Local Control of Entanglement in a Spin Chain

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In this Letter, we show that a control of both static and dynamical properties of entanglement can be achieved by local interactions; therefore methods are required to transfer either the entangled particles or their state at a distance. It has been shown theoretically that spin chains are efficient quantum channels for short distance entanglement transfer [2]; thus the ability to manipulate the propagation of entanglement in a spin chain can be very important and it has already been shown that breaking the translational invariance of the chain can produce very interesting results in this respect [3,4].

In this Letter, we show that a control of both static and dynamical properties of entanglement can be achieved by acting locally to modify the level spacing of some qubits. In the magnetic language, we analyze a system subject to a spatially inhomogeneous magnetic field, that is, a system with “diagonal defects” (“impurities”), [5].

In contrast to the usual case in which local actions cannot affect nonlocal physical quantities, here, due to the spin-spin interaction, the local control of the magnetic field modifies both the distribution of correlations in the ground state and the entanglement propagation along the chain. Indeed, spatial inhomogeneities of the external field lead to an Anderson-like localization of entanglement [6], and this can occur even for a single defect [7], giving rise to a mirrorlike effect in the entanglement propagation [8]. Contrary to the usual description of localization phenomena, we do not conceive the impurity just as a bit of disorder in the system, but rather intend the modification of the local level spacing as a knob to (i) control the content of “static” (ground state) entanglement, and (ii) drive its propagation along the chain.

We consider a 1D XX spin-$\frac{1}{2}$ closed chain, placed in an external magnetic field which is homogeneous everywhere but for two defect sites $l_1$ and $l_2$. This model is described by the Hamiltonian $H = H_0 + H_{\text{def}}$, with

$$H_0 = -\frac{\omega_0}{2} \sum_{i=-N/2}^{N/2} \sigma_z^i - J \sum_{i=-N/2}^{N/2} (\sigma_i^x \sigma_{i+1}^{x+1} + \sigma_i^y \sigma_{i+1}^{y+1}),$$

$$H_{\text{def}} = \frac{1}{2} \sum_{j=1}^{N} \alpha_j \sigma_z^j,$$

where $\omega_0$ is the level spacing of each qubit, except for those residing at sites $l_i$, which have level separation $\omega_0 + \alpha_i$. Henceforth, we shall take $J = 1$ and use the ferromagnetic coupling constant as our energy unit (we have also set $\hbar = 1$). Furthermore, we set to zero the energy of the completely separable eigenstate $|0\rangle^{\otimes N}$, which is the unperturbed ground state for $\omega_0 > 1$, and which is still an energy eigenstate in the presence of defects.

This model can be solved exactly via the Jordan-Wigner (JW) transformation, which maps the spin chain into a spinless fermion system [9]. However, we do not need the general solution here, since we will deal with states having at most one tilted spin (in the JW language, states lying in the single-particle subspace), [10]. In the continuum limit $N \to \infty$, we solve the model (restricted to the single-particle sector of its Hilbert space) using the Green operator technique [7].

To this end, we consider the Green operator $G_0$ describing the homogeneous chain. It is known [7,8] that $G_0$ displays a branch cut on the real axis in the complex energy plane for energies $E \in [\omega_0 - 1, \omega_0 + 1]$. This cut signals a continuous energy band, which survives even in the presence of the defects.

Given $G_0$, the full (single-particle) Green operator is

$$G = G_0 + G_0 T G_0,$$

where the $T$ matrix is given by

$$T = \frac{\sum |l_1 \rangle t_1(l_1) |l_2 \rangle t_2(l_2) G_0(l_2, l_1) T(l_1) + \text{H.c.}}{1 - t_1 t_2 G_0(l_2, l_1) G_0(l_2, l_1)}.$$
The $T$ matrix describes multiple scattering events of the single-particle excitation at the two defects, and it is precisely the resummation of the various scattering amplitudes which gives rise to the denominator in Eq. (4). The existence of zeros of this denominator is crucial since it implies that, besides the energy band discussed above, the model with defects displays some (at most two, in fact) discrete energy levels.

They lay above or below the energy band depending on the values of the defect fields, and their eigenenergies can be obtained analytically in some special cases. For example, for nearest neighbors defects, with distance $d = |l_2 - l_1| = 1$, by setting $E_{\text{loc}} = E_{\text{loc}} - \omega_0$ one gets

$$x_{\text{loc}} = \frac{\alpha_1 \alpha_2}{4} \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} + \sqrt{\left[1 + \left(\frac{(\alpha_1 - \alpha_2)^2}{4}\right)\right] \left[1 - \frac{(\alpha_1 \alpha_2)^2}{4}\right]}$$

Only one of these two solutions (the one with the plus sign in front of the square root) exists if the defect strengths satisfy the relation $\alpha_1^{-1} + \alpha_2^{-1} \geq 1$. That is, the parameter space is divided in two regions, characterized by the number of discrete energy eigenvalues (which can be one or two).

If we restrict ourselves to the ordered phase of the unperturbed chain, $\omega_0 > 1$, and if we consider only $\alpha_i > 0$, the lowest (or the only existing) localized level becomes the fundamental state of the perturbed problem.

Many of these features obtained for nearest neighbors defects are generic and do not depend on their distance. From the analytic properties of the Green operator, [7], one can show that (i) at most two localized states are present, whose position with respect to the energy band depends on the sign of the $\alpha$’s, (ii) a region in the $\alpha_1 - \alpha_2$ plane exists in which only one discrete state is found. However, this one eigenstate region becomes thinner and thinner as $d$ increases, (iii) the localized states always have the structure of single excitation states of the form $|\psi_{\text{loc}}\rangle = \sum_n b_n |n\rangle$. The amplitudes $b_n$, obtained from the residues of $\mathcal{G}_l$, can be expressed in terms of an inverse localization length $\xi = -\ln[-x_{\text{loc}} - \sqrt{x_{\text{loc}}^2 - 1}]$, as a function “bilocalized” around the two defects:

$$b_n = \text{const} \times (K_1 e^{-\xi |l-n|} + K_2 e^{-\xi |n-l|})$$

The coefficients $K_i$ are given by

$$K_i = \left(\frac{\alpha_i}{2 \sqrt{x_{\text{loc}}^2 - 1}}\right)^{1/2}$$

Their ratio indicates the relative weights of the two localization region. For $\alpha_1 \gg \alpha_2$ one gets $|K_2| \ll |K_1|$ and the discrete levels are localized around $l_1$, while for $\alpha_1 \ll \alpha_2$ the localization center is $l_2$ since $|K_2| \gg |K_1|$. For equal defect strengths, $\alpha_1 = \alpha_2$, one finds $K_2 = \pm K_1$, where the upper (lower) sign refers to the lower (higher) of the two eigenstates. Thus, if $\alpha_1 = \alpha_2$, the two discrete states are given by equal-weight coherent superpositions of two localized parts centered on the two impurities. These states are highly entangled and display strong quantum correlations between the defects and their neighborhoods.

Even for quite small values of the defect fields, the localization length is smaller than the intersite spacing (already $\alpha_1 = \alpha_2 = 1$ gives $\xi^{-1} < 0$, for any distance $d = |l_1 - l_2|$). This implies that the two discrete eigenstates are approximately given by the Bell combinations $(|01\rangle \pm |10\rangle)$, with the rest of the chain almost completely factorized in the state $|0\rangle$. It is noteworthy that this structure does not depend on the distance between the two defects, see the inset of Fig. 1, where the concurrence $C_{l_1,l_2} = 2 |b_1 b_2|$ between the defects is shown as a function of $d = |l_1 - l_2|$. Thus, a long distance bipartite entanglement can be obtained in the ground state of the chain. A similar behavior has been found in Ref. [11], with the difference that in our case this is a bulk property rather than a surface effect.

For generic values of the defect amplitudes, these states display entanglement between any pair of spins residing near each of the two defects, with the peculiarity that the pairwise entanglement for two spins around the same defect depends on the value of the local magnetic field at the other defect. This is illustrated in Fig. 1, where it is shown that the entanglement around $l_1$ is modified by

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**FIG. 1** (color online). Ground state concurrence between the $l_1$th and the $n$th spin of the chain. The entanglement can be remotely controlled by changing the local field at the other defect. The line with open circles (boxes) corresponds to $\alpha_1 = \alpha_2 = 1.5$ ($\alpha_1 = 2, \alpha_2 = 1.5$). For equal defect strengths, spin $l_1$ is entangled both with its own neighborhood and with the other defect’s one. In the asymmetric case, quantum correlations only survive within the localization region. The inset shows the concurrence between the two defects as a function of their distance for various values of defect fields. From below to above, the plots correspond to $\alpha_1 = \alpha_2 = 0.25; 0.5; 1; 2$. 

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177210-2
changing the strength of the local field at $l_2$, thus achieving a remote entanglement control.

The bilocal character of the discrete levels strongly affects the transport of entanglement along the chain. In particular, we consider the possibility of using the chain to send one partner of a maximally entangled pair. We assume that the spin at the sender site $s$ is prepared in a singlet state with an external (uncoupled) qubit. The interaction between the spins causes a transfer of entanglement along the chain. Ideally, after a given transmission time $t$, one would like to get a singlet between the external qubit and the one residing at a receiving site $r$. This is expressed in terms of the retarded Green operator as

$$f_{s \rightarrow r} = \sum_{l_{0c}} e^{-i E_{l_{0c}} t} b_s^* b_r + \int_{-\pi}^{\pi} d\theta e^{-i E_l t} g_r(E) g_r^*(E).$$

The first term describes transport mediated by localized states, while the second one gives a spin-wave mediated transfer, [12]. We analyze them separately.

The first contribution is effective only within a region of length $\xi^{-1}$ around the two defect sites. This gives the noteworthy possibility of transmitting from one neighborhood to the other. This effects is illustrated in Fig. 2, where the entanglement is shown to jump from one defect to the other. Indeed, if the sender site coincides with one of the defects (say $l_1$), than the system evolves by performing almost perfect Rabi oscillations between the two discrete localized states, with a Rabi frequency given by $\omega_R = (E_{loc1} - E_{loc2})$. This can be understood by noticing that the initial singlet state between $l_1$ and the external qubit can be approximately written as

$$|\psi_{in}\rangle \approx \frac{1}{\sqrt{2}} \left[ |0_l\rangle |1_{ext}\rangle - \frac{1}{\sqrt{2}} (|\psi_{loc1}\rangle + |\psi_{loc2}\rangle) |0_{ext}\rangle \right].$$

which implies that the concurrence between the $l_i$th qubit and the external one is a harmonic function:

$$C_{l_i}(t) \approx |\cos \omega_R t|, \quad C_{l_i}(t) \approx |\sin \omega_R t|.$$ (7)

The second term in Eq. (6) is an integral over the energy band, parametrized as $E = \omega_0 - \cos \theta$. It contains the state amplitudes of the continuous energy band, which can be written in terms of the retarded Green and $T$ operators:

$$g_n(E) = \langle n | (1 + G_0^+ T^+) | \psi_0(E) \rangle,$$

with the unperturbed states such that $\langle n | \psi_0(E) \rangle = e^{i n \theta}/\sqrt{2 \pi}$. These states represent distorted spin waves of the system. They are the stationary scattering states of the single-particle Hamiltonian and can be constructed, starting from the usual magnon excitations, by including corrections due to multiple scattering at the defects. For moderate values of the $\alpha$'s, the distortion is not that big and the unperturbed plane wave nature can still be recognized. As a result, the energy eigenstates form (approximate) standing waves between the two defects. Their pattern is reflected in the entanglement propagation shown in Fig. 3 where the sender site is located between the two impurities which act as potential barriers for the spin waves, so that entanglement bounces back and forth between these two mirrors. The extreme situations is reached when the defects are next to nearest neighbors to each other, thus realizing an entanglement trap; see Fig. 4. Since the mirrors are not perfect, [8], the trapped entanglement decreases with time, the superimposed time oscillations being due to the $e^{-i E_{loc} t}$ factor of the symmetric discrete eigenstate (the only one that matters, in this case). Once these oscillations are subtracted, the short time behavior of the concurrence is found to be parabolic, with a convexity that decreases as the defect amplitudes are increased. A long-time (residual) trapped entanglement is

FIG. 2 (color online). Rabi oscillations of the entanglement between the two defects for the case $\alpha_1 = \alpha_2 = 1.5$.  

FIG. 3 (color online). Entanglement bouncing between the two defects which act like (nonperfect) mirrors. The location of the impurities is indicated by the two arrows and the defect strengths are $\alpha_1 = \alpha_2 = 1.5$.  

177210-3
also present, which is due to the tails of the localized state. It diminishes with increasing $\alpha$’s as the localization length does, and becomes negligible if $\xi^{-1}$ is much smaller than the site spacing.

The Rabi oscillations illustrated above give a mean to transfer the entanglement reversibly between the defects. This method works independently of their distance, but has the drawback that the Rabi period increases with distance as $T \sim \alpha^d$ (this can be derived by perturbation theory for large $\alpha$’s; see Ref. [13]).

The form of the discrete states, however, suggests another, more effective method to achieve entanglement transfer on demand between the two defects, namely, the adiabatic passage (see [14] for a related proposal in which the coupling strength is varied instead of the local field).

The idea is to change the defect strengths adiabatically, so that the ground state of the system, having the general form given in Eq. (5), is adiabatically changed from the initial state $|\psi_{gs}(i)\rangle \approx |l_1\rangle$ localized at $l_1$, to the final state $|\psi_{gs}(f)\rangle \approx |l_2\rangle$, localized at $l_2$. This can be done by modifying the defect fields from the initial values

$$\alpha_1(i) \gg 1, \quad \alpha_2(i) \ll 1 \Rightarrow b_n(i) \approx \delta_{n,l_1},$$

and the final (reversed) ones

$$\alpha_1(f) \ll 1, \quad \alpha_2(f) \gg 1 \Rightarrow b_n(f) \approx \delta_{n,l_2}.$$ 

If this is done adiabatically, the system always remains in its instantaneous ground state, thus realizing the desired entanglement transfer provided the initial singlet state involves the external and the $l_1$th qubit. To ensure the adiabaticity, the rates of change of the $\alpha$’s have to be much smaller than the difference between the energies of the two discrete levels, $E_{21} = E_{\text{loc}} l_2 - E_{\text{loc}} l_1$. The most dangerous point in this respect (i.e., the smallest $E_{21}$) occurs at the crossing, when $\alpha_1(t) = \alpha_2(t)$. But, since at this point $E_{21} \sim \alpha^d$, if the adiabatic procedure is designed such that the crossing occurs for a very small $\alpha$, than the adiabaticity can be preserved even for transfer times much smaller than the Rabi period. This procedure has the additional advantage of effectively decoupling the receiving site from the rest of the chain after the transfer has been performed due to its large final local field.

To summarize, we have discussed how to manipulate a spin chain with local control fields, showing that the static entanglement can be remotely controlled, and that entanglement propagation can be adjusted to a large extent in order to achieve transfer on demand. One possibility to implement this model in a realistic setup is to use the method proposed in Ref. [15] to “engineer” spin chains with atoms in an optical lattice. The addition of external static local electric or magnetic fields should enable the control of the qubit energy level spacing, which is essential to test our proposal.

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[10] Since we work in the single-particle subspace, properties of entanglement very similar to those derived here are obtained with the XXZ model. For the sake of clarity, we limited ourselves to the simplest case (XX spin chain).