Electric Fields

Please make sure to read assigned material in Knight before lecture so we can build on our discussions.

The electric field is a quantitative way to express how some charged object called the "source" affects other objects that could be charged or neutral. A given source affects all of space by establishing an electric field:

\[ \mathbf{E}(x, y, z, t) = (E_x(x, y, z, t), E_y(x, y, z, t), E_z(x, y, z, t)) \]

This is a vector \( \mathbf{E} \), a triplet of numbers \( (E_x, E_y, E_z) \), associated with every point in space. Some other object, say a point charge at \((x, y, z)\) with charge \(q\), will respond to the source by being accelerated with a force \(\mathbf{F}\) given by

\[ \mathbf{F} = q \mathbf{E} \]

This equation tells us that electric field \(\mathbf{E}\) has physical units of force over charge, newtons over coulombs:

SI unit of electric field is \[ \frac{N}{C} \]

(1) Provide operational way to measure \(\mathbf{E}\): place a tiny
charged object at different locations in space and measure force $\vec{F}$ on object, then define

$$\vec{E}(x_1, y_1, z_1) = \frac{\vec{F} \text{ at } (x_1, y_1, z_1) \text{ on } q}{q}$$

This is hard to carry out, e.g. $q$ might polarize or push nearby objects, changing $\vec{E}$ or device used to measure $\vec{F}$ might affect $\vec{E}$. In practice, because Coulomb's law is known to be correct, easier to calculate $\vec{E}$ directly from known configurations of charges.

Also, $\vec{E}$ is formally an infinite amount of information, can't possibly measure $\vec{E}$ everywhere in finite amount of time.

The relation: $\vec{F} = q\vec{E}$

also makes good sense if we use Coulomb's law with experimental fact of superposition. For example, consider object of interest consisting of $N$ point particles with charges $q_1, q_2, \ldots, q_N$. Then force of all these particles on some other charge $Q$ given by superposition

$$\vec{F}_{\text{on } Q} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \frac{K q_1 q}{r_1^2} \hat{r}_1 + \frac{K q_2 q}{r_2^2} \hat{r}_2 + \frac{K q_n q}{r_n^2} \hat{r}_n$$

$$= Q \left[ \frac{K q_1 \hat{r}_1}{r_1^2} + \frac{K q_2 \hat{r}_2}{r_2^2} + \frac{K q_n \hat{r}_n}{r_n^2} \right] = Q \vec{E}$$
So if we can calculate \( \vec{E} \) fields for various configurations of charge, we can predict ahead of time what forces will appear on charged objects caused by our source.

In some sense, calculating \( \vec{E} \) is trivial technically, especially with a computer. We simply divide any object into small non-overlapping charged regions and add up the contributions to \( \vec{E} \) via Coulomb's law. But it is insightful to compute \( \vec{E} \) for various objects like

- discrete charges
- sphere
- cylinder or wire
- plane
- ring

To get intuition of how \( \vec{E} \) depends on geometry and charge. Surprisingly large number of problems can then be solved by using superposition of these basic objects to calculate \( \vec{E} \).

Let's work out some examples.
1. What is direction of E field at dot? A, B, C, D, or E

2. What is direction of force at dot if dot is electron?
**Electric Field of Point Charge**

Assume point particle with charge \( Q \) is small. This is ok for finite size object like baseball or Earth provided distances are large compared to object size.

Put point particle with charge \( Q \) at origin of \( x-y \) coordinate system. What is force on test probe point particle \( q \) at location \((x, y)\)?

\[
E = \frac{KQq \hat{r}}{r^2} = q \left[ \frac{KQ}{r^2} \hat{r} \right]
\]

So

\[
E = \frac{F}{q} = \frac{KQ}{r^2} \hat{r}
\]

To get an explicit formula, note that \( \hat{r} \) is unit vector pointing from origin to \((x, y)\):

\[
\hat{r} = \frac{(x, y)}{\sqrt{x^2 + y^2}} = \left[ \frac{x}{\sqrt{x^2 + y^2}} \right] \hat{i} + \left[ \frac{y}{\sqrt{x^2 + y^2}} \right] \hat{j}
\]

So we conclude:

\[
\vec{E} = \frac{KQ \hat{r}}{r^2} = \frac{KQ}{(x^2 + y^2)^{3/2}} \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)
\]

\[
\vec{E} = KQ \left( \frac{x}{(x^2 + y^2)^{3/2}}, \frac{y}{(x^2 + y^2)^{3/2}} \right)
\]

If \( Q > 0 \), \( \vec{E} \) points away from origin.
If \( Q < 0 \), \( \vec{E} \) points toward origin.
Gave discussion for points on plane, for 3D, \( \mathbf{E} \) at location \((x, y, z)\) given by direct generalization:

\[
\mathbf{E}(x, y, z) = KQ \left( \frac{x}{(x^2+y^2+z^2)^{3/2}} - \frac{y}{(x^2+y^2+z^2)^{3/2}} - \frac{z}{(x^2+y^2+z^2)^{3/2}} \right)
\]

**Electric Field of Dipole:**

Let's try something more complicated, what is \( \mathbf{E} \) due to dipole consisting of point particles with charge \( \pm Q \), separated by distance \( d \):

\[
\mathbf{\vec{F}}_a = \hat{r}_a Q \\
\mathbf{\vec{F}}_b = \hat{r}_b Q
\]

Can simplify math by choosing coordinate system symmetrically located at \( Q_1 - Q \) as shown. Then superposition and previous formula gives \( \mathbf{E}(x, y, z) \):

\[
\mathbf{E}_{\text{dipole}}(x, y, z) = KQ \left[ \frac{\mathbf{\hat{r}}_a}{r_a^2} - \frac{\mathbf{\hat{r}}_b}{r_b^2} \right]
\]

\begin{align*}
\mathbf{\hat{r}}_a &= (x, y, z) - \left( \frac{d}{2}, 0, 0 \right) \\
\mathbf{\hat{r}}_b &= (x, y, z) - \left( \frac{d}{2}, 0, 0 \right)
\end{align*}

\[
\mathbf{\hat{r}}_a = \left( x - \frac{d}{2}, y, z \right) \\
\mathbf{\hat{r}}_b = \left( x + \frac{d}{2}, y, z \right)
\]

\[
\mathbf{\hat{r}}_a^2 = \left( x - \frac{d}{2} \right)^2 + y^2 + z^2 \\
\mathbf{\hat{r}}_b^2 = \left( x + \frac{d}{2} \right)^2 + y^2 + z^2
\]
This is not only tedious to write but not insightful, how do we get insight from all these symbols? But do you appreciate that one does have to write down such explicit expressions to get precise answers, e.g., from computer code used in an engineering application.

Two popular ways to get partial insight of \( \mathbf{E} \) fields

1. **Draw vector field** by plotting directions of \( \mathbf{E} \), relative magnitudes of \( \mathbf{E} \) at some representative points. Often easy to choose points on lines or curves of symmetry.

2. **Draw field line** from vector field. These are curves that are locally tangent to \( \mathbf{E} \) vectors. Not only difficult to draw or calculate in practice, difficult to judge magnitude, good mainly for orientation of \( \mathbf{E} \).

All visual representations are inferior to mathematical representation \( E(x,y,z) \). \( \mathbf{E} \) is key physical object.

Discuss slides, applets illustrating vector, field lines.
Fields of continuous distributions of charge

For many macroscopic charged objects like our glass rod or dome of the Van de Graaff generator, we can ignore discreteness of charge and of matter, replace charges with a

linear charge density \( \lambda(x) \)

surface charge density \( \sigma(x, y) \)

volume charge density \( \rho(x, y, z) \)

Then use superposition and integration to calculate \( \vec{E} \) at points of interest. Easier to show by several examples.

\( \vec{E} \) field on bisector of charged line segment

Consider rod that we treat as line segment of zero thickness, length \( L \)

Consider charge \( Q \) (many electrons added or removed) uniformly spread out over rod so linear charge density = charge per unit length is

\[
\lambda = \frac{Q}{L}
\]

Let's calculate \( \vec{E} \) at point of symmetry on \( \vec{E} \) bisector of rod

Math is easier if we introduce \( xy \) coordinate system centered on rod (next page).
Trick in all these problems is to divide object into small (infinitesimal) pieces, apply Coulomb's law to each piece. Another important detail is to introduce a label or some coordinate to identify each piece.

So divide rod into adjacent pieces of with dx labeled by x,

piece at x lies between x and x+dx.

Amount of charge on this small piece is "length" x charge

\[ dq = dx \cdot \tau \]

From symmetry, we expect final \( \vec{E} \) at point \((0, y)\) to point \& along y axis since \( \vec{E} \) from chunk at \( x \) has opposite \( x \)-component to chunk of weight \( dx \) at coordinate \(-x\).

\( \vec{E} \) field from single chunk at \( x \) has \( y \)-component \( E \cos \theta \)

claim

\[ E_y(y) = \int_{0}^{1/2} \frac{K \cdot dq \cdot \cos \theta x}{r^2} \quad \text{up} \]

\( \text{From } x \text{ and } -x \text{ contributions along } y \)
From drawing: \[ r^2 = x^2 + y^2 \]
\[ \cos \theta = \frac{y}{\sqrt{x^2 + y^2}} \]

So integral becomes:

\[
E_y(y) = 2K \int_0^{\frac{\sqrt{2}}{2}} \frac{d\theta \cdot \cos \theta}{r^2}
\]

\[
= 2K \int_0^{\frac{\sqrt{2}}{2}} \frac{(dx \cdot x) \cdot y}{(x^2 + y^2)^{3/2}} \frac{dx}{\sqrt{x^2 + y^2}^2}
\]

\[
E_y = 2\pi Ky \int_0^{\frac{\sqrt{2}}{2}} \frac{dx}{(x^2 + y^2)^{3/2}}
\]

Can look up this integral in table, or evaluate by trigonometric substitution, or just evaluate it symbolically via Mathematica

Integrate \[ (x^2 + y^2)^{-3/2}, \] \[ \exists x \in [0, \frac{\sqrt{2}}{2}] \]

and hit shift-return to get

\[
\int_0^{\frac{\sqrt{2}}{2}} \frac{dx}{(x^2 + y^2)^{3/2}} = \frac{1}{y^2 (L^2 + 4y^2)^{1/2}}
\]

so

\[
E(y) = \frac{KL^2}{y (y^2 + \frac{L^2}{2})^{1/2}}
\]
For practice, let's evaluate the integral by trigonometric substitution, at least verify Mathematica is correct.

\[ I = \int_0^{1/2} \frac{dx}{(x^2+y^2)^{3/2}} \]

Let:

\[ x = y \tan \phi \]
\[ x^2 + y^2 = y^2(1 + \tan^2 \phi) = y^2 \sec^2 \phi \]
\[ dx = y \sec^2 \phi \, d\phi \]
\[ (x^2 + y^2)^{3/2} = y^3 \sec^3 \phi \]

\[ x=0 \rightarrow \phi = 0 \]
\[ x = \frac{1}{2} \rightarrow \tan^{-1} \left( \frac{1}{2y} \right) \]
Change bounds of integral:

\[ I = \int_0^{1/2} \frac{dx}{(x^2+y^2)^{3/2}} = \int_0^{\tan^{-1}(1/2y)} \frac{y \cdot \sec^2 \phi \cdot d\phi}{y^3 \sec^3 \phi} \]

\[ = \frac{1}{y^2} \int_0^{1/2y} \frac{\, d\phi}{\sec^2 \phi} = \frac{1}{y^2} \int_0^{1/2y} \cos^2 \phi \, d\phi = \frac{1}{y^2} \int_0^{1/2y} (1 + \sin(2\phi)) \, d\phi \]
\[ = \frac{\sin(1/2y)}{y^2} \]
\[ \Rightarrow \sin(\phi) = \frac{1/2}{\sqrt{y^2 + (1/2)^2}} \]

\[ \Rightarrow \frac{1}{2y^2 \sqrt{y^2 + (1/2)^2}} \]

which gives same answer as Mathematica.
Formula
\[ E(y) = \hat{y} \left( \frac{K \lambda^2}{y (y^2 + \frac{L^2}{4})^{\frac{3}{2}}} \right) \]

again not easy to appreciate for general y, L. Can get some intuition by taking limits in which y large or small, L large or small,

\[ \text{case } y \gg \frac{L}{2} \Rightarrow \sqrt{y^2 + \left( \frac{L}{2} \right)^2} \approx y \Rightarrow E \approx \hat{y} \left( \frac{K \lambda^2}{y^2} \right) \]

This is Coulomb's law applied to point charge with charge \( Q = \lambda \hat{y} \), what we would expect,

\[ \text{case } y \ll \frac{L}{2} \Rightarrow \text{very close to segment compared to its length, same as letting } L \text{ become large for fixed } y \]

now
\[ y^2 + \left( \frac{L}{2} \right)^2 \approx \left( \frac{L}{2} \right)^2 \quad y \ll \frac{L}{2} \]

\[ E \rightarrow \hat{y} \cdot \frac{2K \lambda^2}{y} \text{ independent of } L \]

Electric field of infinite cone with charge density \( \hat{y} \)

\( E \) points away from one by symmetry.

\( E \) decreases more slowly with increasing distance than Coulomb,

\( \frac{1}{y} \) vs \( \frac{1}{y^2} \). Makes sense, you are picking up contributions \( E \) from dq arbitrarily far away.
Second useful case: \( \vec{E} \) on axis of ring of radius \( R \) and linear charge density \( \lambda \).

Can of course evaluate \( \vec{E} \) anywhere in space but expressions complicated except on axis.

Before doing any calculation, can we guess form of \( \vec{E} \)?

By symmetry, \( \vec{E} \parallel \hat{z} \)-axis.

By symmetry, \( \vec{E} = 0 \) at center of ring.

From ring on \( z \)-axis \( \vec{E} \propto \frac{1}{z^2} \), ring looks like point charge with charge \( Q = (2\pi R)\lambda \).

\( \vec{E} \) points away from plane of rings.

Conclude:

\[
\frac{KQ}{z^2} \sqrt{1 + \frac{z^2}{l^2}}
\]

with maximum at location of order \( R \) since that is only length in this problem.

Now let's try working out details:

Label small piece of ring by angle \( \theta \) with \( 0 \leq \theta \leq 2\pi \). Amount of charge on piece between \( \theta \) and \( \theta + d\theta \)

\[
dQ = (Rd\theta)\lambda
\]
\[ E(0, 0, z) = \hat{z} \left[ \int_0^{2\pi} \frac{K \cdot \hat{z} \cdot z}{(z^2 + r^2)^{3/2}} \cdot \frac{z}{\sqrt{z^2 + r^2}} \cos \theta \, d\theta \right] \]

\[ = \hat{z} \left[ \frac{Kz^2}{(z^2 + r^2)^{3/2}} \int_0^{2\pi} \frac{z}{\sqrt{z^2 + r^2}} \cos \theta \, d\theta \right] \]

This is indeed an odd form of $E$ so has the right qualitative form.

You can verify

\[ E_{\text{max}} \text{ occurs for } z = \frac{R}{\sqrt{2}} \approx 0.7R \]

For $\frac{z}{R}$ large compared to $R$, $R^2 + z^2 \approx R^2 \Rightarrow \vec{E} \propto \frac{1}{z^2}$ (Coulomb's law)

For $z$ large compared to $R$, ?

Can use this solution to get \( \vec{E} \) field of charged thin shell with surface charge density \( \sigma \)

\[ E(z) = \hat{z} \left[ \int_0^{2\pi} \frac{K \cdot \left[ \epsilon_0 \sigma \right] \hat{z} \cdot z}{(z^2 + r^2)^{3/2}} \, d\theta \right] \]

\[ = \hat{z} \left[ 2\pi \epsilon_0 Kz \cdot \int_0^{2\pi} \frac{r \, dr}{(r^2 + z^2)^{3/2}} \right] \]

let \( u = r^2 + \frac{z^2}{r^2} \)

\[ du = 2rdr \]

\[ r = 0 \Rightarrow u = \frac{z^2}{a^2} \]

\[ r = R \Rightarrow \frac{z^2}{R^2} \Rightarrow u = \frac{z^2}{R^2} \]
\[ E_z(z) = \hat{z} \left[ \pi \sigma K z \cdot \int_{\frac{z^2}{z^2 + R^2}} \frac{du}{u^{3/2}} \right] \]

\[ E_x(z) = \frac{\hat{x}}{z} \left[ 2\pi \sigma K \cdot \left( 1 - \frac{1}{\sqrt{1 + \left( \frac{z}{R} \right)^2}} \right) \right] \]

\[ \vec{E} \text{ on axis of disk of radius } R, \text{ surface charge density } \sigma \]

\[ (1 + \varepsilon) \approx 1 + \varepsilon \]

\[ (1 + \varepsilon) \approx 1 + \alpha \varepsilon + \ldots \]

\[ \lim_{z \to R} \left( 1 + \frac{(R-z)^2}{z^2} \right) \approx 1 - \frac{1}{2} \left( \frac{z}{R} \right)^2 \]

\[ \Rightarrow \quad E_z \approx \hat{z} \cdot \frac{\sigma K}{z^2} \]

\[ E 
\]

\[ \Rightarrow \quad E \to \frac{\hat{z}}{z} \left[ 2\pi \sigma K \right] \quad \text{independent of } z. \]

\[ E \text{ field of infinite plane, doesn't depend on distance.} \]

Use superposition to get \( \vec{E} \) of dipole of plates

\[ \vec{E} \text{ way of producing uniform } \vec{E} \text{ of known strength} \]
Start with clicker questions to test elementary understanding of electric field, surface charge density \( \sigma \), field lines, dipoles.

See [www.phy.duke.edu/~hsg/162/Files/lectures](http://www.phy.duke.edu/~hsg/162/Files/lectures)

For PowerPoint files I am using for lectures, can review the clicker questions.

Why calculate electric fields \( \vec{E}(xyz) \) due to arrangement of charges?

1. can predict or control motion of charged particles
   - accelerators like electron gun, Large Hadron Collider
     - use \( \vec{E} \) to create beam of electrons or protons
     - use \( \vec{E} \) to deflect beam like this:

```
                  ++++++
                  |
                  |
                screen
                  |
                  |
```

   electron microscope
   cathode ray tube (old TV, oscilloscope)
   electron beam lithography for integrated circuits
   electrostatic spray paint for cars
Study $\vec{E}$ fields to understand forces and torques on non-point objects:

- electronic ink used in e-book readers
- functionality of various biomolecules
  - ion channels in neuronal membranes
  - folding of amino acid chain into enzyme
  - folding, coiling, supercoiling of DNA (how to fit 2m long DNA in 2μ nucleus of mammalian cell)
- friction and triboelectricity: how much of friction is due to separation of charge?

Some important $\vec{E}$ fields:

point charge: $\vec{E} = \frac{kq_e}{r^2} \cdot \frac{\vec{x} - \vec{x}_0}{||\vec{x} - \vec{x}_0||}$

spherical charge

act like point charge $Q = (4\pi \epsilon_0) \frac{\rho}{R}$ at center for $d > R$

spherical charge constant $\rho$ act like point charge

$Q = \left(\frac{4\pi}{3}\right) \rho$ for $r > R$
$|E| = \frac{2\pi k p}{d} \propto d^{-1}$

2. plane charge with surface density $\sigma$

$E = 2\pi k \sigma$ independent of $d$

ring

easy to compute $E(\theta)$ on axis by symmetry

Get $E$ fields of many other objects by superposition and by integration, at least along symmetry axes:

- truncated cone
- hemisphere
- cylinder
- pyramid
- stack of squares
- stack of sine waves
Some non-symmetric objects can be solved by superposition including holes or gaps, which can be thought of as superposing objects with positive and negative charge densities.

What is \( E \) at center of circle with small gap? Think of gap as small line segment of negative charge density \(-\lambda\) added to complete ring...

What is \( E \) at point \( P \) for two half-infinite line charges of linear charge density \( \lambda \)? Some as single infinite line combined with line segment of length \( a \) with charge density \(-\lambda\) so simply add two known solutions for \( \infty \) line and line segment

Can you invent a subtraction problem?

Another example:

- Uniformly charged sphere of radius \( R_e \) with hole of radius \( R_i \). Treat as two concentric spheres:
  - \( R_e \) with density \( \rho \)
  - \( R_i \) with density \( -\rho \)
Note: it is mathematically difficult to calculate the motion of a point particle with charge \( q \) for some general given \( \vec{E} \) field. This leads to a differential equation for the vector position

\[
\vec{X}(t) = (x(t), y(t), z(t))
\]

over some interval of time. The differential equation is none other than Newton's second law:

\[
m \cdot \vec{a} = m \cdot \frac{d^2 \vec{X}}{dt^2} = \sum \text{ force} = q \ \vec{E}(\vec{X}(t))
\]

where \( \vec{E}(\vec{X}(t)) \) means the electric field vector at location \( \vec{X}(t) \) in space at time \( t \), when the particle is at location \( \vec{X}(t) \).

For example, let \( \vec{E}(x,y) \) be the \( \vec{E} \) field of a point source charge \( Q \) at the origin of some coordinate system:

Then we have seen:

\[
\vec{E}(x,y) = \frac{Q}{4\pi \varepsilon_0} \left( \frac{x}{(x^2+y^2)^{3/2}} \frac{y}{(x^2+y^2)^{3/2}} \right)
\]

This leads to two coupled 2nd-order differential equations:
\[ m \cdot \frac{d^2x}{dt^2} = m \left( \frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} \right) = q \, \vec{E} = q \sqrt{KQ \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)} \]

or

\[ m \cdot \frac{d^2x}{dt^2} = KqQ \left( \frac{x}{(x^2 + y^2)^{3/2}} \right) \]

\[ m \cdot \frac{d^2y}{dt^2} = KqQ \left( \frac{y}{(x^2 + y^2)^{3/2}} \right) \]

which with given initial data, say \( x_0, x_0', y_0, y_0' \) at time \( t=0 \), gives a unique solution \( x(t), y(t) \).

Such differential equations are generally too difficult for general \( \vec{E} \) fields, but fortunately are trivial to solve approximately by numerical methods. For example, in Mathematica, you could type something like this:

\[ m = 2 \, \text{kg} \]

\[ q = 10^{-6} \, \text{C} \]  \( \text{(in Coulomb)} \)

\[ Q = 10^{-4} \, \text{C} \]

\[ K = 10^{-10} \, \text{N m} \]  \( \text{(in SI units)} \)

(next page)
Use \texttt{NDSolve} (numerical differential equation solver) of \texttt{Mathematica} like this:

\[
\text{NDSolve}\left[ \begin{array}{l}
\ddot{x}[t] = -KQ_0 \frac{x[t]}{\sqrt{x[t]^2 + y[t]^2}} \\
\ddot{y}[t] = -KQ_0 \frac{y[t]}{\sqrt{x[t]^2 + y[t]^2}} \\
x[0] = 1, \quad x[0] = 0, \quad (\text{initial data}) \\
y[0] = 0, \quad y'[0] = 0. \\
\end{array} \right] \begin{array}{l}
\{x, y\}, \\
\{t, 0, 50\} \\
(\text{variables to integrate}) \\
(\text{range of time}) \\
\end{array}
\]

and \texttt{Mathematica} will essentially instantly compute numerical solutions accurate to about 6 digits; you can then plot the solutions and analyze them mathematically, valuable for getting intuition, precise values.