The Twin Paradox in a Compact Space

Joshua Albert

December 17, 2007

1 Introduction

The twin paradox is a well known puzzle for young physicists to ponder. The setup is simple. Two twins decide to go separate ways, one will stay home on Earth, the other will fly on a spaceship traveling at high speeds away from Earth, and then turn around and come back. Usually a constant velocity during the outward going and return trip is required. The twin on earth (we'll call her Georgette) believes that the moving twin's clock runs slow. This leads to the belief that she should be older than her space-faring twin. In the frame of the astronaut twin (we'll call her Stella), however, the earth appears to be moving, and so clocks on Earth appear to run slow.

This is usually where one leaves the paradox to make it seem as puzzling as possible. The situation appears to have perfect symmetry, yet it must not, as this would lead to a failure of at least one of the twin's calculations. Furthermore, experiments with atomic clocks have confirmed that moving clocks run slow. The resolution is that there is no perfect symmetry–the moving twin must accelerate at some point so as to turn around, while the earthbound twin just stays there.

A useful and instructive exercise is to consider the paradox with Doppler shifted time signals being sent from each twin to the other. This, especially when drawn on a spacetime diagram, can make it more clear exactly how the twins perceive each other's aging. While a Doppler analysis of timing signals can be informative for the traditional twin paradox (see figure 1), it will not help very much in the problem discussed later on in this paper.

Many variations on this problem have been proposed, but often they still come down to an asymmetry between the twins because one of them must accelerate. In an infinite flat universe, there is no way for the twins to meet after Stella leaves without one of the twins accelerating. To get around this, we need a universe where a straight line can go through one point more than once. From this requirement, we are led

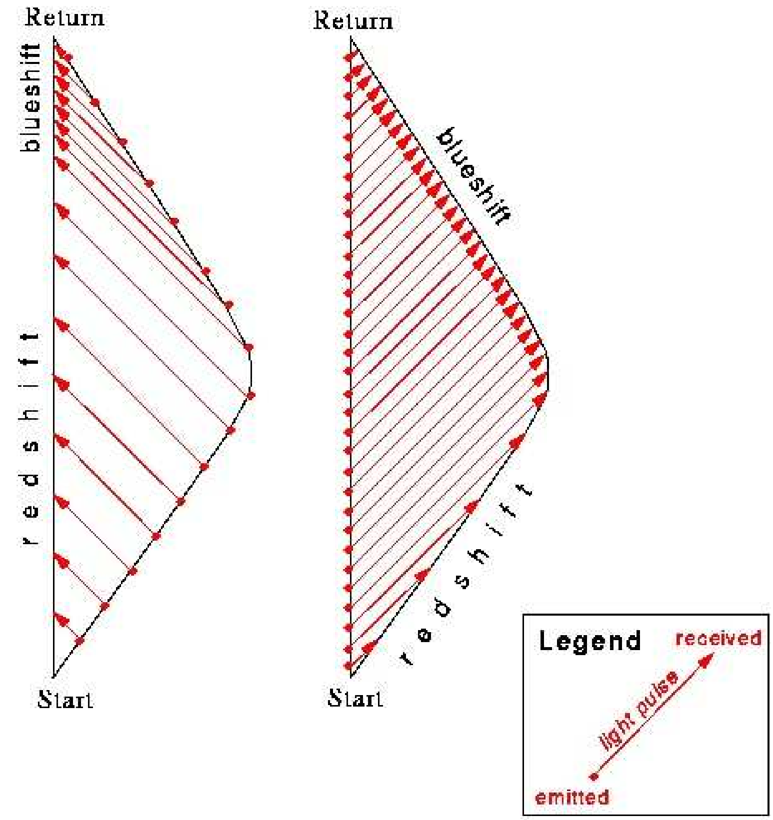

Figure 1: The twin paradox explained through Doppler shifted timing pulses in space-time diagrams. From [3]. I apologize for the image quality.

to a compact space.

2 The Compact Space

2.1 Compact Spaces

This paradox takes place in a space with at least one compact dimension. This means that if one travels far enough in some direction, they will return to their starting point from the other side. This is much like living on the surface of the earth–if you travel East along the equator far enough, you'll return to the starting point from the West. In a universe like ours, the cosmological principle that the universe should be isotropic means that every dimension is either compact or infinite. For purposes of this problem, we will keep motion along one dimension only, so the compactness of the other dimensions does not need to concern us.

2.2 A Simple Condition

We will want our space to function like normal Euclidean space, obeying all the same laws obeyed in our universe. This includes the Lorentz transformations. The Minkowski metric, $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, is still valid. We however, add the following condition [2]:

$$(t, x, y, z) \sim (t, x + nL, y, z) \tag{1}$$

Here, $n \in \mathbb{Z}$, and L is the length of this compact dimension of the universe. This equivalence relation means that those sets of coordinates should be equivalent, so any function of the coordinates, including electromagnetic fields, trajectories of particles, and so on must be cyclic in the x direction with periodicity L.

The trouble arises because the periodicity requirement (1) is not Lorentz invariant. Let's demonstrate this. In a frame S, we define the point $p_0 = (t_0, x_0, y_0, z_0)$. Using the relation (1), we find all points

$$p_n = (t_0, x_0 + nL, y_0, z_0) \tag{2}$$

correspond to that original p_0. We now consider a new frame, $\overline{S}$, which is moving at a velocity v in the x direction with respect to S. We use the following Lorentz transformation:

$$\begin{pmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \gamma(t - \beta x) \\ \gamma(x - \beta t) \\ y \\ z \end{pmatrix} \tag{3}$$

Note that we are using units where t is equivalent to ct, for simplicity. This gives:

$$\bar{p_0} = (\gamma t_0 - \gamma\beta x_0, \gamma x_0 - \gamma\beta t_0, y_0, z_0) \tag{4}$$

and

$$\bar{p_n} = (\gamma t_0 - \gamma\beta x_0 - \gamma\beta nL, \gamma x_0 + \gamma nL - \gamma\beta t_0, y_0, z_0) \tag{5}$$

Then, combining (4) and (5) we get:

$$\bar{p_n} = (\bar{t_0} - \gamma\beta nL, \bar{x_0} + \gamma nL, \bar{y_0}, \bar{z_0}) \tag{6}$$

How odd! We seem to have found that the equivalence relation is different for a different frame. We can write the general relation as

$$(\bar{t}, \bar{x}, \bar{y}, \bar{z}) \sim (\bar{t} - \beta\gamma nL, \bar{x} + \gamma nL, \bar{y}, \bar{x}) \tag{7}$$

which reduces to (1) for $\beta \to 0$. Clearly, we need to have some preferred reference frame in which to measure β. We can define

$$L_{eff} \equiv \gamma L \tag{8}$$

to represent the effective size of the compact dimension in a frame $\overline{S}$. The relevance of this definition should be made apparent by comparing the x behavior in (7) and (1). The preferred frame can be identified as having the smallest L_{eff}.

This certainly mucks up the universe. In any non-preferred frame, a clock at any point p_0 must have a different time than a clock at any p_n (supposing the clocks are working properly). This makes measuring the absolute time everywhere in the universe impossible. In the preferred frame S, a set of clocks could be set up (in typical Einstein fashion) and calibrated to be aligned throughout the universe. However, in $\overline{S}$, the best one can do is to set up aligned clocks throughout some "patch" of the universe which does not extend to meet up with itself around the compact dimension. Effectively, a branch cut is needed, much like one would see in complex analysis. Across this branch cut in space, a "transition function" can be used to realign the time and position with the chosen branch. The analogy to this in complex analysis would have $f_\pm(g(\theta)) = g(\theta \mp 2\pi)$. Thus, we write our transition functions [2].

$$f_\pm(g(t, x, y, z)) = g(t \pm \beta\gamma L, x \mp \gamma L, y, z) \tag{9}$$

where f_+ and f_- are transition functions for passing the branch cut in the positive and negative directions, respectively.

2.3 Generalization

If we would like to generalize this to a space with more than one compact dimension, we can produce a hypertorus with all dimensions compact. The following setup comes directly from [1].

We embed the (3+1) dimensional spacetime in a (4+1) dimensional spacetime with the new coordinate fixed. Thus,

$$x^\alpha = \begin{pmatrix} t \\ x \\ y \\ z \\ q \end{pmatrix}, \tag{10}$$

where q is a fixed value (we can set it to unity) and we let Greek indices run over 0,1,2,3,4. We can now write some matrices to represent the generator from one point to a corresponding one one universe circling away. For example, with

$$T_x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & L_x \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

we can rewrite (1) as

$$x \sim (T_x)^n x. \tag{12}$$

When we consider also the other compact dimensions,

$$T_y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & L_y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, T_z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & L_z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{13}$$

we can form a group of equivalence relations with these three generators. Every element ϕ in this group is the product of some number of generators T_x, T_y, T_z. It is also possible to include some rotations in this group [1] to make a truly mind-bending compact space, but we'll leave that out for now. In general, our equivalence relation is

$$x^\alpha \sim \phi^\alpha_\beta x^\beta. \tag{14}$$

Next, we can perform a Lorentz transformation to get our relation in any frame, the very general form corresponding to (7).

$$x^\alpha \sim \Lambda^\alpha_\beta \phi^\beta_\gamma x^\gamma \tag{15}$$

for

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z & 0 \\ -\gamma\beta_x & 1+\frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & 0 \\ -\gamma\beta_y & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1+\frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 0 \\ -\gamma\beta_z & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1+\frac{(\gamma-1)\beta_z^2}{\beta^2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

We can now define any trajectory taken by the traveling twin, Stella, by defining a ϕ to correspond to the trajectory. On this trajectory, $x_{end} = \phi x_{start}$. See figure 2 for an example of such a trajectory. Then, the Lorentz transformation should provide us with all we need to resolve what each twin observes.

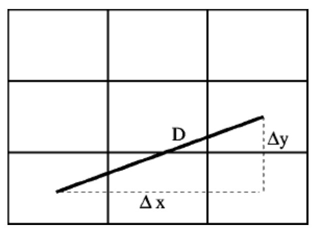

Figure 2: Here is an example trajectory for the twin on a hypertorus with compact x and y dimensions. Each rectangle represents the universe, the tiling simply makes it easier to see the total distance traversed, and each rectangle should relate to each other one through an equivalence relation. The starting point and ending point are equivalent points, given by $x_{end} = T_y T_x^2 x_{start}$. Image from [1].

3 Paradox Resolution

Now that the mathematical description of our universe is all set, we can resolve the paradox. Let us consider the simplest possible case, a trajectory along the x axis only from Earth straight to the nearest corresponding Earth at $x + L_x$. We will examine the Earth frame first, and see things from Georgette's perspective, then we'll examine Stella's. We'll make our calculations using the notation of section 2.2, because it makes things easier to follow.

3.1 Georgette's Calculations

When Stella was born, she came into this world travelling at velocity β directly into a spaceship pre-accelerated for her. Strange stuff. Anyway, Georgette watches Stella travel away at velocity $\beta\hat{x}$. For Georgette, in the preferred frame, the distance that Stella travels is L_x, and there is no time difference between the starting Earth and ending Earth. Thus, the time that Georgette measures between Stella's departure and the high speed high five which will mark Stella's next pass is given by

$$\Delta t_G = L_x/\beta. \tag{17}$$

Georgette, predictably, sees Stella's clock running slow, and so expects Stella to measure

$$\Delta t_S = \Delta t_G/\gamma = L_x/(\gamma\beta). \tag{18}$$

3.2 Stella's Calculations

Stella had an odd childhood. From her perspective, her sister, mother, and home planet have been sailing away from her at a constant velocity $-\beta\hat{x}$ since birth. Stella is not in the universe's preferred frame, so she sees the distance that the Earth travels to reach her again to be $L_{x,eff} = \gamma L_x$. There is also a time adjustment that comes from crossing the branch cut. We can use the f_- function to determine this time difference. What we will find is that Stella observes the Earth that she sees coming towards her from the distance to have gotten a head start!

$$f_-(\overline{t_G}) = \overline{t_G} - (-\beta\gamma L_x) = \overline{t_G} + \overline{t_{headstart}} \tag{19}$$

$$\overline{\Delta t_S} = L_{x,eff}/\beta - \overline{t_{headstart}} = \gamma L_x/\beta - \beta\gamma L_x = L_x/(\gamma\beta) \tag{20}$$

This is in perfect agreement with Georgette's observation from (18). It is also what one would expect from a trip where the distance has been Lorentz contracted by a factor of γ. That's just a nice confirmation that things operate as you'd expect. Finally, Stella sees Georgette's clock running slow. She calculates:

$$\overline{\Delta t_G} = L_{x,eff}/(\beta\gamma) = L_x/\beta. \tag{21}$$

And we are done with this paradox! Both Stella and Georgette agree on each other's ages, and the existence of a preferred reference frame does away with the false symmetry which makes this appear to be a paradox in the first place.

All of these calculations could, of course, be done using the tensor notation from section 2.3, but the results are the same, and this notation is, in this author's opinion, clearer.

4 Tests for a Preferred Frame

While the paradox has been resolved, there is still one other topic to explore in our tour of this universe with a compact dimension. How does one determine whether one's frame is the preferred one?

4.1 Light Pulses

A well known method [2] to test if one is in the preferred frame in a compact universe is to send out a light pulse in either direction along the compact dimension (or, if you don't know where that is, just send it in all directions). While not all frames are equal in this universe, the speed of light is still c. The difference in frames depends on the transition function. Because a non-preferred frame will have a time shift added for every crossing of the universal branch cut, and the opposite time shift added for each crossing in the opposite direction, the light pulses will not return to the sender at the same time. In fact, it is possible to calculate exactly how fast one is moving with respect to the preferred frame using this method.

In the preferred frame (see figure 3a), it is easy to calculate when the light pulses will return to you. Each pulse will return in

$$t = L_x/c, \tag{22}$$

assuming L_x is the relevant compact dimension size. On the other hand, in a non-preferred frame (see figure 3b), the pulse sent in the (+) and (−) directions will have different return times. We compute:

$$t_{\pm} = f_{\pm}(L_{x,eff}/c) = \gamma L_x/c \pm \beta\gamma L_x \tag{23}$$

Solving (23) for β, we get

$$\beta = \frac{t_+ - t_-}{t_+ + t_-}. \tag{24}$$

The downside to this method is that one needs to wait until the light pulse can traverse the universe

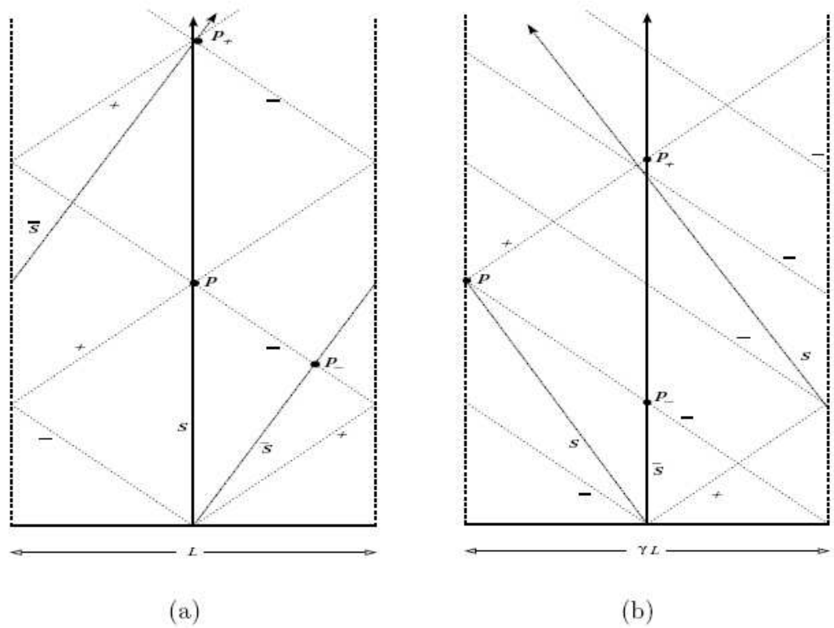

Figure 3: A pair of space-time diagrams showing the twin paradox described in section 3. The preferred frame is shown in (a), and the moving frame in (b). The solid lines indicate the world-lines of the twins, and the dotted lines represent light signals. In both cases, light signals are sent in each direction at $t = 0$, and are eventually received by the twins. It becomes clear that the difference in the times that the light is received is due to the time offset that comes with each wrapping around the universe when you cross the branch cut (at the edge of each diagram). Full credit goes to [2] for this diagram.

before one can get a result. In a universe which remotely resembles our own, this is a terribly long time. Of course, we are dealing with an odd pair of twins, so perhaps they won't mind the wait. In any case, an alternate method has been proposed, using electromagnetic phenomena [2].

4.2 Electric Field

In a flat infinite space, the electric field from a point charge at rest is $1/r^2$. If there is a compact dimension, then the boundary conditions change. We must now deal with the electric field of all the equivalent charges from every point in the universe related to the first by (15). Let's consider, for simplicity, the universe with only one compact dimension. We will have:

$$\vec{E}(x,y,z) = \frac{q}{4\pi\epsilon_0} \sum_{n=-\infty}^{\infty} \frac{(x+nL_x)\hat{x} + y\hat{y} + z\hat{z}}{((x+nL_x)^2 + y^2 + z^2)^{3/2}} \tag{25}$$

This clearly reduces to the usual form as $L_x \to \infty$.

If an observer is watching a charge at rest in a non-preferred frame, this expression changes to

$$\vec{E}(x,y,z) = \frac{q}{4\pi\epsilon_0}\sum_{n=-\infty}^{\infty}\frac{(x+nL_{eff})\hat{x}+y\hat{y}+z\hat{z}}{((x+nL_{eff})^2+y^2+z^2)^{3/2}} \tag{26}$$

If we only consider points on the x-axis, we can simplify this to

$$\vec{E}(x,0,0) = \frac{q}{4\pi\epsilon_0}\sum_{n=-\infty}^{\infty}\frac{\hat{x}}{(x+nL_{eff})^2} \tag{27}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{\pi^2}{L_{eff}^2}csc^2\left(\frac{\pi x}{L_{eff}}\right) \tag{28}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{x^2}+\frac{\pi^2}{3L_{eff}^2}+\cdots\right]\hat{x}. \tag{29}$$

Thus, if an observer is able to measure the electric field to within $(d/L_{eff})^2$ where d is the distance from the test charge, and L_{eff} is the length of the compact dimension, and the observer had knowledge of what the compact dimension's size should be, then a measurement of β should be possible [2].

This observation is questionable in its usefulness, because those conditions are not remotely trivial. Additionally, there is the requirement that the particle be still long enough for the electrostatic limit to apply. This is the condition which makes this measurement technique truly pointless. The formula (26) does not take into account the fact that the fields from all other charges will be retarded by a very significant amount. In order to get a measurement using (27), the particle needs to be at rest for all time! This most certainly will not do. While it is not very useful for any measurements without complete knowledge of a particle's history, the Green's function for the potential of a point charge in this cylindrical universe can be computed, along with the potential:

$$G(\vec{x}(t),\vec{x'}(t')) = \sum_{n=-\infty}^{\infty}\frac{\delta\left(t'-\left[t-\frac{|\vec{x}-(\vec{x'}+n\gamma L\hat{x})|}{c}+n\beta\gamma L/c\right]\right)}{|\vec{x}-(\vec{x'}+n\gamma L\hat{x})|} \tag{30}$$

This is a much more accurate result, although it isn't necessarily more useful than (27).

Other ideas could be considered to measure the initial frame. One might be tempted to assume that, because a light signal sent out returns more often from one side of the universe than the other (see figure 3), that perhaps a Doppler shift could be observed more from one side than from the other. This is a fallacy. Time flows at the same rate regardless of which direction one looks in. The only thing which is different is the time offset.

No method described in these sections allows one to measure whether they are in the preferred frame in a time less than L_{eff}/c. This is a natural manifestation of causality. The best one might be able to do is to utilize some past event which occured at a nearby, known location in space and time, to make a calculation based on the light pulse method. A supernova might be a promising option. Of course, if we could determine these things so easily, why would we keep using twins to test out these crazy ideas?

References

[1] John D. Barrow and Janna Levin. Twin paradox in compact spaces. *PRA*, 63:0044104, 1987.

[2] Aravindhan Sriharan Dhruv Bansal, John Laing. On the twin paradox in a universe with a compact dimension. arXiv:gr-qc/0503070v1, 2007.

[3] Michael Weiss et al. The physics and relativity faq, 2006. http://math.ucr.edu/home/baez/physics/Administrivia/copyright.html.