 3, 5, 7, \dots \end{cases}$$

The corresponding energy eigenvalues are of course still $E_n = n^2 - 1$. Plots of some of the states are shown below.

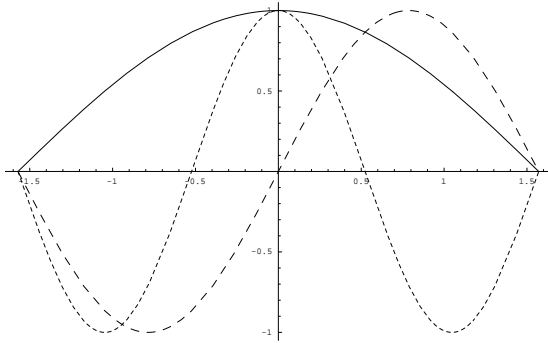


Figure 3: Three lowest energy eigenstates of $H^{(-)}$.

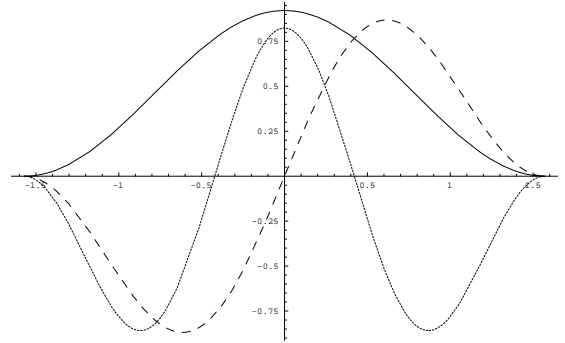


Figure 4: Three lowest energy eigenstates of $H^{(+)}$.

1.5 Free Particle in 3 Dimensions

This section is mostly useful to us as a precursor to the next, where we add a Coulombic potential to the Hamiltonian. It also finds use in the treatment of scattering off a hard sphere.¹ Spherical symmetry of the Hamiltonian allows us to write $\psi_{klm}(r, \theta, \phi) = R_{kl}(r)Y_{lm}(\theta, \phi)$ with $R_{kl}(r)$ satisfying

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2}\right] R_{kl}(r) = \frac{\hbar^2 k^2}{2m} R_{kl}(r)$$

I will suppress all physical values here and instead just focus on the mathematical technique: set $\hbar = 2m = k = 1$. Switching to the function $u = rR(r)$, we get

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}\right] u_l(r) = u_l(r)$$

Now define

$$A_l := \frac{d}{dr} - \frac{l+1}{r}$$

$$H_l^{(-)} := A_l^\dagger A_l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}$$

Direct computation reveals the paired Hamiltonian $H_l^{(+)} = A_l A_l^\dagger$ actually equals $H_{l+1}^{(-)}$! So, given an eigenfunction u_l of $H_l^{(-)}$ with non-zero energy, we get an eigenfunction $A_l u_l$ of $H_{l+1}^{(-)}$ for free: first note that $A_l u_l$

¹If the quantum particles being scattered are photons, this is known as *Mie scattering* and is needed in understanding the appearance of a great variety of substances, like milk.

is an eigenstate of $H_l^{(+)}$ and then that $H_l^{(+)} = H_{l+1}^{(-)}$. Since there is no attractive potential, all states have positive energy.

Let's construct some of the solutions. The base case ($l = 0$) has the solutions

$$-u_0''(r) = 0 \longrightarrow u_0(r) = \sin r \text{ or } \cos r$$

From here, we can repeatedly apply the appropriate A_l to generate all the remaining solutions. The first few are

l	Solution 1	Solution 2
0	$\sin r$	$\cos r$
1	$\frac{\sin r}{r} - \cos r$	$\frac{\cos r}{r} + \sin r$
2	$\left(\frac{3}{r^2} - 1\right) \sin r - \frac{3}{r} \cos r$	$\left(\frac{3}{r^2} - 1\right) \cos r + \frac{3}{r} \sin r$

These are known as the *Riccati-Bessel functions*. Divide through each of these by r to get $R(r)$ and we recognize them as the familiar spherical Bessel functions that we knew all along were the radial wavefunctions.

1.6 Hydrogen Atom

Adapted from a lecture by Prof. Robert Jaffe of MIT

We have already seen in Section 3.1 that the hydrogen spectrum displays a degeneracy of energy levels – E_n is n^2 -fold degenerate – is larger than that which would result purely from rotational invariance of the potential. The Runge-Lenz vector expands this symmetry to one that accounts fully for these degeneracies. Read about this in Appendix A. Supersymmetry is yet another way to account for this larger than expected degeneracy. In fact, we'll see later that supersymmetry suggests the exact form of the Runge-Lenz vector in the same way it suggested the form of the harmonic oscillator's ladder operators. The effective Hamiltonian for the radial wavefunction of an electron in the hydrogen atom is

$$H_l^{\text{coul}} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{1}{r}$$

The label l denotes that this Hamiltonian has had the effect of angular dependence distilled into an effective angular momentum barrier (a "kinetic potential") that depends on l , the quantum number labelling the total orbital angular momentum of the electron. Now, define two operators, also taking labels l , by

$$A_l = \frac{d}{dr} - \frac{l+1}{r} + \frac{1}{2(l+1)}$$

$$A_l^\dagger = -\frac{d}{dr} - \frac{l+1}{r} + \frac{1}{2(l+1)}$$

These operators yield partner Hamiltonians

$$H_l^{(-)} = A_l^\dagger A_l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{1}{r} + \frac{1}{4(l+1)^2}$$

$$= H_l^{\text{coul}} + \frac{1}{4(l+1)^2}$$

$$H_l^{(+)} = A_l A_l^\dagger = -\frac{d^2}{dr^2} + \frac{(l+1)(l+2)}{r^2} - \frac{1}{r} + \frac{1}{4(l+1)^2}$$

$$= H_{l+1}^{\text{coul}} + \frac{1}{4(l+1)^2}$$

One can solve both $A_l|\phi_l\rangle = 0$ and $A_l^\dagger|\phi_l\rangle = 0$ to find that only the former gives a sensible, normalizable result. Thus, $H_l^{(-)}$ that has a zero energy groundstate. This means that SUSY is a good symmetry.

$$\begin{aligned} A_l\phi_l(r) &= \left[\frac{d}{dr} - \frac{l+1}{r} + \frac{1}{2(l+1)} \right] \phi_l(r) = 0 \\ \frac{d\phi_l}{\phi_l} &= \left[\frac{l+1}{r} - \frac{1}{2(l+1)} \right] dr \\ \therefore \phi_l(r) &= Cr^{l+1}e^{-r/2(l+1)} \end{aligned}$$

We can also compute

$$H_l^{(-)}\phi_l = \left[H_l^{\text{coul}} + \frac{1}{4(l+1)^2} \right] \phi_l = 0 \longrightarrow H_l^{\text{coul}}\phi_l = -\frac{1}{4(l+1)^2}\phi_l$$

We now have the functional form and energy in both the shift Hamiltonian $H_l^{(-)}$ and unshifted Hamiltonian H_l^{coul} of this state. The rest of the treatment comes from observing that $H_l^{(-)} = H_l^{\text{coul}} + 1/4(l+1)^2$ and $H_l^{(+)} = H_{l+1}^{\text{coul}} + 1/4(l+1)^2$ together yield

$$H_l^{(+)} = H_{l+1}^{(-)} + \frac{1}{4(l+1)^2} - \frac{1}{4(l+2)^2}$$

We can now relate the isospectral pair $H_l^{(-)}$ and $H_l^{(+)}$ to the isospectral pair $H_{l+1}^{(-)}$ and $H_{l+1}^{(+)}$. By extension,

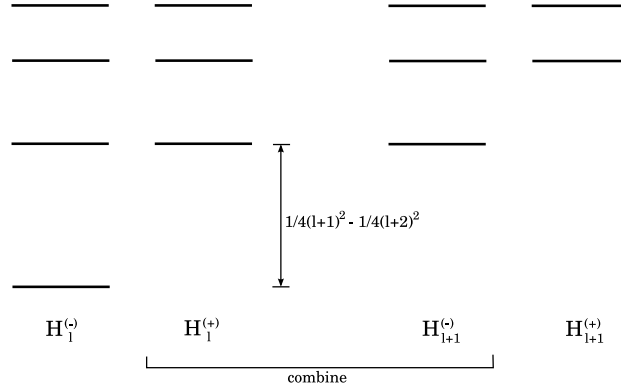


Figure 5: Combining two adjacent pairs of isospectral Hamiltonians

we can combine all pairs of isospectral Hamiltonians, expressing every $H_l^{(+)}$ as a shifted copy of $H_{l+1}^{(-)}$. From above, we have computed the functional form of each of the states along the diagonal bounding the set of states from below. Repeated application of the appropriate A_l^\dagger operator will send us leftward across a row, which will end after the $l = 0$ state is reached. The energy remains the same as we move through the l values. Since it was the original, maximal value of l that appears in the energy eigenvalue of $|\phi_l\rangle$ under H_l^{coul} , we define $n = l_{\text{max}} + 1$ and say

$$E_n = -\frac{1}{4n^2}$$

It is now important to make two notes about the interpretation of this solution, both stemming from conventions set when we defined H_l^{coul} . First, by choosing $\hbar = 2m = e = 1$, we implicitly set the Bohr radius to $a_0 = 2$. With units restored, the exponential in the above radial wavefunction would actually read $\exp[-r/a_0(l+1)]$. Restoring units in the expression for E_n gives the standard result as well. Second, we also solved for $u(r) = rR(r)$, so one must divide by r to get the radial wavefunction.

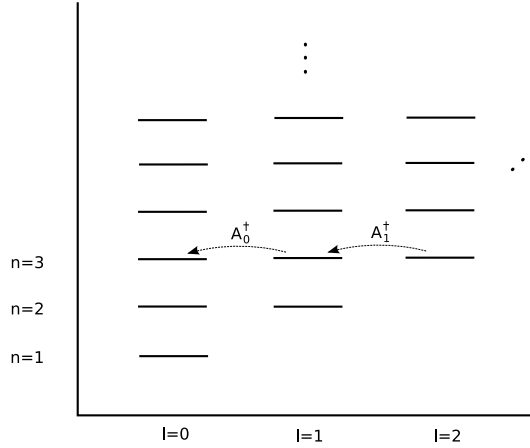


Figure 6: Scheme for obtaining functional form of states

We should now perform an explicit calculation for a simple case, say $l = 2$. The lowest energy state in the $l = 2$ tower can be seen in Figure 6 to be $n = 3$ according to the labelling scheme chosen here. Then, the state has radial distribution given by

$$u_{3,2}(r) = Ar^3 e^{-r/6}$$

Move to the $l = 1$ tower by applying A_1^\dagger .

$$\begin{aligned} u_{3,1}(r) &= A_1^\dagger u_{3,2}(r) \\ &= \left(-\frac{d}{dr} - \frac{2}{r} + \frac{1}{4} \right) Ar^3 e^{-r/6} \\ &= Br^2 \left(1 - \frac{r}{12} \right) e^{-r/6} \end{aligned}$$

Finish the row by finding the $l = 0$ state

$$\begin{aligned} u_{3,0}(r) &= A_0^\dagger u_{3,1}(r) \\ &= \left(-\frac{d}{dr} - \frac{1}{r} + \frac{1}{2} \right) Br^2 \left(1 - \frac{r}{12} \right) e^{-r/6} \\ &= Cr \left(\frac{r^2}{18} - r + 3 \right) e^{-r/6} \end{aligned}$$

With units restored, and converting to $R(r)$, these three become

$$\begin{aligned} R_{3,2}(r) &= r^2 e^{-r/3a_0} \\ R_{3,1}(r) &= r \left(1 - \frac{r}{6a_0} \right) e^{-r/3a_0} \\ R_{3,0}(r) &= \left[\frac{2}{9} \left(\frac{r}{a_0} \right)^2 - 2 \frac{r}{a_0} + 3 \right] e^{-r/3a_0} \end{aligned}$$

up to an omitted normalization on each of these.

2 Supersymmetry in the Context of Quantum Mechanics

One has a supersymmetric quantum mechanical system when the following conditions are met: there is a Hamiltonian H acting on the Hilbert space \mathcal{H} , along with N self-adjoint operators $Q_i = Q_i^\dagger$ satisfying

$$\{Q_i, Q_j\} = H\delta_{ij} \text{ for } i, j = 1, \dots, N$$

We call the Q_i 's the *supercharges* of the theory. Note that by the above $H = 2Q_1^2 = \dots = 2Q_N^2$, so it automatically follows that

$$[Q_i, H] = 0 \text{ for all } i$$

Sometimes the above definition does not involve the self-adjoint condition, instead folding two supercharges into one complex supercharge²

$$\tilde{Q}_i := \frac{1}{\sqrt{2}}(Q_{2i} + iQ_{2i+1})$$

Working it out, we see that these will satisfy a slightly different algebra

$$\begin{aligned} \{\tilde{Q}_i, \tilde{Q}_j\} &= 0 \\ \{\tilde{Q}_i^\dagger, \tilde{Q}_j^\dagger\} &= 0 \\ \{\tilde{Q}_i, \tilde{Q}_j^\dagger\} &= H\delta_{ij} \end{aligned}$$

for all i, j . The two pictures are equivalent.

2.1 Famous $N = 1$ SUSY System: The Pauli Hamiltonian

Given a vector potential $\vec{A}(\vec{x})$, define a self-adjoint supercharge

$$Q = Q^\dagger := \frac{1}{2\sqrt{m}}(\vec{p} - \frac{e}{c}\vec{A}) \cdot \vec{\sigma}$$

We find the corresponding Hamiltonian is supersymmetric.

$$H = 2Q^2 = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2 - \frac{e\hbar}{2mc}\vec{\sigma} \cdot \vec{B}$$

We recognize this as nothing other than a spin- $\frac{1}{2}$ particle in a magnetic field. The interaction term has a Landé g factor of 2. Therefore, *classically* an electron in a magnetic field possesses a supersymmetry as described here. Quantum effects break this.

2.2 $N = 2$ SUSY QM: The Witten Model

At this point, we can make contact with the discussion of the operators A and A^\dagger from the section on isospectral Hamiltonians by identifying one complex supercharge Q with the matrices

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}$$

from which it easily follows that $\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0$ and

$$\{Q, Q^\dagger\} = \begin{pmatrix} H^{(+)} & 0 \\ 0 & H^{(-)} \end{pmatrix}$$

where $H^{(-)} = A^\dagger A$ and $H^{(+)} = AA^\dagger$. We've now folded the two separate systems with identical energy spectra (up to zero modes) into one system living in a Hilbert space $\mathcal{H} = \mathcal{H}^{(+)} \oplus \mathcal{H}^{(-)}$ with a Hamiltonian $H = H^{(+)} \otimes \mathbb{I} + \mathbb{I} \otimes H^{(-)}$.

We look for an operator that will distinguish the two subspaces from one another. Clearly H cannot. Recall that the whole reason we used an anticommutator in $H = \{Q, Q^\dagger\}$ we to put the same sign in from of $H^{(+)}$ and $H^{(-)}$ in their respective subspaces. Had we not done so, we would've gotten

$$[Q, Q^\dagger] = \begin{pmatrix} AA^\dagger & 0 \\ 0 & -A^\dagger A \end{pmatrix} = \begin{pmatrix} H^{(+)} & 0 \\ 0 & -H^{(-)} \end{pmatrix}$$

²The tildes over \tilde{Q} aren't standard notation. I include them only to distinguish from the self-adjoint case. Since one should stick to one convention or another, there's never any ambiguity that requires this extra decoration.

This is exactly what we want. The eigenvalues of this operator are non-negative on the $\mathcal{H}^{(+)}$ subspace and non-positive on the $\mathcal{H}^{(-)}$ subspace. Let's "divide by" the Hamiltonian to scale out any sensitivity to the exact energy levels and call the result

$$W := \frac{[Q, Q^\dagger]}{\{Q, Q^\dagger\}} = \begin{pmatrix} +\mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

This is the *Witten parity operator*. It's eigenstates are the eigenstates of H , but the eigenvalues now differ between the two subspaces of \mathcal{H} . As it stands, it's only well-defined on the orthogonal complement to $\ker H$ but a variety of regularization procedures resolve this issue. The Witten parity operator implements a \mathbb{Z}_2 grading of the Hilbert space. Energy eigenstates are even or odd if they have eigenvalues $+1$ or -1 , respectively, of W . We can also talk about the parity of operators

$$\begin{aligned} [W, E] = 0 &\iff E \text{ is even} \\ \{W, O\} = 0 &\iff O \text{ is odd} \end{aligned}$$

We've noted that whether or not SUSY is a good symmetry is determined by the ground state energy. Define the *Witten index* by

$$\Delta := \text{Tr}(-W) = \dim \ker H^{(-)} - \dim \ker H^{(+)}$$

All states with energy $E > 0$ come in pairs: one eigenstate of $H^{(+)}$ and one of $H^{(-)}$, so such states contribute nothing to Δ . Cases when $\Delta \neq 0$ have at least one zero mode and hence correspond to a system with good SUSY. $\Delta = 0$, however, does *not* mean SUSY is necessarily broken, at least in general.

2.3 Hints of Superspace

Consider a quantum mechanical system with a complex supercharge Q . Since $H = i\frac{\partial}{\partial t}$ is the generator of time translations and $H = \{Q, Q^\dagger\}$ gives a relationship between the supercharge and this generator of translation, we ask if Q is a generator of translations in some coordinate too such that the given anticommutator reduces to a time translation. The fact that this is an *anti*-commutator suggests that Grassmann variables are natural. We posit a Grassmann valued coordinate θ , and another $\bar{\theta}$ for Q^\dagger . We want Q to go like $i\frac{\partial}{\partial \theta}$ but also need to bring time translations into the mix too so that we'll get the right behavior under anticommutation. Hence, we'll need to add terms of the form $\theta\frac{\partial}{\partial t}$ and $\bar{\theta}\frac{\partial}{\partial t}$. After tooling around a bit, we find that

$$\boxed{\begin{aligned} Q &= i\frac{\partial}{\partial \theta} - a\bar{\theta}\frac{\partial}{\partial t} \\ Q^\dagger &= -i\frac{\partial}{\partial \bar{\theta}} + \bar{a}\theta\frac{\partial}{\partial t} \end{aligned}}$$

are the most general linear combinations that satisfy $\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0$ and still mix derivatives in θ and t . We find

$$\begin{aligned} \{Q, Q^\dagger\} &= \left\{ \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \bar{\theta}} \right\} - a\bar{a}\{\theta, \bar{\theta}\}\frac{\partial^2}{\partial t^2} + \left(i\bar{a}\left\{ \frac{\partial}{\partial \theta}, \theta \right\} + ia\left\{ \frac{\partial}{\partial \bar{\theta}}, \bar{\theta} \right\} \right) \frac{\partial}{\partial t} \\ &= i(a + \bar{a})\frac{\partial}{\partial t} = (a + \bar{a})H \end{aligned}$$

We can pick a and \bar{a} to suit our convenience.³

We can now make a (unitary) SUSY transformation of our state: $|\psi\rangle \longrightarrow \exp(\frac{i}{\hbar}G)$ where

$$G = i\xi^*Q - iQ^\dagger\xi = G^\dagger$$

³Some authors seem to pick $a = \bar{a} = 1$ and then have $\{Q, Q^\dagger\} = 2H$ instead.

is the Hermitian generator of this transformation. In order for the state to be SUSY invariant, we need

$$Q|\psi\rangle = Q^\dagger|\psi\rangle = 0$$

This is a property that we require of the ground state if we want SUSY to be a good (ie, unbroken) symmetry. As we've seen, this requires the ground state energy to vanish.

From here, we can define covariant derivatives

$$D := \frac{\partial}{\partial\theta} - i\bar{a}\bar{\theta}\frac{\partial}{\partial t}$$

$$D^\dagger := \frac{\partial}{\partial\bar{\theta}} + ia\theta\frac{\partial}{\partial t}$$

that satisfy

$$\{D, Q^\dagger\} = 0 \quad \text{and} \quad \{D^\dagger, Q\} = 0$$

This notion will be useful later in the context of a field theory and won't be developed further here.

3 Appendix

3.1 Appendix A: Runge-Lenz Vector

Adapted from a lecture by Prof. Tom Mehen of Duke University

Neglecting the all interactions but the Coulomb interaction between the proton and the electron, the non-relativistic hydrogen atom has energy levels given by:

$$E_n = -\frac{13.6 \text{ eV}}{n^2}$$

The operators \hat{L}^2 and \hat{L}_z both generate symmetries of the Hamiltonian, since $[L^2, H] = [L_z, H] = 0$. We call these the Casimir operator and root, respectively, of an $\text{SO}(3)$ symmetry exhibited by this system. In other words, the hydrogen atom in free space exhibits rotational invariance. All Hamiltonian eigenstates related to one another by the symmetries generated by \hat{L}^2 and \hat{L}_z have the same energy eigenvalues. But this implies only a $(2l + 1)$ -fold degeneracy among the energy eigenvalues, whereas they are truly n^2 -fold degenerate. We have an *accidental symmetry* of the system, probably better dubbed a *hidden symmetry*.

This extended symmetry is due to the Runge-Lenz vector

$$\vec{A}_{\text{cl}} = \vec{L} \times \vec{p} + \frac{me^2}{r}\vec{r}$$

$$\vec{A}_{\text{qu}} = \frac{1}{2}(\vec{L} \times \vec{p} - \vec{p} \times \vec{L}) + \frac{me^2}{r}\vec{r}$$

which is a constant of the motion. Note the following commutation relations among the Runge-Lenz vector and the angular momentum vector

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$$

$$[L_i, A_j] = i\hbar \epsilon_{ijk} A_k$$

$$[A_i, A_j] = -i\hbar \cdot 2m|H|\epsilon_{ijk} L_k$$

Note, $H < 0$. This is the set of commutation relations for an $\mathfrak{so}(4)$ algebra! If we define $A_i = \sqrt{2mH}k_i$ then we obtain the more symmetric commutation relations

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$$

$$[L_i, k_j] = i\hbar \epsilon_{ijk} k_k$$

$$[k_i, k_j] = i\hbar \epsilon_{ijk} L_k$$

Owing to the imaginary ratio of k_i to A_i , we have turned this into the set of commutation relations for $\mathfrak{so}(3, 1)$, the algebra of the Lorentz group. Defining $J_i^\pm = \frac{1}{2}(L_i \pm k_i)$, we get

$$\begin{aligned} [J_i^+, J_j^+] &= i\hbar \epsilon_{ijk} J_k^+ \\ [J_i^-, J_j^-] &= i\hbar \epsilon_{ijk} J_k^- \\ [J_i^+, J_j^-] &= 0 \end{aligned}$$

which gives two decoupled copies of $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$. That is, $\mathfrak{so}(3, 1) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$. Then,

$$(\vec{J}^\pm)^2 = \frac{1}{4}(\vec{L}^2 + \vec{k}^2 \pm 2\vec{L} \cdot \vec{k})$$

but $\vec{L} \cdot \vec{k} \propto \vec{L} \cdot \vec{A} = 0$. Thus, $(\vec{J}^+)^2 = (\vec{J}^-)^2$. It is important to note: the algebra does not impose this condition, it is the physics that led us to see these two $\mathfrak{su}(2)$'s have the same magnitude. So, our good quantum numbers are: $(\vec{J}^+)^2$, J_z^+ , J_z^- , and the states are $|j, m^+, m^-\rangle$ with $-j \leq m^+, m^- \leq j$. This yields a $(2j + 1)^2$ -fold degeneracy. Note now that

$$-2mE\vec{k}^2 = \vec{A} \cdot \vec{A} = 2mE(\vec{L}^2 + \hbar^2) + m^2e^4$$

where the \hbar^2 appeared due to the quantum commutator. Solving for the energy E yields

$$\begin{aligned} E &= -\frac{me^4}{2(\vec{L}^2 + \vec{k}^2 + \hbar^2)} \\ &= -\frac{me^4}{2(4\vec{J}^2 + \hbar^2)} \\ &= -\frac{me^4}{2\hbar^2(2j + 1)^2} \end{aligned}$$

Then, $n = 2j + 1$ with degeneracy $(2j + 1)^2 = n^2$. We've recovered the energy levels and degeneracies for the hydrogen atom.

3.2 Appendix B: Runge-Lenz Vector \leftrightarrow Supercharge of Hydrogen

There are a number of sources that show this connection in a variety of ways but none of them really use a style that I like. I haven't made my own yet but here's a start. Our electron has a spin state described by the vector $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$. Then, taking our supercharge Q_l which will correspond eventually to the SUSY ladder operator A_l from our treatment of Hydrogen to be

$$Q_l = \frac{\vec{S} \cdot \vec{A}}{l + 1}$$

we get a corresponding Hamiltonian

$$H = 2Q^2 = \frac{1}{2(l + 1)^2} |\vec{A}|^2 = \frac{1}{2(l + 1)^2} [2H(\vec{L}^2 + 1) + 1]$$

after suppressing physical values. As it stands, the numerator $\vec{L}^2 = l(l + 1)$ doesn't cancel against the denominator $(l + 1)^2$. I'm sure the issue lies in a subtlety regarding incorporating spin into the right hand side. I'll think about this more later. Once that happens, it will resemble our shifted Hamiltonian

$$H_l^{(-)} = H_l^{\text{coul}} + \frac{1}{2(l + 1)^2}$$

It would be nicer still to go from the supercharge to the precise form of the Runge-Lenz vector. I've found this done in papers using a variety of unsatisfactory tricks.