\[
\sqrt{\frac{(1.01)^{\frac{1}{5}}+(0.99)^{\frac{1}{5}}}{2}}
\]

\[
= \left[ \frac{(1+x)^{\frac{1}{5}}+(1-x)^{\frac{1}{5}}}{2} \right]^{1/2}
\]

\[
= \left[ 1 + \frac{1}{5}x + \frac{2}{5}x^2 + \frac{1}{5}x + \frac{2}{5}x^2 \right]^{1/2}
\]

\[
= \left[ 1 + \frac{7}{5}(\frac{x}{5})^2 \right]^{1/2}
\]

\[
\approx \left[ 1 + \frac{7}{5}(\frac{x}{5})^2 \right]^{1/2}
\]

\[
\approx \left[ 1 + \frac{7}{5}(\frac{x}{5})^2 \right]^{1/2} + \frac{1}{2} \left[ \frac{1}{5} \right] (\frac{x}{5})^2
\]

\[
\approx 1 + \frac{7}{5} \cdot \frac{x}{5} \cdot 10^{-4} + \frac{1}{2} \left[ \frac{1}{5} \right] (\frac{x}{5})^2
\]

\[
\approx 1.000014000 \ldots
\]

Students need to show that
Higher-order terms don't contribute to first ten digits. This requires going to next higher order, e.g.

\[
\frac{(1+x)^{\frac{7}{5}} + (1-x)^{\frac{7}{5}}}{2}
\]

\[
\approx 1 + \frac{7}{5}x + \frac{7(7-1)}{2!}x^2 + \frac{7(7-1)(7-2)}{3!}x^3 + \ldots
\]

\[
+ 1 - \frac{7}{5}x + \left(\frac{7}{5}\right)x^2 - \left(\frac{7}{5}\right)x^3 + \ldots
\]

\[
= 2 + \frac{7(7-1)}{5}x^2 + \frac{1}{4.3}\left(\frac{1}{10}\right)x^4 + x^6
\]

\[
\approx 10^q
\]

\[
\approx 10^q
\]

\[
+ \text{smaller than } 10^3
\]
More carefully, if you retain the next higher order, you get

\[
\int \left[ \frac{(1-x)^{\frac{7}{5}} + (1-x)^{\frac{5}{7}}}{2} \right]^{\frac{1}{2}}
\]

\[\geq 1 + \frac{14}{100} x^2 + \frac{14}{10000} x^4 + \ldots\]

\[\frac{14}{10000} \leq 10^{-3} \quad \text{and} \quad x^4 \leq 10^{-8} \quad \text{for} \quad x = 0.1\]

so next nonzero term is smaller than 10^{-11} and so can't affect the first 10 digits.
HW1 Problem 1 Part (b)

\[ 6 + \frac{3}{(1 + x + x^2)^{\alpha}} \]

\[ = 6 + 3(1 + x + x^2)^{-\frac{1}{5}} \quad \rightarrow \quad x = x + x^2 << 1 \quad ; \quad \alpha = -\frac{1}{5} \]

\[ \approx 6 + 3 \left[ 1 + \frac{\left(\frac{1}{5}\right)}{1}(x + x^2) + \frac{\left(\frac{1}{5}\right)\left(-\frac{1}{5} - 1\right)}{1 \times 2}(x + x^2)^2 \right. \]

\[ + \ldots \] \]

\[ \text{can be ignored} \]

\[ \approx 6 + 3 \left[ 1 - \frac{1}{5} x - \frac{1}{5} x^2 + \frac{3}{25} x^2 + \ldots \right] \]

\[ \text{can be ignored since we only care about } x^2 \text{ in this problem} \]

\[ \approx 6 + 3 - \frac{3}{5} x - \frac{6}{25} x^2 + \ldots \]

\[ \square \]
Problem 2:
* To solve this problem, you really can only change one rod once for the entire solution. The reason is that it’s difficult to control the charge on a rod by rubbing it, and so if you try to recharge the same rod later on, you won’t know how much charge is on it.

Step 1: Touch two metal spheres, 1 & 2, together, bringing a negatively charged plastic rod close to one of the metal spheres, and then separating the two spheres without changing the position of the rod & the sphere that it’s close to. By induction & conservation of charge, this will create 2 spheres with exactly equal & opposite charge.

\[ +2 \quad -2 \]

\[ 1 \quad 2 \]

Step 2: Touch 1 with an uncharged sphere 3 & then separate them. Touch 2 with 4 uncharged spheres & then separate them.

\[ + \frac{1}{2} \quad - \frac{1}{2} \]

\[ 1 \quad 3 \quad 2 \]

Step 3: Touch 1 with an uncharged sphere 4, & touch 4 with another uncharged sphere 5.

\[ + \frac{1}{8} \quad + \frac{1}{8} \quad + \frac{1}{8} \]

\[ 1 \quad 4 \quad 5 \]
Step 4: Touch 4 with 4 other uncharged spheres, so each of them has charge of $\pm \frac{\theta}{4}$.

Then touch sphere 1, two $\pm \frac{\theta}{40}$ spheres, & another uncharged sphere, so each of them has a charge of $\pm \frac{3}{40} \theta$. Call one of them sphere 6.

So $\theta_{sphere_6} : \theta_{sphere_2} : \theta_{sphere_5} = \frac{3}{40} \theta : -\frac{\theta}{4} : \frac{\theta}{8}$

$= 3 : 10 : 5$.
Problem 3

(a) First it's important to realize that $E_A$ must be oriented toward $E_1$ and $E_2$ and $E_3$ must also be oriented toward $E_1$ with a negative charge. Electrostatic force and $E_3$ cannot be in equilibrium. Using Coulomb's law, we can write down conditions for $E_1$ and $E_2$ to remain stationary:

\[
\begin{align*}
\frac{kE_1 E_3}{s^2} + \frac{kE_2 E_3}{l^2} &= 0 \quad \text{Equilibrium for } E_1 \\
\frac{kE_3 E_1}{(1-x)^2} + \frac{kE_3 E_2}{l^2} &= 0 \quad \text{Equilibrium for } E_2
\end{align*}
\]

Solving the equations, we obtain two sets of solutions:

\[
\begin{align*}
x &= \frac{1}{2} \\
x &= -1
\end{align*}
\]

\[
\begin{align*}
E_2 &= -\frac{Q}{l} \\
E_3 &= -4.8
\end{align*}
\]

Are both solutions valid? By going back the physical context we can determine that the first set is the only valid solution.

So why does the first set of solutions make sense? $E_1$ feels two forces in this case: from $E_2$ and from $E_3$, with opposite directions. Because $E_3$ is closer to $E_1$ than $E_2$ and has a smaller magnitude than $E_2$, it is possible for the two forces to have equal magnitude and cancel each other. Similar story for $E_2$.

(b) Total force on $E_3$:

\[
F = \frac{kE_1 (4r^2x)}{(4r^2)^2} - \frac{kE_3 (-r)}{(3r^2)^2} = 0
\]

Therefore $E_3$ would not move.
Problem 3

\[ \vec{b}_1 = \hat{x}, \quad \vec{b}_2 = 0, \quad \vec{b}_3 = 4 \hat{a} \]

1. First, it's important to realize that \( \vec{b}_3 \) must be orthogonal with \( \vec{b}_1 \) and \( \vec{b}_2 \), and \( \vec{b}_1 \) must also be orthogonal to \( \vec{b}_3 \) and \( \vec{b}_2 \). \( \vec{b}_2 \) cannot be in equilibrium. Using Coulomb's law, we can write down conditions for \( \vec{b}_1 \) and \( \vec{b}_2 \) to remain stationary:

\[
\begin{align*}
\frac{k \cdot \vec{b}_3}{x^2} + \frac{k \cdot \vec{b}_2}{l^2} &= 0 \quad \text{Equilibrium for } \vec{b}_1 \\
\frac{k \cdot \vec{b}_3}{(1-x)^2} + \frac{k \cdot \vec{b}_2}{l^2} &= 0 \quad \text{Equilibrium for } \vec{b}_2
\end{align*}
\]

Solving the equations, we obtain two sets of solutions:

\[
\begin{align*}
x &= \frac{1}{2} \\
\vec{b}_3 &= -\frac{2}{7} \hat{a} \quad \text{and} \quad x = -1 \\
\vec{b}_3 &= -\frac{4}{9} \hat{a}
\end{align*}
\]

Are both solutions valid? Bringing back the physical context, we can determine that the first set is the only valid solution.

So why does the first set of solutions make sense? \( \vec{b}_3 \) feels two forces in this case: from \( \vec{b}_1 \) and from \( \vec{b}_2 \), with opposite directions. Because \( \vec{b}_1 \) is closer to \( \vec{b}_3 \) and has a smaller magnitude than \( \vec{b}_2 \), it is possible for the two forces to have equal magnitudes and cancel each other. Similar story for \( \vec{b}_2 \).

2. Total force on \( \vec{b}_3 \):

\[
F = \frac{k \cdot (\vec{b}_3 \cdot -\vec{b}_1)}{(1/2)^2} - \frac{k \cdot (\vec{b}_3 \cdot -\vec{b}_2)}{(4/9)^2} = 0
\]

Therefore \( \vec{b}_3 \) would not move.
Once again, we can qualitatively determine that \( \theta_2 \) must be to the left of \( \theta_1 \), and has negative charge.

\[
\theta_2 < 0 \quad \theta_1 = \theta \quad \theta = -4 \theta
\]

Writing down the conditions for \( \theta_2 \) and \( \theta_3 \) to stay stationary:

\[
\begin{align*}
\frac{kQ\theta_2}{(x_3)^2} + \frac{kQ\theta_3}{(x_3)^2} &= 0 \\
\frac{kQ\theta_2}{(x_3)^2} - \frac{kQ\theta_3}{(x_3)^2} &= 0
\end{align*}
\]

Again, we have two solution sets:

\[
\begin{align*}
\theta_2 &= k \quad \text{and} \quad \theta_3 &= -\frac{k}{3}\theta
\end{align*}
\]

And the first set is the only solution set that makes physical sense. Again, \( \theta_2 \) will not move because its net force is zero.

(4) It's important to realize that being at stable equilibrium means the total force would point toward the charge's original position if the charge moves in ANY direction. Therefore, we can simply consider a small displacement for \( \theta_2 \) or \( \theta_3 \) vertical to the line they make. For example, for \( \theta_1 \) in case 1:

\[
\begin{align*}
F_{y1} &= F_{x1} \\
F_{x1} &= \frac{kQ\theta_1}{(x_1)^2} \quad \text{and} \quad F_{y1} &= \frac{kQ\theta_1}{(x_1)^2}
\end{align*}
\]

With an arbitrary vertical displacement \( \alpha \), the x-direction component of \( F_{x1} \) and \( F_{x1} \) do not necessarily balance each other to make the sum of \( F_{x1} \) and \( F_{y1} \) vertical on the x-axis. Similarly, none of the charges in each case is in static equilibrium.
Problem 4

Each point charge is experiencing four forces: two static electric forces from the other charges, force from the string and gravity. This is a 3-D problem, but it's easy to reduce it to 2-D, since the two electric forces combine to be coplanar with tension force and gravity.

Because the three charges form a horizontal equilateral triangle, forces exerted by $F_1$ and $F_2$ on $F_3$ have equal magnitude and make an angle of $50^\circ$. Therefore, their sum is easy to calculate and its direction is along the bisector.

\[
|F_{tot}| = |F_{23}| = \frac{E F_3 d^2}{2}
\]

\[
|F_{tot}| = |F_{13}| \cos 30^\circ + |F_{12}| \cos 30^\circ = \frac{\sqrt{3}}{2} \cdot \frac{E F_3 d^2}{2}.
\]

We can now draw a 2-D free body diagram for $F_3$:

```
\[\begin{align*}
\text{F}_3 & \quad \text{mg} \\
\text{F}_{tot} & \quad \theta
\end{align*}\]
```

We can determine the angle $\theta$ with the given geometry of the structure:

- $AD$ is the bisector of $\angle CAB$ and because $\triangle ABC$ is equilateral, $AD = BC$ and $CD = DB$.
- $G$ is the center of the triangle and $CG$ is the bisector of $\angle ACB$, so $\angle CGD$ is a right triangle with $\angle CGD = 30^\circ$.
- $CD = \frac{1}{2} CB = \frac{1}{2} d$, so $CG = CD / \cos 30^\circ = \frac{d}{2} / \frac{\sqrt{3}}{2} = \frac{d}{3}

Because $G$ is the center of the triangle, $CG$ is a right triangle with $\theta = \angle CGD$. 

We can thus be determined.
\[
\cos \theta = \frac{\text{CG}}{\text{PP}} = \frac{\frac{12}{3} \frac{d}{3}}{\text{L}} = \frac{4}{3} \frac{d}{\text{L}}
\]

Back to free body diagram for \( \theta \) : From the equilibrium condition:

\[
\begin{align*}
T \cos \theta &= F_{\text{total}} \\
T \sin \theta &= mg
\end{align*}
\]

\[
\Rightarrow \tan \theta = \frac{mg}{F_{\text{total}}} = \frac{mg}{\frac{L}{d^2} (1 - \frac{d}{L})} = \frac{mg d^2}{Lk d^2} = \frac{mg}{Lk d^2}
\]

With \( \cos \theta = \frac{4d}{3L} \), we can solve for \( d \):

\[
\theta = (\frac{mg d^2}{3Lk}) \left(1 - \frac{1}{L} \frac{d}{L}\right)^{-\frac{1}{2}}
\]

\( \theta \) can be positive or negative as long as three charges have the same sign.

\[
\Rightarrow \theta = t \left(\frac{mg d^2}{2Lk}\right) \left(1 - \frac{1}{L} \frac{d}{L}\right)^{-\frac{1}{2}}
\]

Plugging in numbers:

\[
\theta = 6 \times 10^{-10} \text{c}
\]

\[
\text{and } \frac{\theta}{d} = 4 \times 10^3
\]
Problem 5

Using the symmetry of the cube, we can simplify the forces by grouping certain vertices. Vertices $B_1$, $B_2$, $B_3$ are at a distance $l$ to vertex $A$ and therefore forces exerted on $A$ by $B_1$, $B_2$, $B_3$ have equal magnitude and their sum is in the direction $(-1, -1, -1)$. We can also group points $C_1$, $C_2$, $C_3$ which are at a distance of $\frac{l}{3}$ to point $A$ and the sum of their forces on $A$ is also along $(-1, -1, -1)$. Finally, vertex $C$ is at a distance $\frac{l}{3}$ to $A$ and exerts a force on $A$ along $-\frac{l}{3}$, once again $(-1, -1, -1)$.

Magnitude of forces exerted by $B$ on $A$:

$$F_{B1} = -\frac{kq^2}{l^2}$$

and:

$$F_{B2} = (0, 0, -\frac{kq^2}{l^2})$$

The component of $F_{B2}$ along $(-1, -1, -1)$ direction is then:

$$F_{B2||} = (0, 0, -\frac{kq^2}{l^2}) \cdot (-1, -1, -1) = \frac{1}{3} \cdot \frac{kq^2}{l^2}$$

and the sum of forces exerted by $B$ on $A$:

$$F_{Btot} = 3 \cdot \frac{1}{3} \cdot \frac{kq^2}{l^2} = \frac{1}{3} \cdot \frac{kq^2}{l^2} (-1, -1, -1)$$

Similarly, we can have the sum of forces exerted by $C$ on $A$:

$$F_{C1} = -\frac{kq^2}{l^2}$$

and:

$$F_{C2} = (-\frac{kq^2}{l^2}, 0, -\frac{kq^2}{l^2})$$

and:

$$F_{Ctot} = 3 \cdot \frac{1}{3} \cdot \frac{kq^2}{l^2} (-1, -1, -1) = \frac{1}{3} \cdot \frac{kq^2}{l^2} (-1, -1, -1)$$

Then:

$$F_{Ctot} = \frac{kq^2}{3l^2}$$

So, the sum of all forces is:

$$F_{final} = F_{Btot} + F_{Ctot} + F_{B2} = \left(1 + \frac{1}{3} + \frac{1}{3}\right) \frac{kq^2}{l^2} (-1, -1, -1)$$
Problem 6

(a) Setting $\mathbf{r}$ to be the positive direction for forces:

$$F_{\text{total}} = -\frac{k\alpha}{(x-\frac{a}{2})^2} + \frac{k\alpha}{(x+\frac{a}{2})^2}.$$

(b) $L + \frac{a}{2} > L - \frac{a}{2}$, so

$$\frac{k\alpha}{(L+\frac{a}{2})^2} > \frac{k\alpha}{(L-\frac{a}{2})^2}, \quad (L+\frac{a}{2}) - (L-\frac{a}{2}) < 0$$

$\Rightarrow F_{\text{total}} < 0$

$\Rightarrow$ Away from $+\alpha$

(c) Turn the expression of $F_{\text{total}}$ into the form $(x+\alpha)^2$:

$$F_{\text{total}} = -\frac{k\alpha}{L^2} \left( (L+\frac{a}{2})^2 - (L-\frac{a}{2})^2 \right)$$

Note that it's important to realize $\frac{1}{(x-\frac{a}{2})^2} - \frac{1}{(x-\frac{a}{2})^2}$

It's extremely important to factor out $1^2$ to get the $L^2$ in $(x+\alpha)^2$. Now because $|\frac{a}{L}| \ll 1$, we can use the approximation:

$$F_{\text{total}} \approx -\frac{k\alpha}{L^2} \left( 1 + \frac{3a}{2L} - 1 - \frac{3a}{2L} \right)$$

$$= -\frac{k\alpha}{L^2} \left( -\frac{3a}{2L} \right)$$

$$= \frac{3k\alpha a}{L^3}$$
The radius of Earth is approximately $6.4 \times 10^6$ m, and the radius of the moon is approximately $1.8 \times 10^6$ m. The distance between the moon and the earth is about $4.0 \times 10^8$ m, which is two orders greater than the sizes of the earth and the moon. Therefore, it is reasonable to treat them as point charges.

b) First we calculate the magnitude of the force needed to keep the moon orbiting around the earth:

$$F = M_{\text{moon}} a = M_{\text{moon}} \frac{v_{\text{moon}}^2}{d}$$

where $M_{\text{moon}} = 7.3 \times 10^{22}$ kg $\approx 10^{23}$ kg

$v_{\text{moon}} = 1.0 \times 10^3$ m/s $\approx 10^3$ m/s

$d = 4 \times 10^8$ m $\approx 10^8$ m

And the force, based on our assumption, will be provided by static electricity force:

$$F = k \frac{Q^2}{d^2}$$

where $k = 9 \times 10^9$ N m$^2$/C$^2$ $\approx 10^{10}$ N m$^2$/C$^2$

Now we can calculate $Q$:

$$Q = \sqrt{\frac{m v^2 d}{k}} = \sqrt{\frac{10^{22} \cdot (10^3)^2 \cdot 10^8}{10^{10}}} \approx 10^{13}$$

Number of electrons in the moon: Suppose that the moon consists of entirely Si O₂ (molar mass = 60). $N_e = \frac{M_{\text{moon}}}{60} \times N_A \times (14 + 2 \times 8) \approx 10^{46}$

Based on our assumption, there would be $N = \frac{Q}{2} \approx 10^{38}$ electrons on moon, which is small compared to $N_e$. 