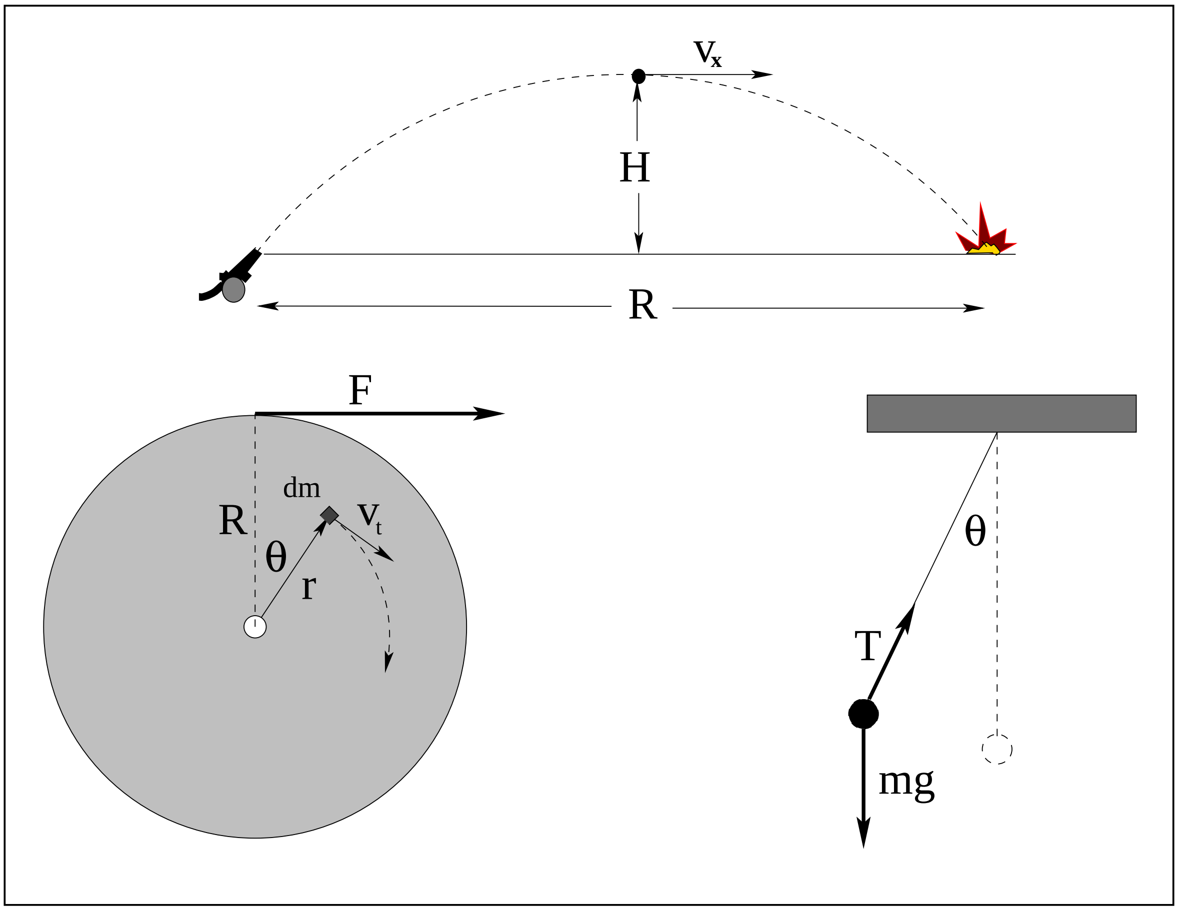
Introductory Physics 1
Mechanics and Applications
Non-Interactive Self-Guided Learning Problems



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Contents

Chapter 1

Preface

First, note well that ***The problems in this collection are provided 'as is' without any guarantee of being correct!*** That's not to suggest that they are all broken – on the contrary, most of them are well-tested and correct, but as problems are added and taken away errors can and do creep in (and are usually corrected as soon as I can do so after they are pointed out by clever students or other faculty who use this collection, but that can take some time). So use them at your own risk, and please feel free to bring any errors you discover to my attention.

This collection of problems is intended to be primarily used by students of serious, calculus based college-level physics to study *on their own* or with limited tutorial mentoring (if they have a mentor handy, a class teaching asssistant, a professor, a friend who is a physics major). It will be most useful to students who *also* have access to a moderately sound calculus based college-level physics *textbook* to use as a primary source for learning the concepts and seeing examples of problems solving, or as a standalone resource to support an actual course taught by any professor/teacher from any textbook.

As the title suggests, it contains ***self***-guided learning problems. These are problems (mostly with solutions provided) you can use to *test your knowledge* of the material you are studying somewhere else. They are best attacked *after* you have prepared for a chapter, attended lecture or other form of presentation of the material in the chapter, completed (in my classes, at least) in-class problems worked in teams intended to immediately reinforce the material covered in the pre-class prep and the lecture, and done the homework problems for the chapter on your own or working with friends or a tutor as your class rules permit. They are an excellent way for you to see if you have *mastered* the homework before tackling a quiz on the chapter, or to use to study for exams and review your conceptual understanding, your knowledge of dimensions and scaling, and above all, your problem solving skills.

Chapter 2

Math Review Problems

The following problems are useful for co-teaching math skills or for students to review on their own.

2.1 Math

2.1.1 Short Answer Problems

Problem 1. problems-1/math-sa-binomial-expansion.tex

Write down the **binomial expansion** for the following expressions, given the conditions indicated. FYI, the binomial expansion is:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

where x can be positive or negative and where n is any real number and only converges if $|x| < 1$. Write at least the first *three non-zero* terms in the expansion:

a) For $x > a$:
$$\frac{1}{(x+a)^2}$$

b) For $x > a$:
$$\frac{1}{(x+a)^{3/2}}$$

c) For $x > a$:
$$(x+a)^{1/2}$$

d) For $x > a$:
$$\frac{1}{(x+a)^{1/2}} - \frac{1}{(x-a)^{1/2}}$$

e) For $r > a$:
$$\frac{1}{(r^2 + a^2 - 2ar\cos(\theta))^1/2}$$

Problem 2. problems-1/math-sa-differentiate-expressions.tex

Evaluate the following expressions, where $\frac{d}{dt}$ means "differentiate with respect to t":

a) $\dfrac{d}{dt}\sin(\omega t) = \boxed{}$

b) $\dfrac{d}{dt}\cos(\omega t) = \boxed{}$

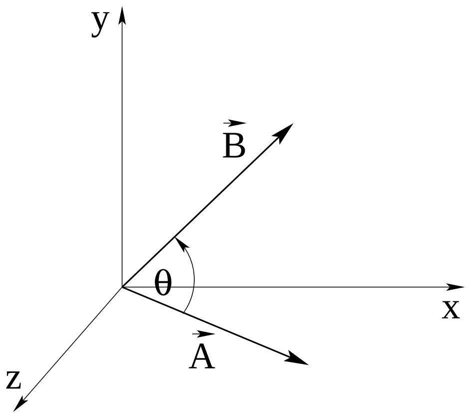
c) $\dfrac{d}{dt}\ln(at) = \boxed{}$

d) $\dfrac{d}{dt}\left(at^5 + bt^2 + c\right) = \boxed{}$

e) $\dfrac{d}{dt}e^{\lambda t} = \boxed{}$

f) $\dfrac{d}{dt}\left(1 + at^2\right)^3 = \boxed{}$

Problem 3. problems-1/math-sa-evaluate-vector-products.tex



a) Express the dot product in terms of its Cartesian components e.g. $\vec{\boldsymbol{A}} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}$:

$\vec{A}\cdot\vec{B} =$

b) Express the dot product in terms of the magnitudes A, B and θ:

$\vec{A}\cdot\vec{B} =$

c) Express the *magnitude* of cross product in terms of the magnitudes A, B and θ:

$\left|\vec{A}\times\vec{B}\right| =$

d) Express the cross product in terms of its Cartesian components e.g. $\vec{\boldsymbol{A}} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}$ (this has a lot of terms):

$\vec{A}\times\vec{B} =$

Problem 4. problems-1/math-sa-general-arithmetic.tex

Solve the following short problems:

a)

$$\frac{10 * 2^4}{5} \quad = \quad \boxed{}$$

b)

$$7.1 * 3. \times 10^3 + 12. \quad = \quad \boxed{}$$

c)

$$\frac{\sqrt{3}}{2} * \frac{1}{\sqrt{6}} \quad = \quad \boxed{}$$

d)

$$\cos(\pi/6)/9 \quad = \quad \boxed{}$$

e)

$$\sin(30^\circ) * 3.14 \quad = \quad \boxed{}$$

Problem 5. problems-1/math-sa-integrate-expressions.tex

Evaluate the following indefinite and definite integrals:

a) $\displaystyle\int_0^{\pi/2} \sin(\theta)\ d\theta = \boxed{}$

b) $\displaystyle\int \cos(\omega t)\ dt = \boxed{}$

c) $\displaystyle\int x^n\ dx = \boxed{}$

d) $\displaystyle\int_a^b \frac{1}{x}\ dx = \boxed{}$

e) $\displaystyle\int (-gt + v_0)\ dt = \boxed{}$

Problem 6. problems-1/math-sa-simple-series.tex

Evaluate the first *three nonzero terms* for the **Taylor Series** for the following expressions. Recall that the radius of convergence for the binomial expansion (another name for the first taylor series in the list below) is $|x| < 1$ – this gives you two ways to consider the expansions of the form $(x + a)^n$.

a) Expand about $x = 0$:

$(1 + x)^{-2} \approx$

b) Expand about $x = 0$:

$e^x \approx$

c) For $x > a$ (expand about x or use the binomial expansion after factoring):

$(x + a)^{-2} \approx$

d) Estimate $0.9^{1/4}$ to within 1% without a calculator, if you can. Explain your reasoning.

Problem 7. problems-1/math-sa-solve-simple-equations.tex

Solve for t. Your answer should be an equation, although you may give a number answer for the last one as ***well*** as the algebraic answer if you have a calculator handy. You may find $\ln(2) \approx 0.693$ to be a useful thing to know if not.

a) $v_0 t - x_0 = 0 \qquad t =$ $\boxed{}$

b) $-\frac{1}{2}gt^2 + v_0 t = 0 \qquad t =$ $\boxed{}$

c) $-\frac{1}{2}gt^2 + v_0 t + x_0 = 0 \qquad t =$ $\boxed{}$

d) $A/2 = Ae^{-t} \qquad t =$ $\boxed{}$

(for $A = 5$).

Problem 8. problems-1/math-sa-solve-simultaneous-equations-soln.tex

Solve the following system of simultaneous equations for a and T. ***Show your work*** and give ***algebraic*** answers in terms of m_1, m_2, θ and g:

Probably the easiest way is to add the two equations to eliminate T on the spot:

$$\begin{array}{rcl} m_1 g \sin(\theta) - T & = & m_1 a \\ +T - m_2 g & = & m_2 a \\ \hline m_1 g \sin(\theta) - m_2 g & = & (m_1 + m_2) a \end{array}$$

so:

$$a = \boxed{\frac{m_1 g \sin(\theta) - m_2 g}{m_1 + m_2}}$$

We then back-substitute this into the second equation (rearranged and put over a common denominator for a useful cancellation and factorization) for:

$$T = \boxed{m_2 g + m_2 a = \frac{m_1 m_2 g}{m_1 + m_2} (\sin(\theta) + 1)}$$

Problem 9. problems-1/math-sa-solve-simultaneous-equations.tex

Solve the following system of simultaneous equations for a and T. ***Show your work*** and give ***algebraic*** answers in terms of m_1, m_2, θ and g:

$$m_1 g \sin(\theta) - T = m_1 a$$
$$T - m_2 g = m_2 a$$

$a =$ $\boxed{}$

$T =$ $\boxed{}$

Problem 10. problems-1/math-sa-solve-simultaneous-equations-soln.tex

Solve the following system of simultaneous equations for a and T. ***Show your work*** and give ***algebraic*** answers in terms of m_1, m_2, θ and g:

Probably the easiest way is to add the two equations to eliminate T on the spot:

$$\begin{aligned} m_1 g \sin(\theta) - T &= m_1 a \\ +T - m_2 g &= m_2 a \\ \hline m_1 g \sin(\theta) - m_2 g &= (m_1 + m_2) a \end{aligned}$$

so:

$$a = \boxed{\frac{m_1 g \sin(\theta) - m_2 g}{m_1 + m_2}}$$

We then back-substitute this into the second equation (rearranged and put over a common denominator for a useful cancellation and factorization) for:

$$T = \boxed{m_2 g + m_2 a = \frac{m_1 m_2 g}{m_1 + m_2} (\sin(\theta) + 1)}$$

Problem 11. problems-1/math-sa-sum-two-vectors-1.tex

Suppose vector $\vec{\boldsymbol{A}} = -4\hat{\boldsymbol{x}} + 6\hat{\boldsymbol{y}}$ and vector $\vec{\boldsymbol{B}} = 9\hat{\boldsymbol{x}} + 6\hat{\boldsymbol{y}}$. Then the vector $\vec{\boldsymbol{C}} = \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}$:

a) is in the first quadrant (x+,y+) and has magnitude 17.

b) is in the fourth quadrant (x+,y-) and has magnitude 12.

c) is in the first quadrant (x+,y+) and has magnitude 13.

d) is in the second quadrant (x-,y+) and has magnitude 17.

e) is in the third quadrant (x-,y-) and has magnitude 13.

Problem 12. problems-1/math-sa-sum-two-vectors-soln.tex

1) Adding the two vectors componentwise, we get the vector:

$$\vec{\boldsymbol{C}} = (3-7)\hat{\boldsymbol{x}} + (6-3)\hat{\boldsymbol{y}} = -4\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}}$$

Thus:

a) is in the first quadrant (x+,y+) and has magnitude 7.

b) is in the second quadrant (x-,y+) and has magnitude 7.

c) is in the second quadrant (x-,y+) and has magnitude 5.

d). is in the fourth quadrant (x+,y-) and has magnitude 5.

e). is in the third quadrant (x-,y-) and has magnitude 6.

2) Again, we simply subtract the vectors componentwise:

$$\vec{\boldsymbol{C}} = -13\hat{\boldsymbol{x}}$$

So:

a) is in the x-direction and has magnitude 17.

b) is in the y-direction and has magnitude 13.

c) is in the $-y$-direction and has magnitude 12.

d) is in the x-direction and has magnitude 5.

e) is in the $-x$-direction and has magnitude 13.

Problem 13. problems-1/math-sa-sum-two-vectors.tex

a) Suppose vector $\vec{\boldsymbol{A}} = 3\hat{\boldsymbol{x}} + 6\hat{\boldsymbol{y}}$ and bector $\vec{\boldsymbol{B}} = -7\hat{\boldsymbol{x}} - 3\hat{\boldsymbol{y}}$. Then the vector $\vec{\boldsymbol{C}} = \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}$:

A) is in the first quadrant (x+,y+) and has magnitude 7.
B) is in the second quadrant (x-,y+) and has magnitude 7.
C) is in the second quadrant (x-,y+) and has magnitude 5.
D) is in the fourth quadrant (x+,y-) and has magnitude 5.
E) is in the third quadrant (x-,y-) and has magnitude 6.

b) Suppose vector $\vec{\boldsymbol{A}} = -4\hat{\boldsymbol{x}} + 6\hat{\boldsymbol{y}}$ and vector $\vec{\boldsymbol{B}} = 9\hat{\boldsymbol{x}} + 6\hat{\boldsymbol{y}}$. Then the vector $\vec{\boldsymbol{C}} = \vec{\boldsymbol{A}} - \vec{\boldsymbol{B}}$:

A) is in the x-direction and has magnitude 17.
B) is in the y-direction and has magnitude 13.
C) is in the $-y$-direction and has magnitude 12.
D) is in the x-direction and has magnitude 5.
E) is in the $-x$-direction and has magnitude 13.

Problem 14. problems-1/math-sa-sum-two-vectors-soln.tex

1) Adding the two vectors componentwise, we get the vector:

$$\vec{\boldsymbol{C}} = (3-7)\hat{\boldsymbol{x}} + (6-3)\hat{\boldsymbol{y}} = -4\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}}$$

Thus:

a) is in the first quadrant (x+,y+) and has magnitude 7.

b) is in the second quadrant (x-,y+) and has magnitude 7.

c) is in the second quadrant (x-,y+) and has magnitude 5.

d). is in the fourth quadrant (x+,y-) and has magnitude 5.

e). is in the third quadrant (x-,y-) and has magnitude 6.

2) Again, we simply subtract the vectors componentwise:

$$\vec{\boldsymbol{C}} = -13\hat{\boldsymbol{x}}$$

So:

a) is in the x-direction and has magnitude 17.

b) is in the y-direction and has magnitude 13.

c) is in the $-y$-direction and has magnitude 12.

d) is in the x-direction and has magnitude 5.

e) is in the $-x$-direction and has magnitude 13.

Problem 15. problems-1/math-sa-taylor-series.tex

Evaluate the first *three nonzero terms* for the **Taylor series** for the following expressions. Expand about the indicated point:

a) Expand about $x = 0$:
$$(1+x)^n \approx$$

b) Expand about $x = 0$:
$$\sin(x) \approx$$

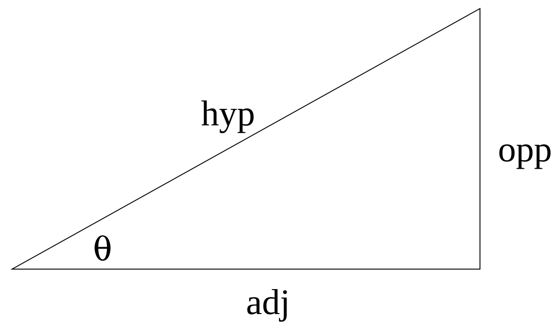
c) Expand about $x = 0$:
$$\cos(x) \approx$$

d) Expand about $x = 0$:
$$e^x \approx$$

e) Expand about $x = 0$ (note: $i^2 = -1$):
$$e^{ix} \approx$$

Verify that the expansions of both sides of the following expression match:
$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

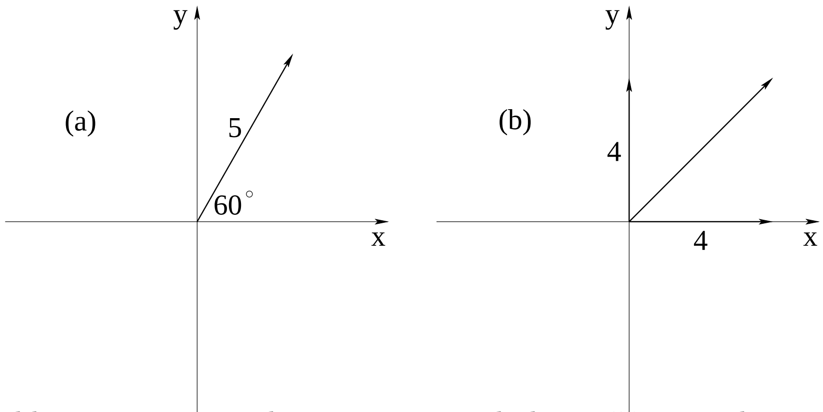
Problem 16. problems-1/math-sa-trig-basic.tex



Fill in the following in terms of the marked sides. For example, one of the answers below might be (but probably isn't) $\frac{\text{hyp}}{\text{opp}}$:

$\sin(\theta) =$ $\cos(\theta) =$ $\tan(\theta) =$

Problem 17. problems-1/math-sa-vector-components.tex



Two simple problems in vector analysis are presented above. You may leave your answers in terms of radical fractions (e.g. $\sqrt{7}/13$) where appropriate. You may not use calculators!

a) Find the **cartesian** coordinate components (X, Y) of the vector given.

$X =$ $\boxed{}$

$Y =$ $\boxed{}$

b) Find the **polar** coordinate components (V, θ) of the vector given.

$|V| =$ $\boxed{}$

$\theta =$ $\boxed{}$

2.1.2 Regular Problems

Problem 18. problems-1/math-pr-elliptical-trajectory.tex

The position of a particle as a function of time is given by:

$$\vec{x}(t) = x_0 \cos(\omega t)\hat{x} + y_0 \sin(\omega t)\hat{y}$$

where $x_0 > y_0$.

a) What is $\vec{v}(t)$ for this particle?

b) What is $\vec{a}(t)$ for this particle?

c) Draw a generic plot of the trajectory function for the particle. What kind of shape is this? In what direction/sense is the particle moving (indicate with arrow on trajectory)?

d) Draw separate plots of $x(t)$ and $y(t)$ on the same axes.

Chapter 3

Essential Laws, Theorems, and Principles

The questions below directly" review basic physical laws and concepts. They are the stuff that one way or another a student should know" going into any exam or quiz following the lecture(s) in which they are covered. Note that there aren't really all that many of them, and a lot of them are actually easily derived from the most important ones.

IMO *every student* should memorize, internalize, learn, *know* the principles, laws, and and theorems covered in this section (and perhaps a few that haven't yet been added). These are things upon which all the rest of the solutions are based.

Short Problem 1.

problems-1/true-facts-angular-momentum-conservation.tex

When is the angular momentum of a system conserved?

Short Problem 2.

problems-1/true-facts-archimedes-principle.tex

What is Archimedes' Principle? (Equation with associated diagram or *clear and correct* statement in words.)

Short Problem 3.

problems-1/true-facts-bernoullis-equation.tex

What is Bernoulli's equation? What does it describe? Draw a small picture to illustrate.

Short Problem 4.

problems-1/true-facts-coefficient-of-performance.tex

How is the coefficient of performance of a refrigerator defined? Draw a small diagram that schematically indicates the flow of heat and work between reservoirs.

Short Problem 5.

problems-1/true-facts-conditions-static-equilibrium.tex

What are the two conditions for a rigid object to be in static equilibrium?

Condition 1:

Condition 1:

Short Problem 6.

problems-1/true-facts-coriolis-force.tex

What "force" makes hurricanes spin counterclockwise in the northern hemisphere and clockwise in the southern hemisphere?

Short Problem 7.

problems-1/true-facts-definition-of-decibel.tex

One measures sound intensity in decibels. What is a decibel? (Equation, please, and define and give value of all constants.)

Short Problem 8.

problems-1/true-facts-doppler-shift-moving-source.tex

What is the equation for the Doppler shift, specifically for the frequency f' heard by a stationary observer when a source emitting waves with speed u at frequence f_0 is approaching at speed u_s?

Short Problem 9.

problems-1/true-facts-equipartition-theorem.tex

What is the Equipartition Theorem?

Short Problem 10.

problems-1/true-facts-four-forces-of-nature.tex

Name the four fundamental forces of nature as we know them now.

a)

b)

c)

d)

Short Problem 11.

problems-1/true-facts-generalized-work-energy.tex

What is the *Generalized* Work-Mechanical-Energy Theorem? (Equation only. This is the one that differentiates between conservative and non-conservative forces.)

Short Problem 12.

problems-1/true-facts-heat-capacity-monoatomic-gas.tex

What is the heat capacity at constant volume C_V of N molecules of an ideal monoatomic gas? What is its heat capacity at constant pressure C_P?

Short Problem 13.

problems-1/true-facts-heat-engine-efficiency.tex

What is the algebraic definition of the efficiency of a heat engine? Draw a small diagram that schematically indicates the flow of heat and work between reservoirs.

Short Problem 14.

problems-1/true-facts-inelastic-collision-conservation.tex

What is conserved (and what *isn't*) in an inelastic collision?

Short Problem 15.

problems-1/true-facts-integral-definition-moment-of-inertia.tex

Write the *integral* definition of the moment of inertia of an object about a particular axis of rotation. Draw a picture illustrating what "dm" is within the object relative to the axis of rotation.

Short Problem 16.

problems-1/true-facts-kepler1.tex

What is Kepler's First Law?

Short Problem 17.

problems-1/true-facts-kepler2.tex

What is Kepler's Second Law and what physical principle does it correspond to?

Short Problem 18.

problems-1/true-facts-kepler3.tex

What is Kepler's Third Law?

Short Problem 19.

problems-1/true-facts-momentum-conservation.tex

Under what condition(s) is the linear momentum of a system conserved?

Short Problem 20.

problems-1/true-facts-n1.tex

What is Newton's First Law?

Short Problem 21.

problems-1/true-facts-n2.tex

What is Newton's Second Law?

Short Problem 22.

problems-1/true-facts-n3.tex

What is Newton's Third Law?

Short Problem 23.

problems-1/true-facts-newtons-law-gravitation.tex

What is Newton's Law for Gravitation? Draw a picture showing the coordinates used (for two pointlike masses at arbitrary positions), and indicate the value of G in SI units.

Short Problem 24.

problems-1/true-facts-parallel-axis-theorem.tex

Write the parallel axis theorem for the moment of inertia of an object around an axis parallel to one through its center of mass. Draw a picture to go with it, if it helps.

Short Problem 25.

problems-1/true-facts-pascals-principle.tex

What is Pascal's principle? A small picture would help.

Short Problem 26.

problems-1/true-facts-perpendicular-axis-theorem.tex

Write the perpendicular axis theorem for a mass distributed in the $x - y$ plane. Draw a picture to go with it, if it helps.

Short Problem 27.

problems-1/true-facts-toricellis-law.tex

What is Toricelli's Law (for fluid flow) and what is the condition required for it to be approximately true?

Short Problem 28.

problems-1/true-facts-venturi-effect.tex

What is the Venturi Effect?

Short Problem 29.

problems-1/true-facts-wave-equation-string.tex

Write the wave equation (the differential equation) for waves on a string with tension T and mass density μ. Identify all parts.

Short Problem 30.

problems-1/true-facts-work-energy.tex

What is the Work-Kinetic Energy Theorem?

Short Problem 31.

problems-1/true-facts-youngs-modulus.tex

What is the definition of Young's modulus Y? Draw a picture illustrating the physical situation it describes and define all terms used in terms of the picture.

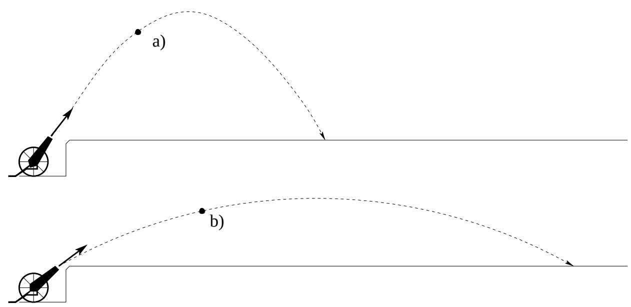
Chapter 4

Dynamics

4.1 Kinematics

4.1.1 Multiple Choice Problems

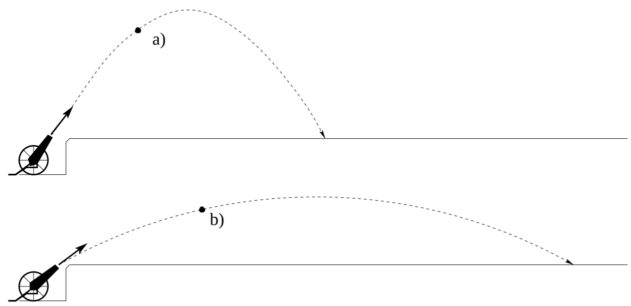
Problem 19. problems-1/kinematics-mc-cannonball-timed-trajectories-icp.tex



Two cannons fire projectiles into the air along the trajectories shown. Neglect the drag force of the air. Which one is in the air longer? Justify your answer in some (very terse) way.

- ☐ Cannonball $\boldsymbol{a}$ is in the air longer.
- ☐ Cannonball $\boldsymbol{b}$ is in the air longer.
- ☐ Cannonballs $\boldsymbol{a}$ and $\boldsymbol{b}$ are in the air the same amount of time.
- ☐ We cannot tell which is in the air longer without more information than is given in the picture.

Problem 20. problems-1/kinematics-mc-cannonball-timed-trajectories-icp-soln.tex



Two cannons fire projectiles into the air along the trajectories shown. Neglect the drag force of the air. Which one is in the air longer? Justify your answer in some (very terse) way.

☒ Cannonball ***a*** is in the air longer. ☐ Cannonball ***b*** is in the air longer.

☐ Cannonballs ***a*** and ***b*** are in the air the same amount of time.

☐ We cannot tell which is in the air longer without more information than is given in the picture.

Solution: The motion in x and y are ***independent*** for 2D trajectory problems. You can therefore ***ignore the x motion altogether*** when you assess the answer to this question. So, considering ***only how high each cannonball goes***, which one is in the air longer?

Hopefully, given this hint, you correctly selected cannonball a). This is the easy way.

Now let's do it "the hard way". Suppose a cannonball has an initial upward component of its velocity of v_{0y}. Then:

$$y(t_g) = v_{0y}t_g - \frac{1}{2}gt_g^2 = 0 \quad \Rightarrow \quad t_g = 0, \frac{2v_{0y}}{g}$$

are the times the cannonball is at $y = 0$ (the ground) and:

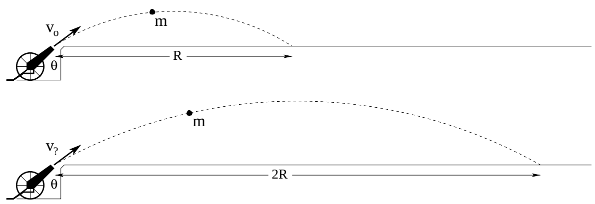
$$v_y(t_H) = 0 = v_{0y} - gt_H \quad \Rightarrow \quad t_H = \frac{v_{0y}}{g} = \frac{t_g}{2}$$

We relate this to the maximum height H as:

$$H = v_{0y}t_H - \frac{1}{2}gt_H^2 = \frac{v_{0y}^2}{2g} \quad \Rightarrow \quad v_{0y} = \sqrt{2gH} \quad \Rightarrow \quad \boxed{t_g = 2 \times \frac{\sqrt{2gH}}{g} = 2\sqrt{\frac{2H}{g}}}$$

Or (to conclude), the time the cannonball stays in the air scales monotonically with $\sqrt{H}$, so *explicitly,* the higher the cannonball goes, the longer it is in the air before it hits the ground, ***independent of its range*** because v_{0x} can be selected independent of v_{0y} and hence H. You can literally make the cannonball have any range you like (within reason, limited by the curvature of the Earth, relativity theory, drag etc) as long as you give it *some* positive v_{0y} so $t_g > 0$ in $R = v_{0x}t_g$.

Problem 21. problems-1/kinematics-mc-range-of-cannon-on-plain-review.tex

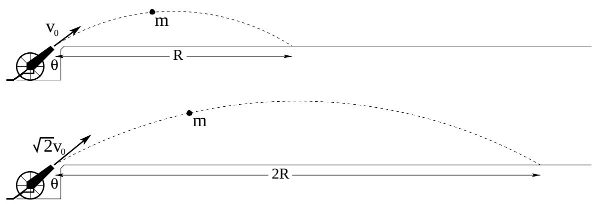


A cannon sits on a horizontal plain. When it fires a cannonball of mass m at speed v_0 at an angle θ relative to the ground it has a range R (neglecting friction and drag). Suppose one wishes to fire at a target a distance $2R$ away without altering the the elevation angle θ. The initial speed of the cannonball as it leaves the cannon in terms of v_0 must then be:

☐ $\sqrt{2}v_0$ ☐ $2v_0$ ☐ $3v_0$ ☐ $4v_0$

☐ We cannot tell because the answer depends upon θ.

Problem 22. problems-1/kinematics-mc-range-of-cannon-on-plain-review-soln.tex



A cannon sits on a horizontal plain. When it fires a cannonball of mass m at speed v_0 at an angle θ relative to the ground it has a range R (neglecting friction and drag). Suppose one wishes to fire at a target a distance $2R$ away without altering the the elevation angle θ. The initial speed of the cannonball as it leaves the cannon in terms of v_0 must then be:

☒ $\sqrt{2}v_0$ ☐ $2v_0$ ☐ $3v_0$ ☐ $4v_0$

☐ We cannot tell because the answer depends upon θ.

Solution: This is a conceptual problem; we'll use scaling to answer it. Note that the time the cannonball is in the air is proportional to v_0. Note that the distance in x that the cannonball travels when it is in the air is also proportional to v_0. We therefore expect:

$$R \propto v_0^2 \quad \Rightarrow \quad 2R \propto (x \times v_0)^2 \quad \Rightarrow \quad \boxed{x = \sqrt{2}}$$

Note that this reasoning only takes a few seconds. If you want to fill in details, or check your answer with a full solution, you would note that (after solving the simple kinematics of constant acceleration near-Earth gravitation):

$$y(t_f) = v_0 \sin\theta\, t_f - \frac{1}{2}gt_f^2 = 0 \quad \Rightarrow \quad \boxed{t_f = 0,\ \frac{2v_0\sin\theta}{g}}$$

We obviously want the second time as the time the cannonball *lands*; $t_f = 0$ is when it is *fired.* Then:

$$x(t_f) = R = v_0 \cos\theta\, t_f \quad \Rightarrow \quad \boxed{R = \left(\frac{2\sin\theta\cos\theta}{g}\right) v_0^2}$$

verifying explicitly our verbal conclusion that $R \propto v_0^2$ above.

Problem 23. problems-1/kinematics-mc-falling-time-mars-review.tex

Surface gravity on Mars is roughly 1/3 that of the Earth. Suppose you drop a rock (initially at rest) from a height H_m on Mars and it takes a time t_g to hit the ground. From what height H_e do you need to drop the mass on Earth so that it hits the ground in the same amount of time?

- ☐ $H_e = \sqrt{3} H_m$
- ☐ $H_e = H_m/3$
- ☐ $H_e = 9H_m$
- ☐ $H_e = H_m/\sqrt{3}$
- ☐ $H_e = 3H_m$

Problem 24. problems-1/kinematics-mc-falling-time-mars-review-soln.tex

Surface gravity on Mars is roughly 1/3 that of the Earth. Suppose you drop a rock (initially at rest) from a height H_m on Mars and it takes a time t_g to hit the ground. From what height H_e do you need to drop the mass on Earth so that it hits the ground in the same amount of time?

- ☐ $H_e = \sqrt{3} H_m$
- ☐ $H_e = H_m/3$
- ☐ $H_e = 9H_m$
- ☐ $H_e = H_m/\sqrt{3}$
- ☒ $H_e = 3H_m$

Solution: Let's answer this using ***scaling***. On Mars (where the acceleration is g_m) we know that:

$$H_m = \frac{1}{2} g_m t_g^2$$

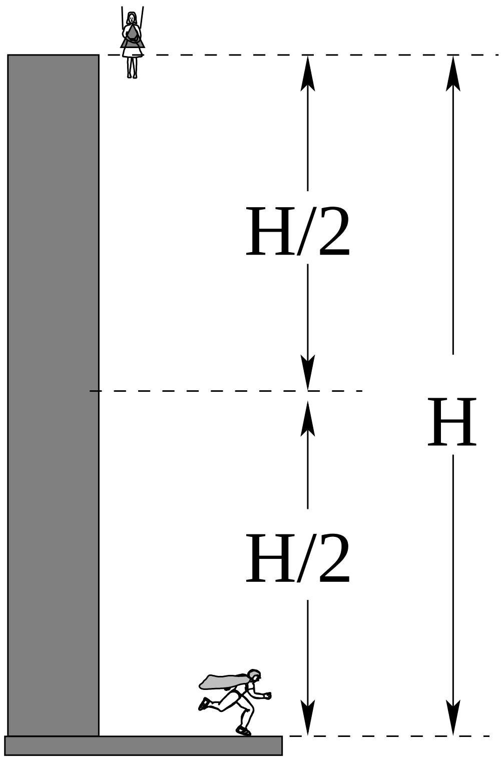
(where $g_e = 3g_m$). We need:

$$H_e = \frac{1}{2} g_e t_g^2 = \frac{1}{2} 3 g_m t_g^2 = 3 \left(\frac{1}{2} g_m t_g^2 \right) \quad \Rightarrow \quad \boxed{H_e = 3H_m}$$

Note that there are *several other algebraically equivalent ways* (for example, substitution) to arrive at the same conclusion, all correct, but the virtue of this one is that we never have to explicitly evaluate t_g on either Mars or the Earth. You can almost do it in your head!

4.1.2 Short Answer Problems

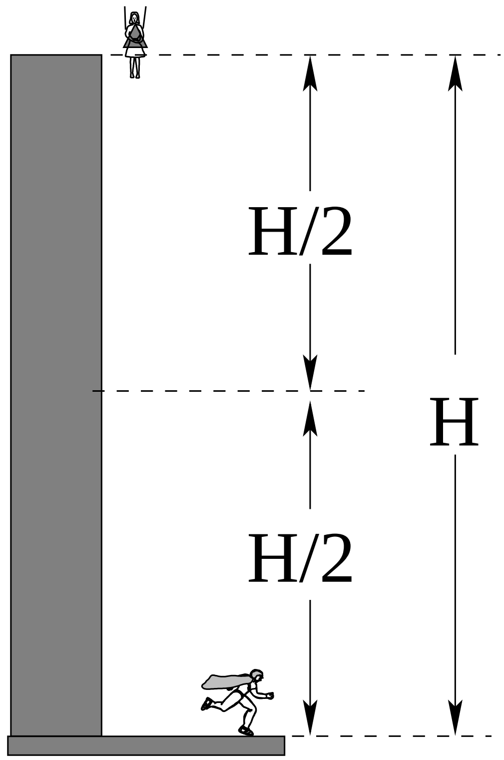
Problem 25. problems-1/kinematics-sa-green-laser-falls-review.tex



The Green Lantern's daughter, Green Laserbeam, steps off of a tall building to follow her dad to the ground. She falls freely (from rest) to the ground, falling the last **half** of the total distance in a time t_2. Find the ***ratio*** of t_2 to the ***total time it takes for her to reach the ground*** $t_{\rm tot}$. Your answer should be a number (and may have e.g. square roots in it). Hint: $t_2 = t_{\rm tot} - t_1$.

$$\frac{t_2}{t_{\rm tot}} = \boxed{}$$

Problem 26. problems-1/kinematics-sa-green-laser-falls-review-soln.tex



The Green Lantern's daughter, Green Laserbeam, steps off of a tall building to follow her dad to the ground. She falls freely (from rest) to the ground, falling the last **half** of the total distance in a time t_2. Find the ***ratio*** of t_2 to the ***total time it takes for her to reach the ground*** $t_{\rm tot}$. Your answer should be a number (and may have e.g. square roots in it). Hint: $t_2 = t_{\rm tot} - t_1$.

$$\frac{t_2}{t_{\rm tot}} = \boxed{1 - \frac{\sqrt{2}}{2} \approx 0.3}$$

Solution: We use constant acceleration kinematics several times. Note the following:

$$\frac{H}{2} = \frac{1}{2} g t_1^2 \quad \Rightarrow \quad t_1 = \sqrt{\frac{H}{g}}$$

$$H = \frac{1}{2} g t_{\rm tot}^2 \quad \Rightarrow \quad t_{\rm tot} = \sqrt{\frac{2H}{g}}$$

$$t_2 = t_{\rm tot} - t_1 = (\sqrt{2} - 1)\sqrt{\frac{H}{g}} \quad \Rightarrow \quad \boxed{\frac{t_2}{t_{\rm tot}} = \frac{\sqrt{2}-1}{\sqrt{2}} = \frac{2-\sqrt{2}}{2} = 1 - \frac{\sqrt{2}}{2}}$$

4.1.3 Ranking Problems

Problem 27. problems-1/kinematics-ra-scaling-fall-times-mongo-bongo-review.tex

A mass m is used to perform kinematics experiments on two distant planets, *Mongo* and *Bongo*. You expect that the near-surface gravitational force on Mongo or Bongo, like that of Earth, is given by:

$$Fy = -mg_i \qquad (i = \text{M,B})$$

You observe that when it is dropped from 10 meters above the ground on Mongo, it takes 2 seconds to reach the surface. On the other hand, when it is dropped from 5 meters above the ground on Bongo, it takes 1 second to reach the surface. Rank the surface gravity of Mongo relative to Bongo below (put a symbol $<, >, =$ in the provided box):

$$g_M \ \boxed{} \ g_B$$

Problem 28. problems-1/kinematics-ra-scaling-fall-times-mongo-bongo-review-soln.tex

A mass m is used to perform kinematics experiments on two distant planets, *Mongo* and *Bongo*. You expect that the near-surface gravitational force on Mongo or Bongo, like that of Earth, is given by:

$$Fy = -mg_i \qquad (i = \text{M,B})$$

You observe that when it is dropped from 10 meters above the ground on Mongo, it takes 2 seconds to reach the surface. On the other hand, when it is dropped from 5 meters above the ground on Bongo, it takes 1 second to reach the surface. Rank the surface gravity of Mongo relative to Bongo below (put a symbol $<, >, =$ in the provided box):

$$g_M \quad \boxed{<} \quad g_B$$

Solution: Let's form the ratios:

$$\frac{H_M}{H_B} = 2 = \frac{\frac{1}{2} g_M t_M^2}{\frac{1}{2} g_B t_B^2} = \frac{g_M}{g_B} \times 4$$

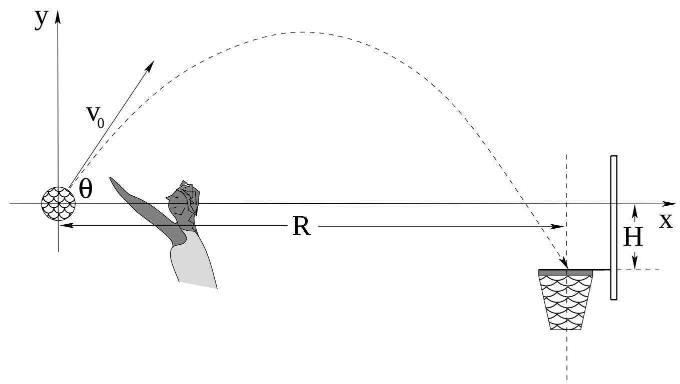
or:

$$g_M = \frac{1}{2} g_B$$

4.1.4 Regular Problems

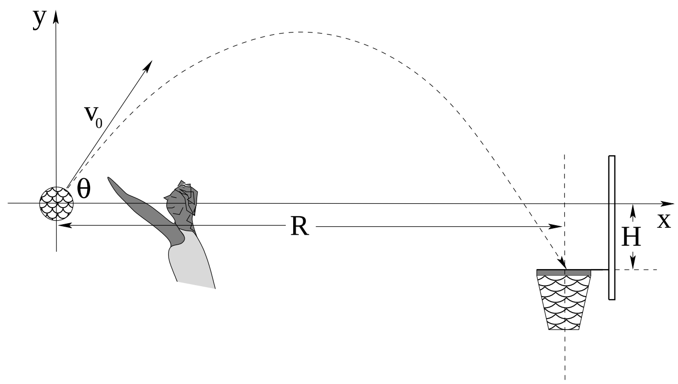
Problem 29. problems-1/kinematics-pr-2D-basketball-trajectory-review.tex

A basketball player shoots a jump hook at a (horizontal) distance R from the basket, releasing the ball at a height H above the rim as shown. To shoot over his opponent's outstretched arm, he releases the basketball at an angle θ with respect to the horizontal.



Find v_0, the ***speed*** he must release the basketball with (in terms of H, R, g and θ) for the ball to go through the hoop "perfectly" as shown. Assume that his release is on line and undeflected, at initial speed v_0 and that the acceleration of the basketball is $\vec{a} = -g\hat{j}$, ignoring drag.

Problem 30. problems-1/kinematics-pr-2D-basketball-trajectory-review-soln.tex



First, note that

$$a_x = 0, v_{0x} = v_0 \cos(\theta), x_0 = 0$$

and

$$a_y = -g, v_{0y} = v_0 \sin(\theta), y_0 = 0$$

define the initial conditions of two ***independent*** 1D constant acceleration problems.

Integrate $a_x = 0$ twice to get:

$$x(t) = v_0 \cos(\theta) t$$

Integrate $a_y = -g$ twice to get:

$$y(t) = -\frac{1}{2} g t^2 + v_0 \sin(\theta) t$$

Next, find the ***time*** t_b that the basketball reaches the horizontal position of the hoop:

$$R = v_0 \cos(\theta) t_b \Rightarrow t_b = R/(v_0 \cos(\theta))$$

This must also be the time that the ball has exactly the height of the hoop:

$$-H = -\frac{1}{2} g t_b^2 + v_0 \sin(\theta) t_b$$

$$-H = -\frac{gR^2}{2v_0^2 \cos^2(\theta)} + R \tan(\theta)$$

$$\frac{gR^2}{2v_0^2 \cos^2(\theta)} = R \tan(\theta) + H$$

And finally, we solve for v_0:

$$\boxed{v_0 = \sqrt{\frac{gR^2}{2(R \sin(\theta) \cos(\theta) + H \cos^2(\theta))}}}$$

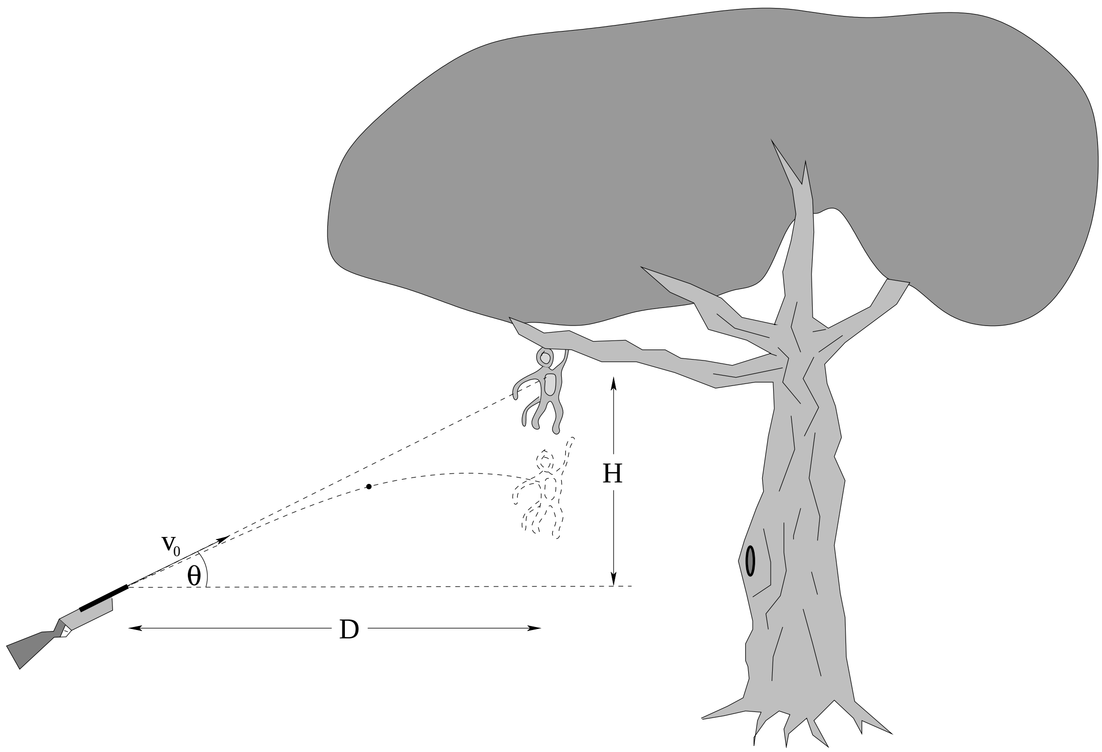
After doing the algebra, check the dimensions. Are they OK?

Check "common sense" – does the solution vary the way you expect? Well, if g goes up, he must shoot the ball faster to overcome gravity on (say) Jupiter. Makes sense. If H goes up, must shoot faster even here on Earth to reach the hoop. Makes sense.

Note that solution doesn't tell us whether a shot at the ***given*** angle will hit the rim, but if θ points directly at center of hoop ($\tan(\theta) = -H/R$) then v_0 has to become "infinite" for ball to travel in a straight line to the target. There are ***no solutions*** for angles less than this as we can tell because the solution speed becomes imaginary! This too makes "sense".

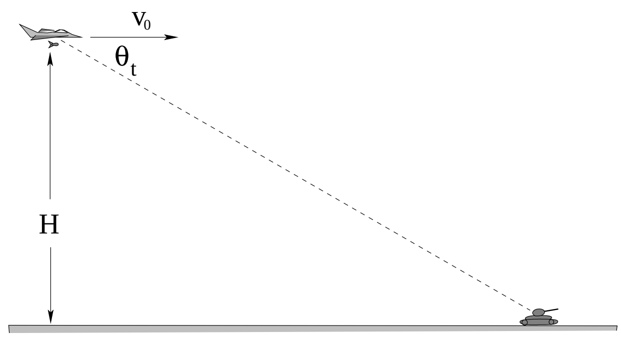
Note Well: There is a second, much more painful solution that involves finding the time that the basketball reaches the right height H ***first***, then substituting it into equation for R. **This solution can work**, but it is not easy. The trick is to isolate the radical on one side of the equals sign, square both sides (to make the radical go away), and then solve for v_0. Done perfectly, it will give you precisely the same answer obtained above.

Problem 31. problems-1/kinematics-pr-monkey-gun.tex



A hunter aims his gun directly at a monkey in a distant tree. Just as she fires, the monkey lets go and drops in free fall towards the ground. Show that the bullet hits the monkey.

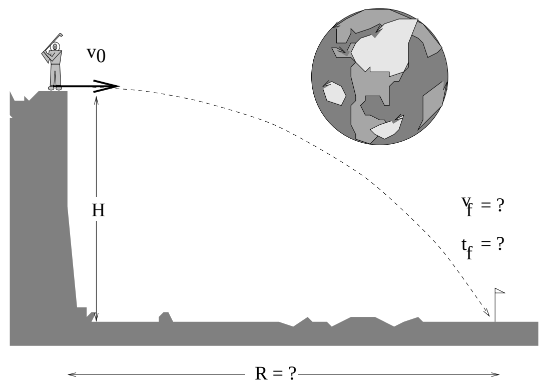
Problem 32. problems-1/kinematics-pr-bombing-run-trajectory-review.tex



A bomber flies with a constant horizontal velocity v_0. It wishes to target a dummy tank on the ground in a practice bombing run. The bombardier will drop a bomb (of mass m) when the view angle of the tank relative to the bomber is θ_t. The bomber's height is H. Assume that there are no drag forces.

What should the angle θ_t be at the instant of release if the bomber wishes to hit the tank?

Problem 33. problems-1/kinematics-pr-golf-on-moon-icp.tex



An astronaut on the moon hits a golf ball of mass m horizontally from a tee H meters above the plane as shown. The initial speed of the ball is v_0 in the x-direction only. The gravitational force law for the moon is:

$$\vec{\boldsymbol{F}}_m = -m\frac{g}{6}\hat{\boldsymbol{j}}$$

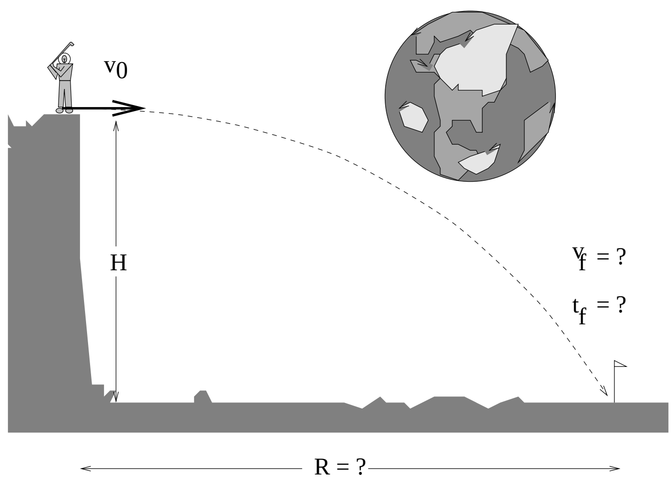
Note that there are no drag forces as the moon is in a vacuum, and that the lunar plane is flat on the scale of this picture. Use Newton's second law to answer the following questions:

a) How long does it take the ball to reach the ground?

b) How far from the base of the cliff where the tee is located does the ball strike?

c) How fast is the ball going when it hits the ground?

Problem 34. problems-1/kinematics-pr-golf-on-moon-icp-soln.tex



a) $t_g = \sqrt{\dfrac{2H}{g'}} = \sqrt{\dfrac{12H}{g}}$

b) $R = x(t_g) = v_0 t_g = v_0\sqrt{\dfrac{2H}{g'}}$

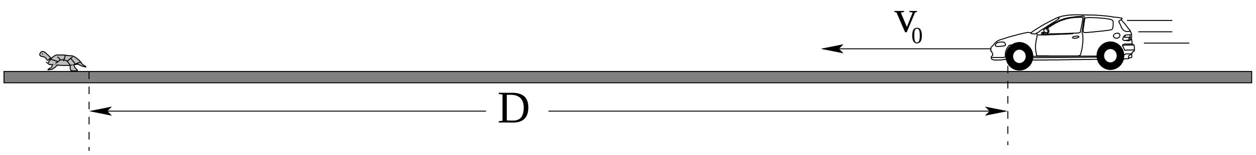
c) This one is a bit tricky (and gets much easier with energy conservation later):

$v_x(t_g) = v_0$

$v_y(t_g) = \sqrt{\dfrac{2Hg}{6}}$

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_0^2 + \dfrac{2Hg}{6}}$

Problem 35. problems-1/kinematics-pr-stopping-before-a-turtle.tex



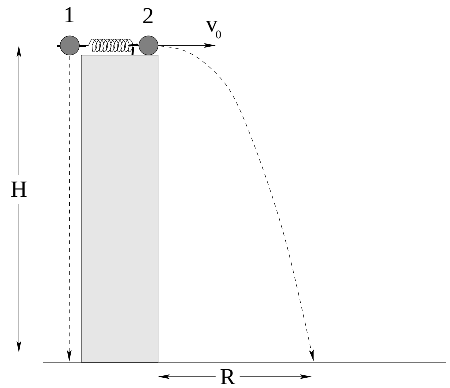
A distance of D meters ahead of your car you see a box turtle sitting on the road. Your car is traveling at a speed of v_0 meters per second straight at the turtle (along the straight road).

a) What is the (algebraic) magnitude of the *minimum* acceleration your car must have in order to stop before hitting the turtle? What is its direction?

b) How long does it take to stop your car at this acceleration?

c) Evaluate your algebraic answers for $D = 50$ m, $v_0 = 20$ m/sec (about 45 mph). If your car's maximum braking acceleration magnitude is $a = 5$ m/sec^2, do you hit the turtle?

Problem 36. problems-1/kinematics-pr-two-falling-balls.tex



When a trigger is pulled at time $t = 0$, a compressed spring simultaneously drops ball 1 and hits identical ball 2 so that it is shot out to the right as initial speed v_0 as shown. The two balls then independently fall a height H. Answer the following questions, assuming that the balls fall only under the influence of gravity. (Neglect drag forces, and express all answers in terms of the givens, in this case H and v_0 and (assumed) gravitational acceleration g.).

a) Which ball strikes the ground first (or do they strike at the same time)? Prove your answer by finding the time that each ball hits the ground.

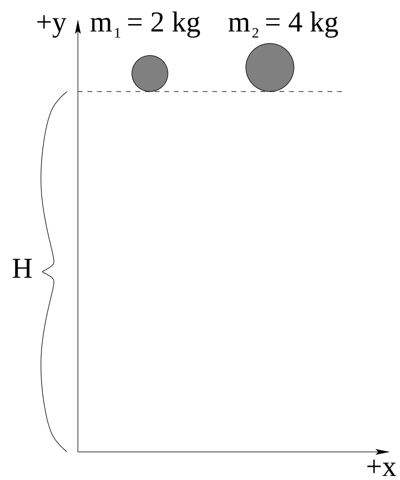
b) Which ball is travelling faster when it hits the ground (or do they hit at the same speed)? Prove your answer by finding an expression for the speed each ball has when it hits the ground.

c) Find an expression for R, the horizontal distance ball 2 travels before hitting the ground.

4.2 Newton's Laws

4.2.1 Multiple Choice Problems

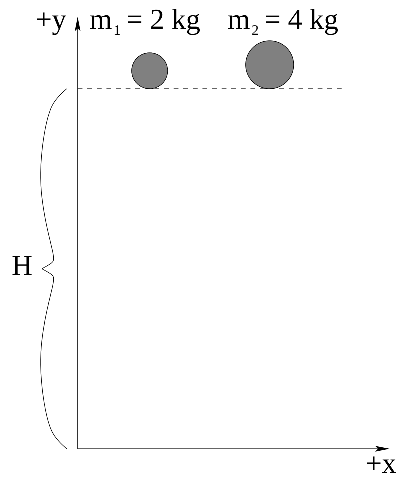
Problem 37. problems-1/force-mc-acceleration-two-masses-1.tex



A mass $m_1 = 2$ kg and a mass of $m_2 = 4$ kg are both dropped from rest from the same height at the same time. Mark the true statements with an "X" below (there can be more than one). **Neglect drag forces.**

- ☐ While the two masses are falling, the force acting on m_1 and the force acting on m_2 are equal in magnitude.
- ☐ While the two masses are falling, the acceleration of m_1 and the acceleration of m_2 are equal in magnitude.
- ☐ Mass m_2 will strike the ground first.
- ☐ Mass m_1 will strike the ground first.
- ☐ The two masses will strike the ground at the same time.

Problem 38. problems-1/force-mc-acceleration-two-masses-1-soln.tex



We recall that for near-Earth gravity (Newton's Second Law):

$$F_y = -mg = ma_y$$

The masses are different: $m_1 g \neq m_2 g$. The accelerations are the same, in both cases g because m *cancels* in Newton's Second Law.

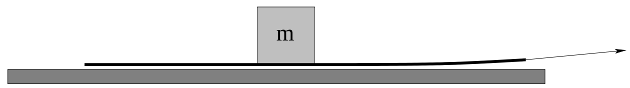
If the accelerations are the same and the initial conditions (in y) are the same, the motion (in y) must be the same! In fact:

$$y_1(t) = y_2(t) = H - \frac{1}{2}gt^2$$

This is what Galileo observed in the (possibly apocryphal) experiment of dropping two different masses off of the leaning tower of Pisa. They struck the ground at the same time (within a small difference we can attribute to the neglected drag forces)!

- ☐ While the two masses are falling, the force acting on m_1 and the force acting on m_2 are equal in magnitude.
- ☒ While the two masses are falling, the acceleration of m_1 and the acceleration of m_2 are equal in magnitude.
- ☐ Mass m_2 will strike the ground first.
- ☐ Mass m_1 will strike the ground first.
- ☒ The two masses will strike the ground at the same time.

Problem 39. problems-1/force-mc-block-on-paper-icp.tex



A block of mass m is resting on a long piece of smooth paper. The block has coefficient of static and kinetic friction μ_s, μ_k with the paper, respectively. You jerk the paper horizontally so it ***slides*** out from under the block quickly in the direction indicated by the arrow without sticking. Which of the following statements about the force acting on and acceleration of the ***block*** are true?

a) $F = \mu_s mg$, $a = \mu_s g$, both to the right.

b) $F = \mu_k mg$ to the right, $a = \mu_k g$ to the left.

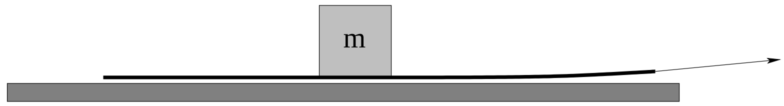
c) $F = \mu_k mg$, $a = \mu_k g$ both to the left.

d) $F = \mu_k mg$ to the left, $a = \mu_k g$ to the right.

e) $F = \mu_k mg$, $a = \mu_k g$ both to the right.

f) None of the above.

Problem 40. problems-1/force-mc-block-on-paper-icp-soln.tex



Explanation: Friction opposes the ***relative sliding direction*** between the two surfaces. As the paper is pulled out from under the block, it tries to pull the block along with it as it slides back! You can easily verify this with a simple experiment in which you mark the position of a "block" (e.g. cell phone, quarter, keys) on a sheet of paper ***relative to the table it sits on*** with your finger and then pull the sheet out from under it sharply. You will see tht the block is displaced ***in the direction the paper was pulled*** and does *not* move backwards in absolute terms.

Students are often confused by this because their mental point of view "automatically" jumps into the accelerating frame of the paper. Relative to the *paper*, the block appears to slide ***backwards***. But this is an illusion, the visualization of a pseudoforce in an accelerating frame, just as no "force" pushes you back into your car seat when you accelerate, the car seat pushes you forward so that you keep up with the car!

The answer therefore is:

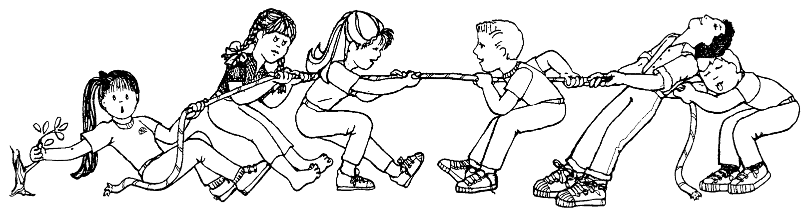
a) $F = \mu_s mg$, $a = \mu_s g$, both to the right. b) $F = \mu_k mg$ to the right, $a = \mu_k g$ to the left.

c) $F = \mu_k mg$, $a = \mu_k g$ both to the left. d) $F = \mu_k mg$ to the left, $a = \mu_k g$ to the right.

e) $F = \mu_k mg$, $a = \mu_k g$ both to the right. f). None of the above.

Problem 41. problems-1/force-mc-N3-pick-list-1-icp.tex

The following sentences each describe two specific forces exerted by objects in a physical situation. Circle the letter of the sentences where those two forces form a ***Newton's Third Law*** force pair. ***More than one sentence or no sentences at all*** in the list may describe a Newton's Third Law force pair.



a) In an evenly matched tug of war (where the rope does not move); team one pulls the rope to the left with some force and team two pulls the rope to the right with an equal magnitude force in the opposite direction.

b) Gravity pulls me down; the normal force exerted by a scale I'm standing on pushes me up with an equal magnitude force in the opposite direction.

c) The air surrounding a helium balloon pushes it up with a buoyant force; the balloon pushes the air down with an equal magnitude force in the opposite direction.

Problem 42. problems-1/force-mc-N3-pick-list-1-icp-soln.tex

Newton's Third Law states:

$$\vec{\boldsymbol{F}}_{AB} = -\vec{\boldsymbol{F}}_{BA}$$

or, in words, if object B exerts a force (with a named force law) on object A, then object A exerts and equal and opposite force on object B. Note well: ***A* and *B* must be the same** and **the (named) force must be the same**.

A common mistake is to interpret a problem statement with object A exerting a force on object C, object B exerting an opposite force on object C, and concluding that $\vec{\boldsymbol{F}}_{AC}$ and $\vec{\boldsymbol{F}}_{BC}$ are a Newton's Third Law pair.

Therefore the only correct statement is c).

Problem 43. problems-1/force-mc-N3-pick-list.tex

Identify the Newton's Third Law pairs from the following list of forces (more than one could be right). Explain *why* each answer that you come up with is correct.

- [ ] Gravity exerts a force on a block sliding down an incline. Kinetic friction and the normal force exert a force up the incline on the block.
- [ ] Static friction prevents a block from sliding down an angled plank; the block exerts a equal magnitude force of static friction down the incline of the plank.
- [ ] A block is dragged at constant speed across the ground by a rope with tension T. Kinetic friction pulls back with an equal and opposite force.
- [ ] A person stands on a scale that reads the magnitude of the normal force pressing on its top surface. Gravity pulls the person down with a force equal to their weight read by the scale.
- [ ] Expanding gases push a bullet out of a gun. The bullet pushes back on the expanding gases with an equal but opposite force.

Problem 44. problems-1/force-mc-N3-pick-list-soln.tex

Identify the Newton's Third Law pairs from the following list of forces (more than one could be right). Explain *why* each answer that you come up with is correct.

☐ Gravity exerts a force on a block sliding down an incline. Kinetic friction and the normal force exert a force up the incline on the block.

☒ Static friction prevents a block from sliding down an angled plank; the block exerts a equal magnitude force of static friction down the incline of the plank.

☐ A block is dragged at constant speed across the ground by a rope with tension T. Kinetic friction pulls back with an equal and opposite force.

☐ A person stands on a scale that reads the magnitude of the normal force pressing on its top surface. Gravity pulls the person down with a force equal to their weight read by the scale.

☒ Expanding gases push a bullet out of a gun. The bullet pushes back on the expanding gases with an equal but opposite force.

Explanations: In order to be an N3 pair, ***both*** the name of the named force law ***and*** the two objects interacting via the force have to be the same, in which case N3 says the magnitudes have to be the same and in the opposite direction.

In the first case, there are *three* forces, *none* of which form an interaction pair.

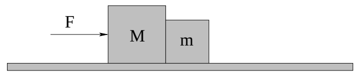
In the second case, it is static friction, same magnitude, opposite direction. Check!

In the third case, it is tension and kinetic friction, each of which has a *different* interaction partner and name. So even though they happen to be equal and opposite, they aren't an N3 pair.

In the fourth case, it is normal force and gravity, each of which has a *different* interaction partner and name. So even though they happen to be equal and opposite, they aren't an N3 pair.

In the fifth case, gas pushes on the bullet via pressure at the point of contact, so *the bullet pushes back on the gas, also* via pressure at the surface of contact. Even without knowing the name or origin of the macroscopic force (gas pressure), you can tell this is an N3 pair from the problem statement!

Problem 45. problems-1/force-mc-pushing-two-blocks.tex



The figure shows two blocks of mass M and m that are being pushed along a horizontal frictionless surface by a force of magnitude F as shown. What is the magnitude of the force that the block of mass M exerts on the block of mass m?

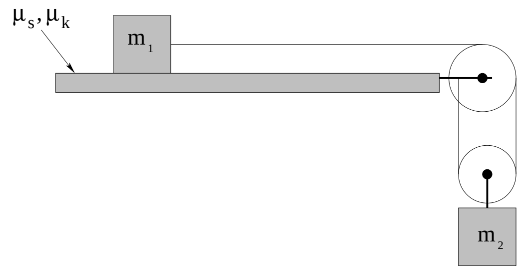
a) F

b) $m\frac{F}{M}$

c) $m\frac{F}{(M+m)}$

d) $M\frac{F}{(M+m)}$

Problem 46. problems-1/force-mc-sliding-block-and-tackle.tex



A block of mass m_1 sits on a rough table. The coefficient of static and kinetic friction between the mass and the table are μ_s and μ_k, respectively. Another mass m_2 is suspended as indicated in the figure above (where the pulleys are massless and the string is massless and unstretchable). What is the maximum mass m_2 for which the blocks remain at rest?

a) $m_2 = 2m_1\mu_k$

b) $m_2 = m_1\mu_k/2$

c) $m_2 = m_1/\mu_s$

d) $m_2 = 2m_1\mu_s$

e) $m_2 = m_1\mu_s/2$

Problem 47. problems-1/force-mc-terminal-velocity-vsq-2bs.tex

Two spherical objects, both with mass m, are falling freely under the influence of gravity through air. The air exerts a drag force on the two spheres in the opposite direction to their motion with *magnitude* $F_1 = b_1 v_1^2$ and $F_2 = b_2 v_2^2$ respectively, with $\boldsymbol{b_2 = 2b_1}$.

Suppose the terminal speed for object 1 is v_t. Then the terminal speed of object 2 is:

a) $2v_t$

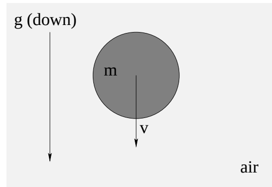
b) $\sqrt{2}v_t$

c) v_t

d) $\frac{\sqrt{2}}{2}v_t$

e) $v_t/2$

Problem 48. problems-1/force-mc-terminal-velocity-vsq.tex



In the figure above, a spherical mass m is falling freely under the influence of gravity through air. The air exerts a ***turbulent/quadratic drag force*** on the sphere in the opposite direction to its motion with drag coefficient c. After a (long) time, the falling mass approaches a constant *terminal speed* v_t, where:

a) $v_t = \frac{F_d}{c}$

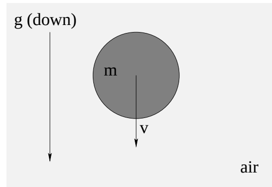
b) $v_t = \frac{mg}{c}$

c) $v_t = \left(\frac{mg}{c}\right)^2$

d) $v_t = \sqrt{\frac{mg}{c}}$

e) $v_t = \left(\frac{F_d}{m}\right) t$

Problem 49. problems-1/force-mc-terminal-velocity-vsq-soln.tex



In the figure above, a spherical mass m is falling freely under the influence of gravity through air. The air exerts a ***turbulent/quadratic drag force*** on the sphere in the opposite direction to its motion with drag coefficient c. After a (long) time, the falling mass approaches a constant *terminal speed* v_t, where:

Solution: Terminal velocity means *no acceleration*, which means in turn *no net force*:

$$F_{\text{tot}} = mg - cv_t^2 = ma = 0$$

or

$$cv_t^2 = mg$$

or

$$v_t = \sqrt{\frac{mg}{c}}$$

Hence:

a) $v_t = \frac{F_d}{c}$

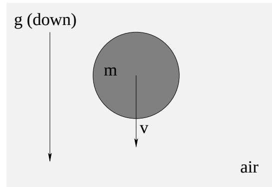
b) $v_t = \frac{mg}{c}$

c) $v_t = \left(\frac{mg}{c}\right)^2$

d) $v_t = \sqrt{\frac{mg}{c}}$

e). $v_t = \left(\frac{F_d}{m}\right) t$

Problem 50. problems-1/force-mc-terminal-velocity-v.tex



In the figure above, a spherical mass m is falling freely under the influence of gravity through air. The air exerts a drag force on the sphere in the opposite direction to its motion of *magnitude* $F_d = bv$ (where the *drag coefficient* b is determined by the shape of the object and its interaction with the air). After a (long) time, the falling mass approaches a constant *terminal speed* v_t, where:

a) $v_t = \frac{F_d}{b}$

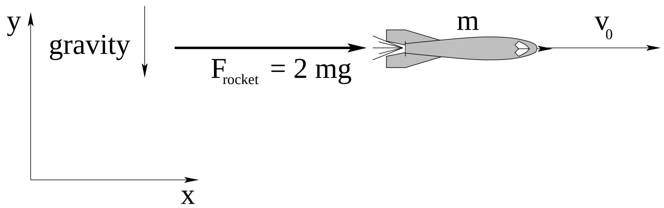
b) $v_t = \frac{mg}{b}$

c) $v_t = \left(\frac{mg}{b}\right)^2$

d) $v_t = \sqrt{\frac{mg}{b}}$

e) $v_t = \left(\frac{F_d}{m}\right) t$

Problem 51. problems-1/force-mc-two-constant-forces.tex



In the figure above, a rocket engine exerts a ***constant*** force $\vec{F} = 2mg\,\hat{x}$ to the right on a ***freely falling mass*** near the surface of the earth. The object is ***initially moving at velocity*** $\boldsymbol{v_0}$ ***to the right*** ($+x$ direction). No drag or frictional forces are present – consider only the two forces of gravity and the rocket engine. The object:

a) Moves in a straight line with an acceleration of magnitude $3g$.

b) Moves in a straight line with an acceleration of magnitude $\sqrt{5}g$.

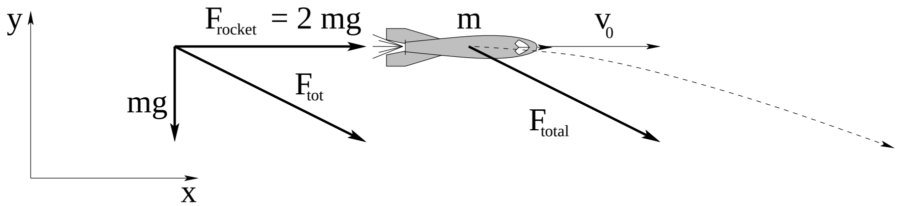
c) Moves in a parabolic trajectory with an acceleration of magnitude $3g$.

d) Moves in a parabolic trajectory with an acceleration of magnitude $\sqrt{5}g$.

e) We cannot determine the trajectory and/or the magnitude of the acceleration from the information given.

Sketch your best guess for the trajectory of the particle in on the figure above as a dashed line with an arrow.

Problem 52. problems-1/force-mc-two-constant-forces-soln.tex



Solution: The vector sum of the forces is:

$$\vec{\boldsymbol{F}}_{\rm tot} = 2mg\hat{\boldsymbol{x}} - mg\hat{\boldsymbol{y}} \quad \Rightarrow \quad \vec{\boldsymbol{a}} = (2\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}})g \quad \Rightarrow \quad \boxed{a = |\vec{\boldsymbol{a}}| = \sqrt{2^2 + 1^2)}g = \sqrt{5}g}$$

so the acceleration is down and to the right. It has a nonzero initial velocity component perpendicular to this direction so it moves in a parabolic, not a linear, trajectory:

a) Moves in a straight line with an acceleration of magnitude $3g$.

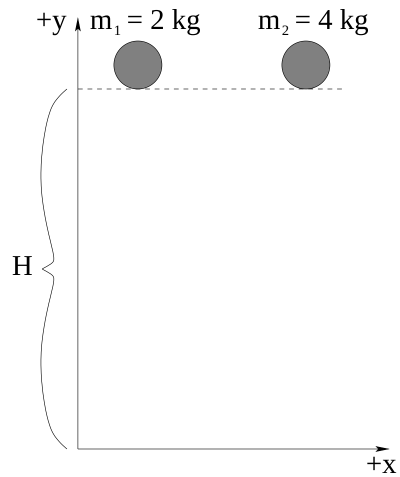
b) Moves in a straight line with an acceleration of magnitude $\sqrt{5}g$.

c) Moves in a parabolic trajectory with an acceleration of magnitude $3g$.

d) Moves in a parabolic trajectory with an acceleration of magnitude $\sqrt{5}g$.

e). We cannot determine the trajectory and/or the magnitude of the acceleration from the information given.

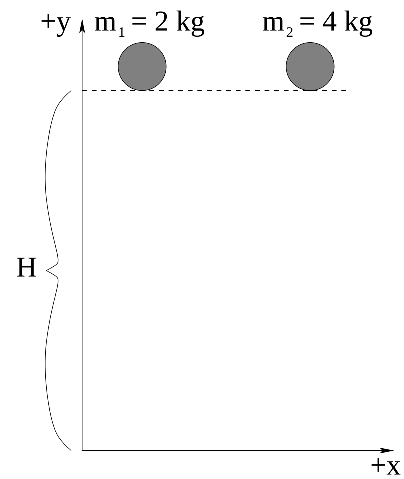
Problem 53. problems-1/force-mc-two-masses-falling-drag-icp.tex



A mass $m_1 = 2$ kg and a mass of $m_2 = 4$ kg have identical size, shape, and surface characteristics, and are both dropped from rest from the same height $H \approx 50$ meters at the same time. ***Air resistance (drag force) is present!*** Place a T/F in each box below as required:

- ☐ Initially, the acceleration of both masses is the same.
- ☐ The 2 kg mass hits the ground first.
- ☐ The 4 kg mass hits the ground first.
- ☐ Both masses hit the ground at the same time.
- ☐ Just before they hit, the acceleration of the heavier mass is greater.

Problem 54. problems-1/force-mc-two-masses-falling-drag-icp-soln.tex



Suppose that $a = g - cv^2/m$ (down) for either mass (quadratic drag). The answer will not depend on this – you'll get the same answer for linear drag.

Both masses initially accelerate with $a = g$ when $v = 0$, but *after* they've fallen a short distance, the larger mass will ***always*** have the larger acceleration down, although after a very long time both accelerations will approach zero as terminal velocity is reached.

Note that the *terminal* velocity of the lighter mass will always be smaller than the terminal velocity of the heavier one:

$$\text{quadratic: } v_t = \sqrt{\frac{mg}{c}} \quad \text{or} \quad \text{linear: } v_t = \frac{mg}{b}$$

Either way, the heavier mass will consistently move faster than the lighter mass, and hence will reach the ground first.

It's good to learn to use intuition, rather than computation, to answer questions like this – it's much faster! With masses of 2 kg and 4 kg that are relatively close together, the answer may not be obvious. But what if one ball was coated styrofoam and had a mass of 0.1 kg and the other was coated tungsten and had a mass of 100 kg? The first would *float* slowly down, almost instantly reaching terminal velocity. The other would fall like a proverbial rock, almost minimally affected by drag over a very long distance. Sometimes it helps to think about extreme cases when looking at problems like this that are hard to resolve when things are close.

$\boxed{\text{T}}$ Initially, the acceleration of both masses is the same.

$\boxed{\text{F}}$ The 2 kg mass hits the ground first.

$\boxed{\text{T}}$ The 4 kg mass hits the ground first.

$\boxed{\text{F}}$ Both masses hit the ground at the same time.

$\boxed{\text{T}}$ Just before they hit, the acceleration of the heavier mass is greater.

Problem 55. problems-1/force-mc-inertial-reference-frames.tex

Newton's Second Law states that $\vec{F}_{\rm tot} = m\vec{a}$ where $\vec{F}_{\rm tot}$ is the total force exerted on a given mass m by *actual forces* of nature or force rules that idealize actual forces of nature (such as Hooke's Law, normal forces, tension in a string), but only if one defines $\vec{a}$ in an **inertial reference frame**.

Select the ***best*** (most complete and accurate) explanation for the inertial reference frame requirement below:

- ☐ It is too difficult to solve for the acceleration of a mass in a non-inertial reference frame.
- ☐ In non-inertial reference frames, the sum of the actual forces acting on a mass is no longer equal to the mass times its acceleration in the frame.
- ☐ The Earth's surface is "the" reference inertial reference frame; we use it as the basis for physics in all other frames moving at constant velocity relative to the Earth.
- ☐ Inertial reference frames allow one to use pseudoforces when forces alone are not enough.
- ☐ Because the inertia/mass of an object cannot be measured in a non-inertial reference frame, Newton's Second Law doesn't hold there.

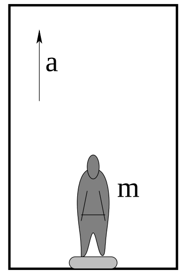
Problem 56. problems-1/force-mc-inertial-reference-frames-soln.tex

Newton's Second Law states that $\vec{\boldsymbol{F}}_{\rm tot} = m\vec{\boldsymbol{a}}$ where $\vec{\boldsymbol{F}}_{\rm tot}$ is the total force exerted on a given mass m by *actual forces* of nature or force rules that idealize actual forces of nature (such as Hooke's Law, normal forces, tension in a string), but only if one defines $\vec{\boldsymbol{a}}$ in an **inertial reference frame**.

Select the ***best*** (most complete and accurate) explanation for the inertial reference frame requirement below:

- ☐ It is too difficult to solve for the acceleration of a mass in a non-inertial reference frame.
- ☒ In non-inertial reference frames, the sum of the actual forces acting on a mass is no longer equal to the mass times its acceleration in the frame.
- ☐ The Earth's surface is "the" reference inertial reference frame; we use it as the basis for physics in all other frames moving at constant velocity relative to the Earth.
- ☐ Inertial reference frames allow one to use pseudoforces when forces alone are not enough.
- ☐ Because the inertia/mass of an object cannot be measured in a non-inertial reference frame, Newton's Second Law doesn't hold there.

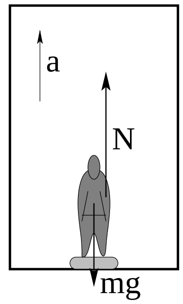
Problem 57. problems-1/force-mc-weight-in-elevator-icp.tex



In the figure above, a person of mass m is standing on a scale in an elevator (near the Earth's surface) that is accelerating upwards with acceleration a. What does the scale read?

- ☐ mg.
- ☐ ma.
- ☐ $m(g + a)$.
- ☐ $m(g - a)$.
- ☐ We cannot tell what the scale would show without more information.

Problem 58. problems-1/force-mc-weight-in-elevator-icp-soln.tex



From the force diagram and the given information, we know that:

$$F_{\text{tot}} = N - mg = ma \quad \text{(up)}$$

Scales measure not weight but the normal force N. Solving for N,

$$N = mg + ma = m(g + a) = mg'$$

Hence:

☐ mg. ☐ ma. ☒ $m(g + a)$. ☐ $m(g - a)$.

☐ We cannot tell what the scale would show without more information.

Problem 59. problems-1/force-mc-coriolis-dropped-mass-at-equator.tex

A dense mass m is dropped "from rest" from a high tower built at the equator. As the mass falls, it to a person standing on the ground appears to be deflected as it falls to the:

a) East.

b) West.

c) North.

d) South.

e) Cannot tell from the information given.

Problem 60. problems-1/force-mc-coriolis.tex

The Earth is a rotating sphere, and hence is not really an inertial reference frame. Select the true answers from the following list for the ***apparent*** behavior of e.g. naval projectiles or freely falling objects:

a) A naval projectile fired due North in the northern hemisphere will be (apparently) deflected East (spinward).

b) A naval projectile fired due South in the northern hemisphere will be (apparently) deflected East (spinward).

c) A bomb dropped from a helicopter hovering over a fixed point on the surface in the northern hemisphere will be (apparently) deflected West (antispinward).

d) A bomb dropped from a helicopter hovering over a fixed point on the surface in the northern hemisphere will be (apparently) deflected East (spinward).

e) An object placed at (apparent) "rest" on the surface of the Earth in the Northern hemisphere experiences an (apparent) force to the North.

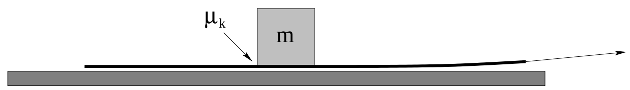
f) An object placed at (apparent) "rest" on the surface of the Earth in the Northern hemisphere experiences an (apparent) force to the South.

g) The true weight of an object measured with a spring balance in a laboratory on the equator is a bit larger than the measured weight.

h) The true weight of an object measured with a spring balance in a laboratory on the equator is a bit smaller than the measured weight.

4.2.2 Short Answer Problems

Problem 61. problems-1/force-sa-block-on-paper.tex

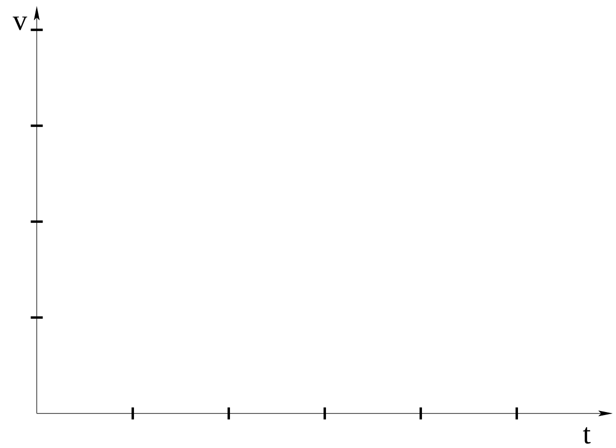


A block of mass m is resting on a long piece of smooth paper. The block has a coefficient of kinetic friction μ_k with the paper. You pull the paper horizontally out from under the block quickly in the direction indicated by the arrow.

a) Draw the ***direction*** of the frictional force acting on the ***block***.

b) What is the ***magnitude and direction*** of the acceleration of the block?

Problem 62. problems-1/force-sa-free-fall-cliff-icp.tex

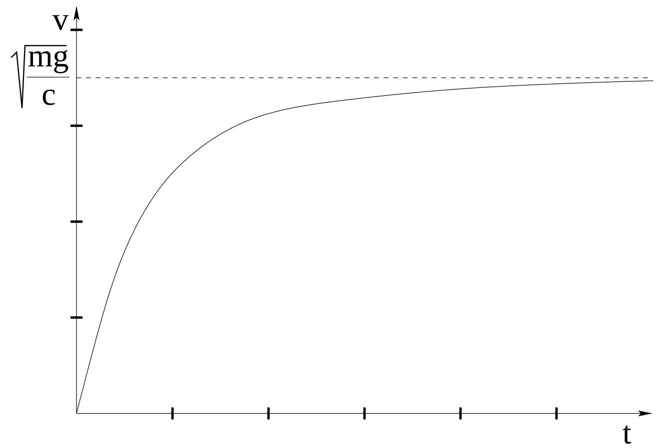


A ball of mass m is dropped from rest over the edge of a very tall (kilometer high) cliff. It experiences a drag force opposite to its velocity of $F_d = -cv^2$ where c is the quadratic drag coefficient.

a) On the axes above, *qualitatively* plot its downward ***speed*** as function of time.

b) What is its approximate speed when it hits after falling a ***long*** time/distance?

Problem 63. problems-1/force-sa-free-fall-cliff-icp-soln.tex



Note that the terminal velocity (down) is given by:

$$F = mg - cv_t^2 = 0 \quad \text{or} \quad \boxed{v_t = \sqrt{\frac{mg}{c}}}$$

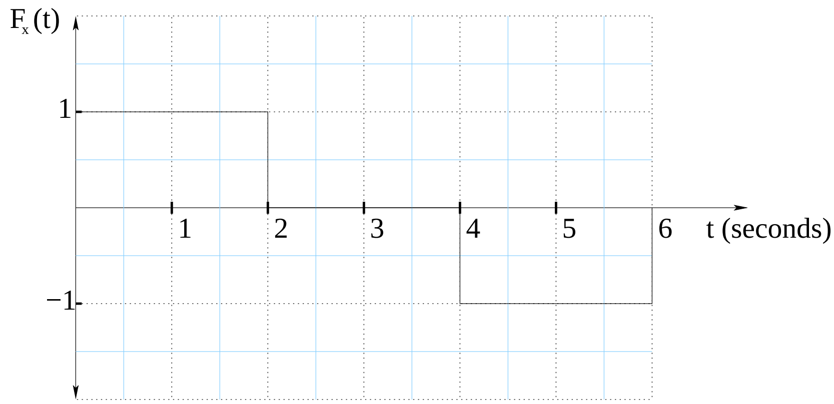
This determines the dashed asymptote at the top and is the answer to part b)

We know that right after it is released, the drag force is zero, so the slope of $v(t)$ must be g (again, positive down). In between, $v(t)$ has to smoothly start with slope g and bend over to approach v_t. In the textbook it is shown that the actual function is:

$$v(t) = \sqrt{\frac{mg}{c}} \tanh\left(\sqrt{\frac{gc}{m}} t\right)$$

(which is what is actually plotted above) but you **don't need to know this**, or know what a hyperbolic tangent function looks like, to get the ***qualitatively*** correct shape, asymptotically approaching the easily found terminal velocity after initially falling with an acceleration (slope of $v(t)$!) of g.

Problem 64. problems-1/force-sa-kinematic-graphs-1-icp.tex



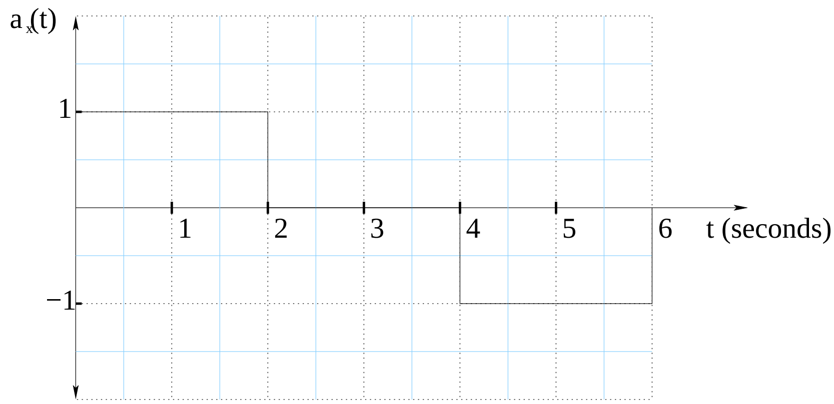
The graph above represents the force in the positive x direction $F_x(t)$ applied to a mass $m = 1$ kg as a function of time in seconds. The mass begins at rest at $x = 0$. The force F is given in Newtons, the position x is given in meters.

a) What is the acceleration of the mass during the time interval from $t = 0$ to $t = 6$ seconds (sketch a curve)?

b) How fast is the mass going at the end of 6 seconds?

c) How far has the mass travelled at the end of 6 seconds?

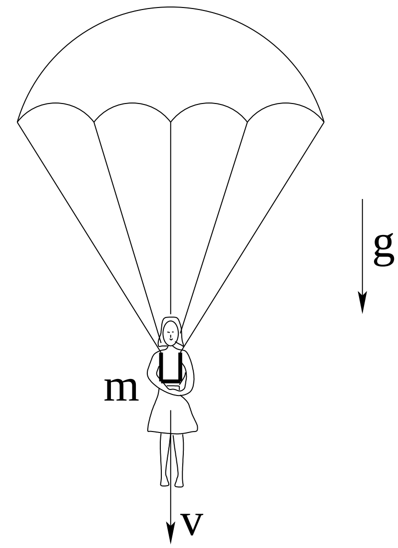
Problem 65. problems-1/force-sa-kinematic-graphs-1-icp-soln.tex



Since m is 1 kg, $a_x = F_x/m$ has the same graph, just different units on the vertical scale. Since it starts at rest, the speed in the positive x direction is just **the area under the $\boldsymbol{a_x}$ curve**. At the end of six seconds, then, it is at rest again, as the positive and negative areas (visibly) cancel. The distance travelled is more difficult, but is $x(6) = 2 + 4 + 2 = 8$ meters, where each number represents a two second integral of the velocity/speed in the positive x-direction.

Problem 66. problems-1/force-sa-parachutist.tex

A skydiver of mass m jumps from a helicopter and immediately opens her parachute (so that her initial downward speed with the parachute open is basically zero.) The parachute exerts a ***quadratic*** drag force (proportional to v^2) with a drag coefficient b.

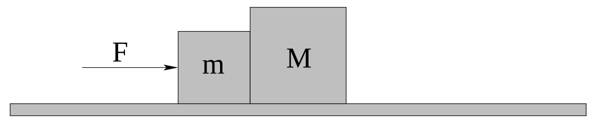


a) Draw a free body diagram showing the forces acting on the skydiver a short time later when her downward speed is v. Write down an expression for the *magnitude* of her acceleration at this speed.

b) Find her terminal (asymptotic) speed as she falls over a very long distance.

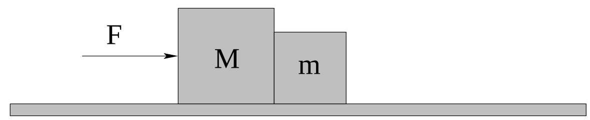
c) If her terminal speed needs to be reduced by a factor of 2 for her to land safely, by what factor must b be increased?

Problem 67. problems-1/force-sa-pushing-two-blocks-2.tex



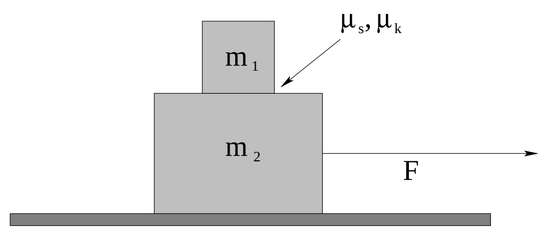
The figure shows two blocks of mass M and m that are being pushed along a horizontal frictionless surface by a force of magnitude F as shown. What is the ***magnitude*** of the (contact/normal) force that the block of mass m exerts on the block of mass M?

Problem 68. problems-1/force-sa-pushing-two-blocks.tex



The figure shows two blocks of mass M and m that are being pushed along a horizontal frictionless surface by a force of magnitude F as shown. What is the magnitude of the (contact/normal) force that the block of mass M exerts on the block of mass m?

Problem 69. problems-1/force-sa-two-blocks-friction.tex

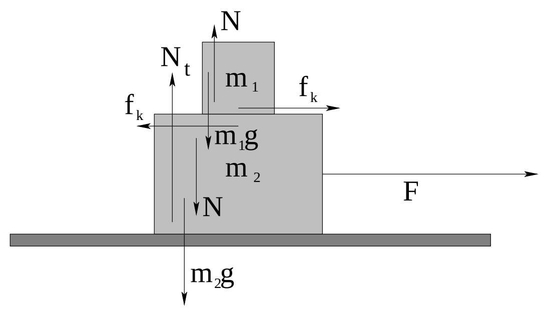


A block of mass m_1 is placed on a larger block of mass $m_2 > m_1$, where there is a coefficient of static friction $\mu_s = 0.25$ and a coefficient of kinetic friction $\mu_k = 0.2$ for the surface in contact between the blocks. Both blocks are on a frictionless table. A force of magnitude $F = 3(m_1 + m_2)g$ is applied to the ***bottom*** block only.

a) Is the ***magnitude*** of the acceleration of the lower block greater than, less than, or equal to the magnitude of the accleration of the upper block?

b) Find the acceleration of the ***top block only*** (magnitude and direction).

Problem 70. problems-1/force-sa-two-blocks-friction-soln.tex



a) Solving this problem requires insight and careful reasoning.

Suppose that the two blocks "moved as one". Then the acceleration of the system would be:

$$a = \frac{F}{(m_1 + m_2)} = \frac{3(m_1 + m_2)g}{(m_1 + m_2)} = 3g$$

That means that the force of static friction acting to the right on the upper block would have to be $f_{s1} = 3m_1 g > \mu_s m_1 g = f_s^{\max}$. This exceeds the maximum static friction force one can exert on the upper block, so that the upper block slides and they do ***not*** move as one.

Be careful: the upper block slides backwards ***relative to the lower block***, but they both accelerate *forward* (to the right) with the acceleration of the lower block being (as we shall see) much greater than the acceleration of the upper one.

b) This (preliminary) reasoning leads us to the force diagram above, which results in two independent solutions, one for m_1 and one for m_2. We observe nothing interesting in the y-direction except that $N = m_1 g$ and $N_t = (m_1 + m_2)g$ as expected.

This suffices to tell us the magnitude of the force of *kinetic* friction between the two blocks:

$$f_k = \mu_k N = \mu_k m_1 g$$

which acts (as drawn) to the right on block m_1, to the left on block m_2:

$$F - f_k = F - \mu_k m_1 g = m_2 a_2$$

$$f_k = \mu_k m_1 g = m_1 a_1$$

where $a_1 \neq a_2$, or:

$$a_2 = \frac{F - \mu_k m_1 g}{m_2} = \frac{3m_2 g + (3 - \mu_k)m_1 g}{m_2}$$

(to the right) and:

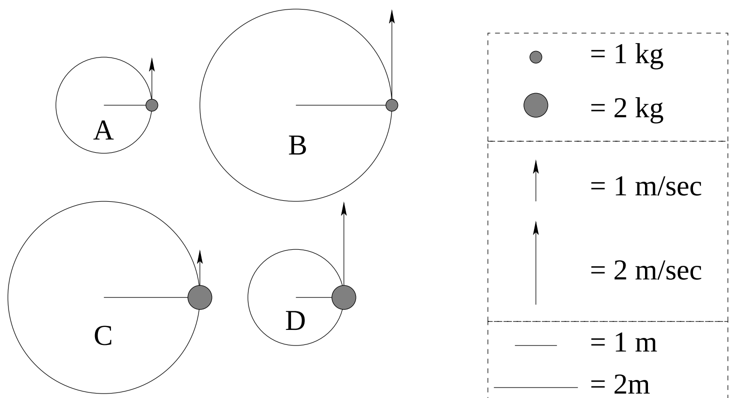
$$a_1 = \mu_k g = 0.2g \approx 2 \text{ m/sec}^2$$

(also to the right).

As promised, $a_2 \gg a_1$.

4.2.3 Ranking Problems

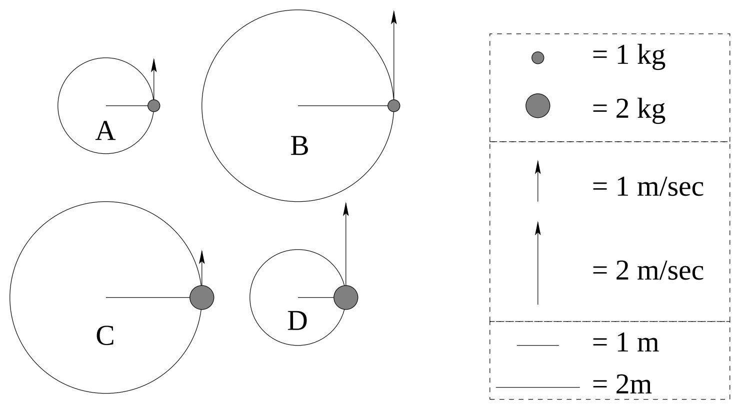
Problem 71. problems-1/force-ra-circular-motion-tension.tex



In the four figures above, you are looking down on a mass sitting on a frictionless table being whirled on the end of a string. The mass, length of string, and speed of the mass in each figure are indicated in the key on the right.

Rank the tension in the string in each of the four figures above, from ***lowest to highest***. Equality is a possibility. An example of a possible answer is thus: $A < B = C < D$.

Problem 72. problems-1/force-ra-circular-motion-tension-soln.tex



In the four figures above, you are looking down on a mass sitting on a frictionless table being whirled on the end of a string. The mass, length of string, and speed of the mass in each figure are indicated in the key on the right.

Rank the tension in the string in each of the four figures above, from ***lowest to highest***. Equality is a possibility. An example of a possible answer is thus: $A < B = C < D$.

Solution: In each figure, Newton's Second Law (N2) is just

$$T = ma_c = \frac{mv^2}{r}$$

(T pulling towards the center of the circle, of course, with a_c the centripetal acceleration.) We note that m, v, and r all vary by a factor of 2, somewhat irregularly. Let's let:

$$m_A = m = 1 \text{ kg} \qquad v_A = v = 1 \text{ m/sec} \qquad r_A = r = 1 \text{ m} \quad \Rightarrow] \quad T_A = \frac{mv^2}{r}$$

Then we can just use ***scaling***:

$$T_B = \frac{m(2v)^2}{2r} = 2T_A$$

$$T_C = \frac{(2m)v^2}{2r} = T_A$$

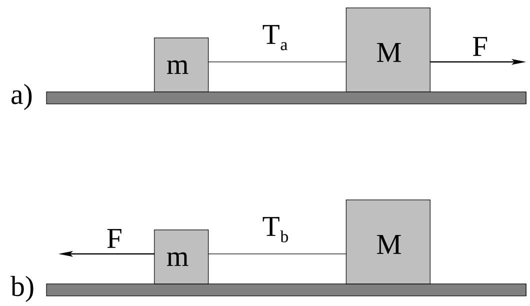
$$T_D = \frac{(2m)(2v)^2}{r} = 8T_A$$

so that:

$$\boxed{A = C < B < D} \quad \text{or} \quad \boxed{T_A = T_C < T_B < T_D}$$

as you prefer.

Problem 73. problems-1/force-ra-tension-between-two-unequal-blocks-friction.tex



In the figure above a block of mass m is connected to a block of mass $M > m$ by a string. Both blocks sit on a smooth surface with a coefficient of kinetic friction μ_k between either block and the surface. In figure **a)**, a force of magnitude F (large enough to cause both blocks to slide) is exerted on block M to pull the system to the **right**. In figure **b)**, a force of (the same) magnitude F is exerted on block m to pull the system to the **left**.

Circle the true statement:

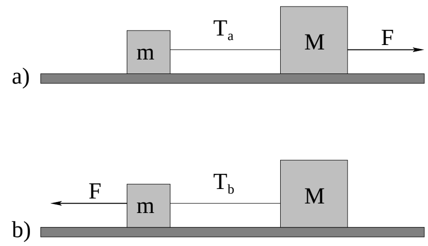
a) The tension $T_a > T_b$.

b) The tension $T_a < T_b$.

c) The tension $T_a = T_b$.

d) There is not enough information to determine the relative tension in the two cases.

Problem 74. problems-1/force-ra-tension-between-two-unequal-blocks-icp.tex



In the figure above a block of mass m is connected to a block of mass $M > m$ by a string. Both blocks sit on a frictionless floor. In a), a force of magnitude F is exerted on block M to pull the system to the right. In b), a force of (the same) magnitude F is exerted on block m to pull the system to the left. Circle the true statement:

a) The tension $T_a > T_b$.

b) The tension $T_a < T_b$.

c) The tension $T_a = T_b$.

d) There is not enough information to determine the relative tension in the two cases.

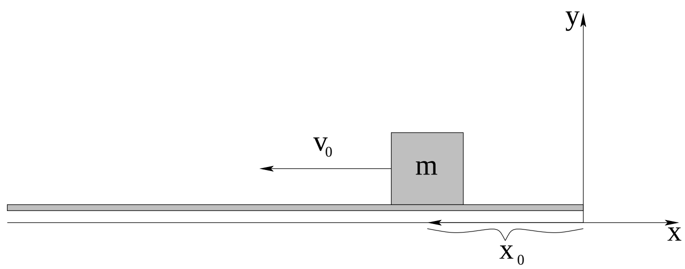
Problem 75. problems-1/force-ra-tension-between-two-unequal-blocks-icp-soln.tex

In both cases the acceleration is the same. The force on the *trailing* mass is always that mass times the acceleration, and is exerted only by the string. Hence:

b) $T_a < T_b$.

4.2.4 Regular Problems

Problem 76. problems-1/force-pr-a-constant-simplest-icp.tex



A block of mass m sits on a horizontal frictionless table as shown. A constant force $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{x}}$ in the $+x$-direction (to the right) is applied to it. The mass is initially moving to the left with speed v_0, and starts a the position x_0 as shown.

a) Draw a ***force diagram*** for the mass m onto the figure above. This should include ***all*** the forces, including those that cancel.

b) Write down an expression for the acceleration $\vec{\boldsymbol{a}}$ of the mass.

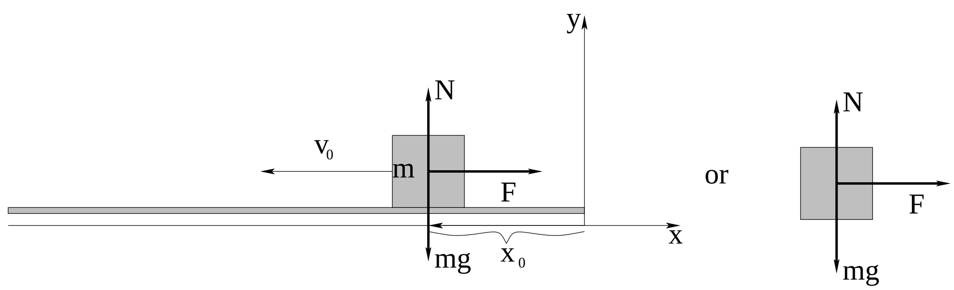
c) ***Integrate*** the acceleration one time to find $\vec{\boldsymbol{v}}(t)$.

d) ***Integrate*** the velocity one time to find $\vec{\boldsymbol{x}}(t)$.

e) How long will it take to bring the particle to rest (where infinity is a possible answer)?

f) Where will it be when it comes to rest?

Problem 77. problems-1/force-pr-a-constant-simplest-icp-soln.tex



A block of mass m sits on a horizontal frictionless table as shown. A constant force $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{x}}$ in the $+x$-direction (to the right) is applied to it. The mass is initially moving to the left with speed v_0, and starts a the position x_0 as shown.

a) Draw a ***force diagram*** for the mass m onto the figure above. This should include ***all*** the forces, including those that cancel.

b) Write down an expression for the acceleration $\vec{\boldsymbol{a}}$ of the mass.

c) ***Integrate*** the acceleration one time to find $\vec{\boldsymbol{v}}(t)$.

d) ***Integrate*** the velocity one time to find $\vec{\boldsymbol{x}}(t)$.

e) How long will it take to bring the particle to rest (where infinity is a possible answer)?

f) Where will it be when it comes to rest?

For a), see above.

For b) in the y-direction:

$$\sum_y F_y = N - mg = ma_y = m0 = 0$$

$$a_y = 0$$

and nothing interesting happens in the y-direction. The block doesn't jump into the air or fall through the solid table! In the x-direction:

$$\sum_x F_x = F = ma_x$$

$$a_x = F/m$$

Acceleration is a ***vector*** so we must specify its ***magnitude and direction*** in some way or some coordinate frame. Any of the following are acceptable ways:

$$\vec{\boldsymbol{a}} = F/m\hat{\boldsymbol{x}} = (F/m, 0) = (|\vec{\boldsymbol{a}}| = F/m, \theta_{\vec{\boldsymbol{a}}} = 0)(\text{polar}) = F/m \text{ “to the right”}$$

or just $a_x = F/m, a_y = 0$ as obtained above.

For part c), $a_y = 0 = dv_y/dt$ so $v_y =$ a constant. But its initial y-velocity is zero, so $v_y = 0 = dy/dt$. Thus y is a constant. But the initial value for y is 0 in a reasonable coordinate system and in any event it does not change, so we'll choose coordinates where boring old $y = 0$ throughout.

x is more interesting:

$$\begin{aligned} a_x = \frac{F}{m} &= \frac{dv_x}{dt} \\ dv_x &= \left(\frac{F}{m}\right) dt \\ v_x = \int dv_y &= \left(\frac{F}{m}\right) \int dt = \left(\frac{F}{m}\right) t + v_{0x} \end{aligned}$$

hence (using the ***given*** fact that $v_x(0) = -v_0$):

$$v_x = \left(\frac{F}{m}\right) t - v_0 = \frac{dx}{dt}$$

For part d) we now repeat this process for dx:

$$x = \int dx = \int \left\{ \left(\frac{F}{m}\right) t - v_0 \right\} dt = \frac{1}{2}\left(\frac{F}{m}\right) t^2 - v_0 t - x_0$$

where we used the ***given*** fact that $x(0) = -x_0$ in the coordinate system shown.

Again, we could express both of these ***vector*** answers in any acceptable way. I'll use cartesian coordinates:

$$\begin{aligned} \vec{v}(t) &= \left\{ \left(\frac{F}{m}\right) t - v_0] \right\} \hat{x} \\ x(t) &= \left\{ \frac{1}{2}\left(\frac{F}{m}\right) t^2 - v_0 t - x_0 \right\} \hat{x} \end{aligned}$$

For part e) we try to solve:

$$v_x(t_s) = \left(\frac{F}{m}\right) t_s - v_0 = 0$$

for the particular time t_s that the block's velocity will equal zero. We get:

$$t_s = \frac{m v_0}{F}$$

At that time its y-coordinate will remain zero (of course) but its x-coordinate will be:

$$\begin{aligned} x(t_s) &= \frac{1}{2}\left(\frac{F}{m}\right) t_s^2 - v_0 t_s - x_0 \\ &= \frac{1}{2}\left(\frac{F}{m}\right) \left(\frac{m v_0}{F}\right)^2 - v_0 \left(\frac{m v_0}{F}\right) - x_0 \\ &= -\frac{1}{2}\left(\frac{m v_0^2}{F}\right) - x_0 \end{aligned}$$

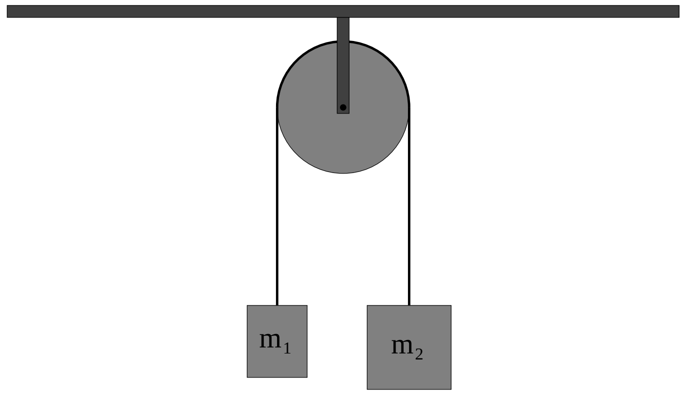
Note well! This last result can be made familiar to us by noting that the acceleration is constant, so that $v_f^2 - v_0^2 = 2a_x\Delta x$, with $a_x = F/m$. Hence:

$$x_f - (-x_0) = \Delta x = -v_0^2/(2F/m) = -\frac{1}{2}\left(\frac{mv_0^2}{F}\right)$$

and $x_f = x(t_s)$ as expected. There is yet another way to do it using work and energy.

This example solution has been worked in *more* detail than would usually be required on a problem, but I would still recommend that you start out working homework and additional examples from this guide at exactly this level. After a bit some of the steps will be so obvious and easy and boring (like the discussion of the *nothing interesting* that happens in y above) that you can safely omit them or cover them with a phrase like "$a_y = 0$ so y is unchanged", but wait for that to happen and pay attention to the stated requirements of your class's particular grader(s).

Problem 78. problems-1/force-pr-atwoods-machine-icp.tex



In the figure above Atwood's machine is drawn – two masses m_1 and m_2 hanging over a massless frictionless pulley, connected by a massless unstretchable string.

a) Draw free body diagrams (isolated diagrams for each object showing just the forces acting on that object) for the two masses in the figure above.

b) Convert each free body diagram into a statement of Newton's Second Law for that object.

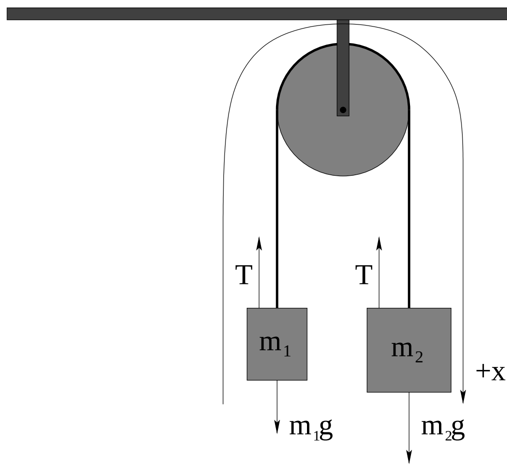
c) Find the acceleration of the system and the tensions in the string on *both* sides of the pulley in terms of m_1, m_2, and g.

d) Suppose mass $m_2 > m_1$ and the system is released from rest with the masses at equal heights. When mass m_2 has descended a distance H, find the speed of the masses.

Problem 79. problems-1/force-pr-atwoods-machine-icp-soln.tex

In the figure above Atwood's machine is drawn – two masses m_1 and m_2 hanging over a massless frictionless pulley, connected by a massless unstretchable string.

a) I didn't isolate the two objects because a force diagram like this is also acceptable. The advantage of the diagram the way I draw it is that it is also used it to specify my "around the corner" coordinate system in which the accelerations of the two masses are the same including the same sign.



b) Using this coordinate frame (so positive is up on the left and down on the right): $T - m_1 g = m_1 a$

$m_2 g - T = m_2 a$

c) Add these two equations to eliminate T, solve for a, and back-substitute in either equation to solve for T. We will do this a ***lot*** this semester, so **practice this!**

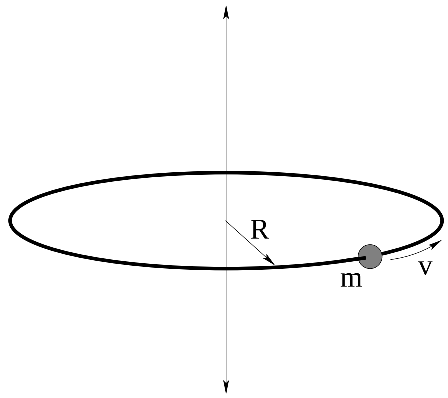
$$a = \frac{(m_2 - m_1)g}{(m_1 + m_2)}$$

$$T = m_1(a + g) = \frac{(2m_1 m_2)g}{m_1 + m_2}$$

d) Lots of ways to get this, soon we'll have even more:

$$v = \sqrt{\frac{2H(m_2 - m_1)g}{(m_1 + m_2)}}$$

Problem 80. problems-1/force-pr-bead-on-hoop.tex



A bead of mass m is threaded on a metal hoop of radius R. There is a coefficient of kinetic friction μ_k between the bead and the hoop. It is given a push to start it sliding around the hoop with initial speed v_0. The hoop is located on the space station, so you can ignore gravity.

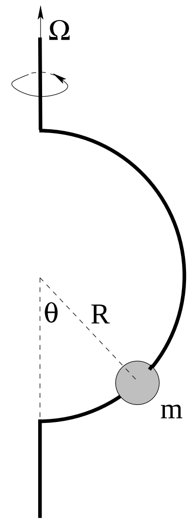
a) Find the normal force exerted by the hoop on the bead as a function of its speed.

b) Find the dynamical frictional force exerted by the hoop on the bead as a function of its speed.

c) Find its speed as a function of time. This involves using the frictional force on the bead in Newton's second law, finding its *tangential* acceleration on the hoop (which is the time rate of change of its speed) and solving the equation of motion.

All answers should be given in terms of m, μ_k, R, v (where requested) and v_0.

Problem 81. problems-1/force-pr-bead-on-semicircular-hoop.tex

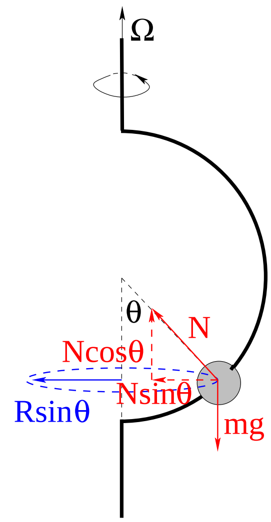


A small frictionless bead is threaded on a semicircular wire hoop with radius R, which is then spun on its vertical axis as shown above at angular velocity Ω.

a) Find the angle θ where the bead will remain stationary relative to the rotating wire as a function of R, g, and Ω.

b) From your answer to the previous part, it should be apparent that there is a ***minimum*** angular velocity Ω_{min} that the hoop must have before the bead moves up from the bottom at all. What is it? (Hint: Think about where the previous answer has solutions.)

Problem 82. problems-1/force-pr-bead-on-semicircular-hoop-soln.tex



A small frictionless bead is threaded on a semicircular wire hoop with radius R, which is then spun on its vertical axis as shown above at angular velocity Ω.

a) Find the angle θ where the bead will remain stationary relative to the rotating wire as a function of R, g, and Ω.

b) From your answer to the previous part, it should be apparent that there is a ***minimum*** angular velocity $\Omega_{\rm min}$ that the hoop must have before the bead moves up from the bottom at all. What is it? (Hint: Think about where the previous answer has solutions.)

Solution: The bead has to be in a sort of *dynamical equilibrium*, moving in a circle of radius $R\sin\theta$ The only two forces acting are gravity (down) and the normal force between the bead and the wire (directed in towards the center of the semicircle, ***not*** the circle of motion). Finally, the tangential speed of the bead at this radius is $v_t = \Omega R\sin\theta$.

For a): Start by decorating the figure with forces (see above) decomposed into vertical (where the total force component must be zero) and center-directed (where the force component must equal ma_c):

$$\text{Vertical:} \quad N\cos\theta - mg = 0 \quad \Rightarrow \quad N\cos\theta = mg$$

$$\text{Central:} \quad N\sin\theta = m\frac{v_t^2}{R\sin\theta} = m\Omega^2 R\sin\theta$$

Rearrange and take the ratio of these two equations to eliminate N,$\sin\theta$ (and incidentally, m):

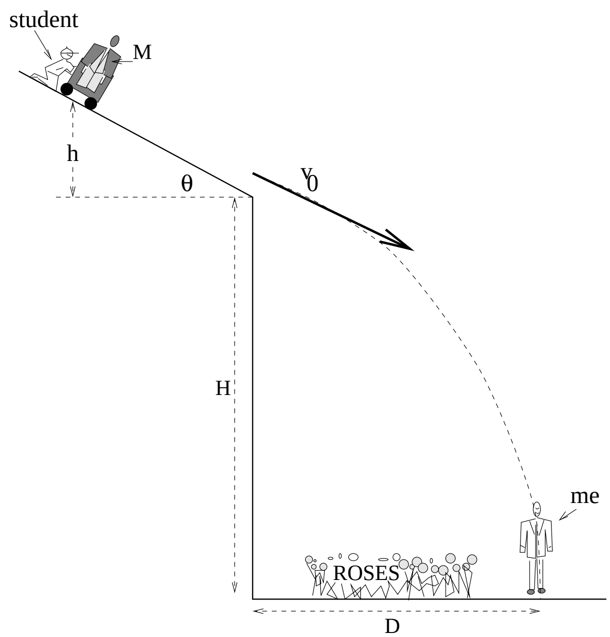
$$\frac{\cancel{N}\cancel{\sin\theta}}{\cancel{N}\cos\theta} = \frac{\cancel{m}\Omega^2 R\cancel{m}\cancel{\sin\theta}}{g} \quad \Rightarrow \quad \theta = \boxed{\cos^{-1}\left(\frac{g}{R\Omega^2}\right)}$$

For b): Note that the ***domain*** of $\cos^{-1}$ is $[-1, 1]$. This means that:

$$\frac{g}{R\Omega^2} \le 1 \quad \Rightarrow \quad \boxed{\Omega > \sqrt{\frac{g}{R}}}$$

in order for a stable circular “orbit” angle $\theta > 0$ to exist.

Problem 83. problems-1/force-pr-dropping-washington-duke-trajectory.tex



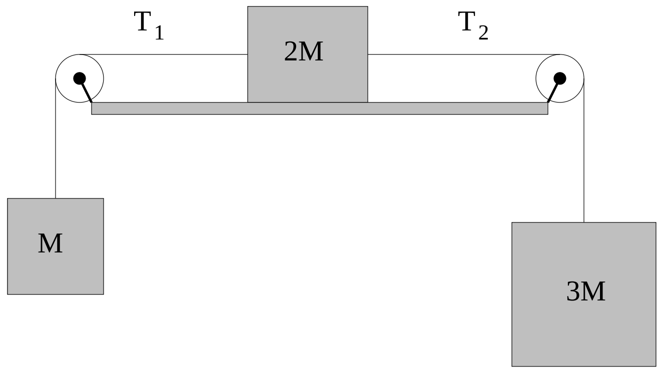
A physics student irritated by the personal mannerisms of their physics professor decides to rid the world of him. The student plans to drop a large, massive object (the statue of Washington Duke, actually, recently stolen by pranksters from his fraternity), mounted on nearly frictionless casters, from a tall building of height H with a smooth roof sloped at the angle θ as shown. However, the student (being a thoughtful sociopath) wants to make sure that the mass M will make it *over the roses* to the path a distance D from the base of the building and needs to know how far to let the statue roll down the roof to get the right speed.

Unfortunately, the student isn't very good at physics and comes to *you* for help. Since they don't want to tell you *which* building or *which* path they want to use (you might be able to testify against them!) they want you to find (in **two steps**, each counting as a separate problem) a *general formula* for the requisite distance.

a) Help them out. Start by finding v_0 in terms of H, M, D, θ and g (the gravitational constant) that will drop M on RGB assuming *no friction or drag forces.* (That way I'm still pretty safe).

b) Now that you know the speed (or rather, assuming that you know the speed, as the case may be) find h (the vertical distance the statue must roll down, released from rest, to come off with the right speed). Explicitly show that your overall answer (in which v_0 should NOT appear) has the right units. If you were clueless in problem 4) you may leave v_0 in your answer but should still try to find SOME combination of the letters H, M, D, θ and g that has the right units and varies the way you expect the answer to (more height H means *smaller* h, for example, so it probably belongs on the bottom).

Problem 84. problems-1/force-pr-flat-plane-three-blocks.tex



Three blocks of mass M, $2M$ and $3M$ are drawn above. The middle block ($2M$) sits on a frictionless table. The other two blocks are connected to it by massless unstretchable strings that run over massless frictionless pulleys. At time $t = 0$ the system is released from rest. Find:

a) The acceleration of the middle block sitting on the table.

b) The tensions T_1 and T_2 in the strings as indicated.

Problem 85. problems-1/force-pr-flat-plane-two-blocks.tex

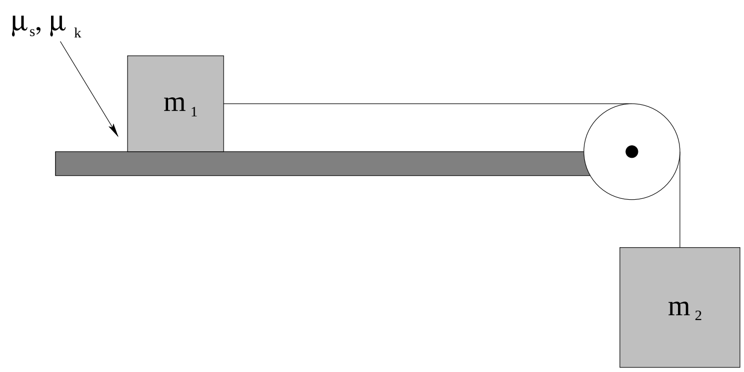
A mass m_1 is attached to a second mass m_2 by an Acme (massless, unstretchable) string. m_1 sits on a frictionless table; m_2 is hanging over the ends of a table, suspended by the taut string from an Acme (frictionless, massless) pulley. At time $t = 0$ both masses are released.

a) Draw the force/free body diagram for this problem.

b) Find the acceleration of the two masses.

c) How fast are the two blocks moving when mass m_2 has fallen a height H (assuming that m_1 hasn't yet hit the pulley)?

Problem 86. problems-1/force-pr-flat-plane-two-blocks-friction.tex



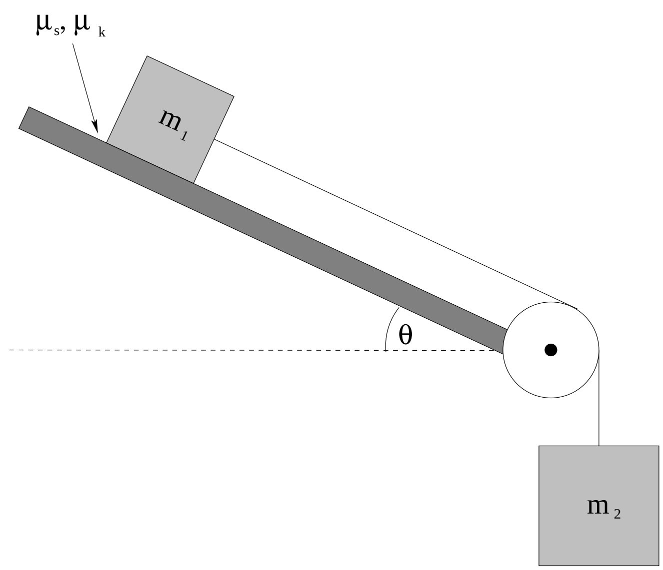
A mass m_1 is attached to a second mass m_2 by a massless, unstretchable string. m_1 sits on a rough table with coefficients of static and kinetic friction μ_s and μ_k respectively. m_2 is hanging over the end of the table, suspended by the taut string from a frictionless, massless pulley.

At time $t = 0$ both masses are released from rest. Answer the following two questions:

a) What is the minimum mass $m_{2,\text{min}}$ such that the two masses begin to move?

b) Suppose $m_2 > m_{2,\text{min}}$. Determine how fast the two blocks are moving when mass m_2 has fallen a height H (assuming that m_1 hasn't yet hit the pulley)?

Problem 87. problems-1/force-pr-inclined-atwoods-machine-downhill-friction-icp.tex



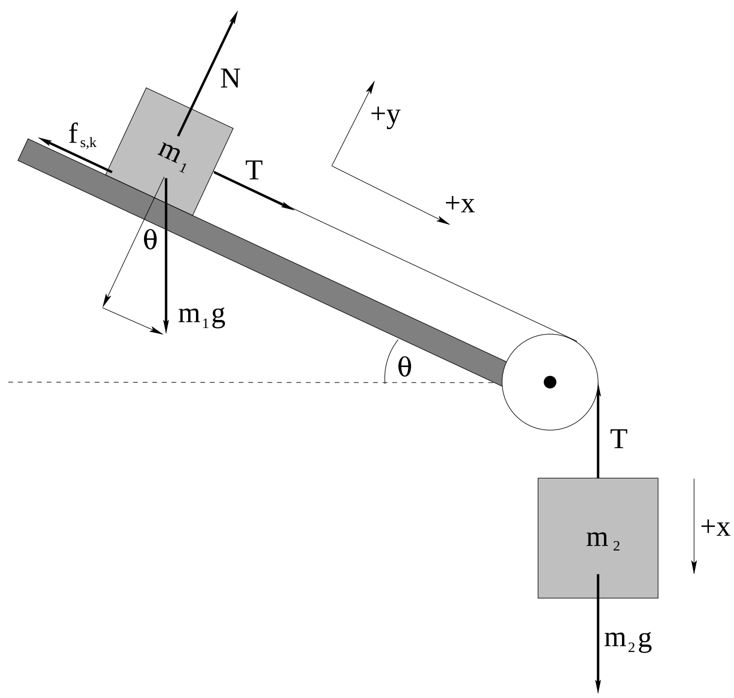
A block of mass m_1 sits on a plane inclined at the angle θ as shown. It is connected with a massless, unstretchable string running over a massless, frictionless pulley to m_2, which is hanging over a drop to the ground. The two masses are released initially from rest. The inclined plane has coefficients of static and kinetic friction with m_1 of μ_s and μ_k respectively.

a) Draw separate free-body diagrams for each mass m_1 and m_2, and select (indicate on your figure) an appropriate coordinate system for each diagram;

b) Find the minimum mass $m_{2,\mathrm{min}}$ such that the two masses begin to move;

c) If $m_2 > m_{2,\mathrm{min}}$ (so that the block definitely slides), determine the magnitude of the acceleration of the blocks.

Problem 88. problems-1/force-pr-inclined-atwoods-machine-downhill-friction-icp-soln.tex



a) Draw separate free-body diagrams for each mass m_1 and m_2, and select (indicate on your figure) an appropriate coordinate system for each diagram;

See above. It is also correct (and many textbooks advise or require it) to draw two separate figures, one for each mass, separate from the provided diagram, and indicate the forces on those, but as long as the result is clear (as it is on the figure above) it really doesn't matter.

b) Find the minimum mass $m_{2,\text{min}}$ such that the two masses begin to move;

After decomposing all forces in the coordinate system selected (in the case above only $m_1 g$ needs it), writing Newton's Second Law for each coordinate direction for each mass, using $f_{s,\text{max}} = \mu_s N$ in a force balance expression (because the maximum uphill static friction will balance the larges mass m_2 that **doesn't** quite move, and solving for m_2 we get:

$$m_{2,\text{min}} = m_1(\mu_s \cos\theta - \sin\theta)$$

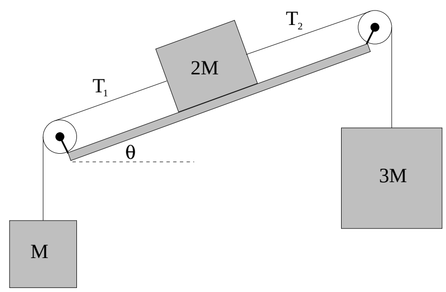
c) If $m_2 > m_{2,\text{min}}$ (so that the block definitely slides), determine the magnitude of the acceleration of the blocks.

Now we change to $f_k = \mu_k N$ and solve the two x-equations simultaneously to get:

$$a_x = \frac{m_1 g(\sin\theta - \mu_k \cos\theta) + m_2 g}{m_1 + m_2}$$

Note that this question could easily have asked for the tension T as well as or instead of a_x. How would you get it?

Problem 89. problems-1/force-pr-inclined-plane-three-blocks.tex



Three blocks of mass M, $2M$ and $3M$ are drawn above. The middle block ($2M$) sits on a frictionless table tipped at an angle θ with the horizontal as shown. The other two blocks are connected to it by massless unstretchable strings that run over massless frictionless pulleys. At time $t = 0$ the system is released from rest. Find:

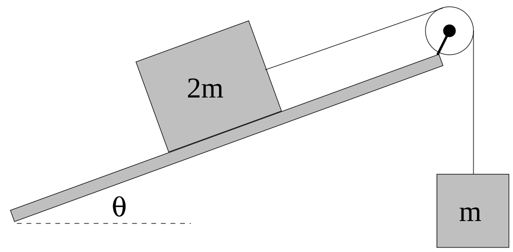
a) The magnitude of the acceleration of the middle block sitting on the table.

b) The tensions T_1 and T_2 in the strings as indicated.

c) Suppose $\theta = 30^\circ$. Which way will the system of blocks accelerate?

☐ Down on the right ☐ Down on the left ☐ They won't move.

Problem 90. problems-1/force-pr-inclined-plane-two-blocks-30deg-icp.tex



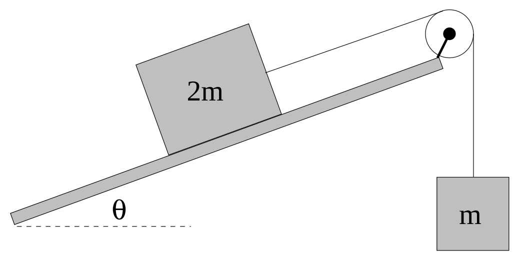
A block of mass $2m$ sits on a frictionless incline held at an angle θ relative to the horizontal as shown in the figure above. It is connected by a massless, unstretchable string that runs over a frictionless, massless pulley to a block m hanging over a drop. The two blocks are initially ***at rest***.

a) For what angle θ_0 will this system will be in ***force balance*** (and hence remain stationary).

b) If the incline is lifted from this angle to a new (given) angle $\theta > \theta_0$, what is the subsequent direction of motion for both blocks? Indicate the direction on the figure above for each block.

c) At this angle θ, find the magnitude of the acceleration a and the magnitude of the tension T in the string.

Problem 91. problems-1/force-pr-inclined-plane-two-blocks-30deg-icp-soln.tex



A block of mass $2m$ sits on a frictionless incline held at an angle θ relative to the horizontal as shown in the figure above. It is connected by a massless, unstretchable string that runs over a frictionless, massless pulley to a block m hanging over a drop. The two blocks are initially ***at rest***.

a) $\theta_0 = 30^\circ$

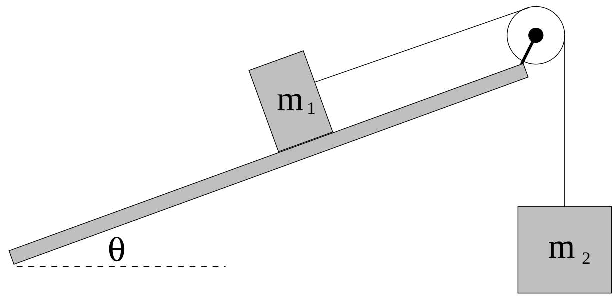
b) $2m$ slides down the incline, m rises.

c)

$$a = \frac{2g\sin(\theta) - g}{3}$$

$$T = \frac{2mg}{3}(\sin(\theta) + 1)$$

Problem 92. problems-1/force-pr-inclined-plane-two-blocks.tex



Two blocks of mass m_1 and $m_2 > m_1$ are drawn above. The block m_1 sits on a frictionless inclined plane tipped at an angle θ with the horizontal as shown. Block m_2 is connected to m_1 by a massless unstretchable string that runs over a massless, frictionless pulley to hang over a considerable drop. At time $t = 0$ the system is ***released from rest***.

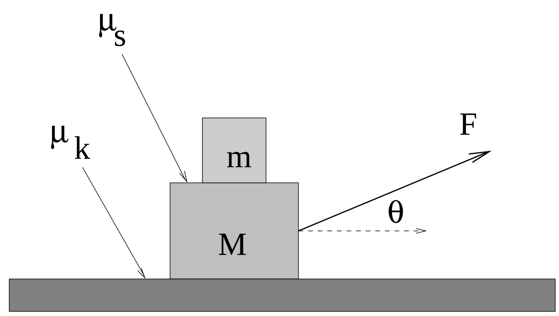
a) Draw a force/free body diagram for the two masses.

b) Find the magnitude of the acceleration of two masses.

c) Find the tension T in the string.

d) When mass m_2 has fallen a height H, how fast are the two masses moving?

Problem 93. problems-1/force-pr-pulling-rough-blocks-on-rough-table.tex



A rope at an angle θ with the horizontal is pulled with a force vF. It pulls, in turn, two blocks, the bottom with mass M and the top with mass m. The coefficients of friction are μ_s between the top and bottom block (assume that they **do not slide for the given force $\vec{F}$**) and μ_k between the bottom block and the table. Remember to show (and possibly evaluate) *all* forces acting on *both* blocks, including internal forces between the blocks.

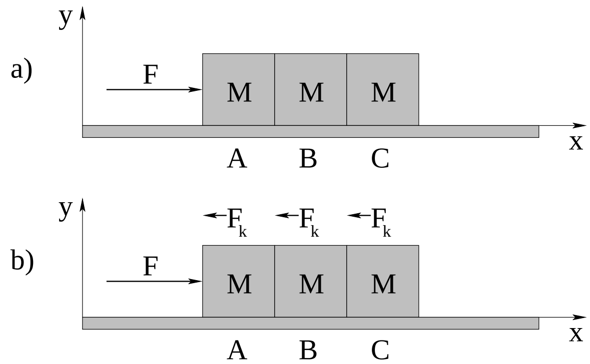
a) Draw a "free body diagram" for **each** mass shown, that is, draw in and label all *real* forces acting on it;

b) Apply **Newton's Second Law** in appropriate coordinates to **each** mass shown;

c) Solve for the acceleration(s) of **each** mass shown and evaluate **all** unknown forces (such as a normal force or the tension in a string) in terms of the given quantities.

Don't forget that the acceleration is a **vector** and must be given as a magnitude and a direction (for example, "along the plane to the right" is ok) or in vector components.

Problem 94. problems-1/force-pr-pushing-three-blocks.tex



Blocks A, B, and C each have mass M and are sitting on a smooth horizontal surface. A horizontal force with magnitude F is applied to block A on the left in the x-direction as shown.

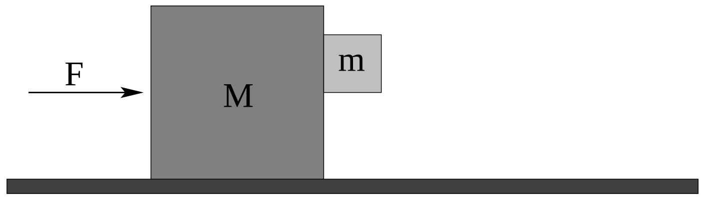
a) Initially, assume that the horizontal surface is frictionless. Determine:

- The acceleration of the system of blocks.
- The normal contact force N_{AB} between block A and block B.
- The normal contact force N_{BC} between block B and block C.

b) Now, assume that in addition to the force F the horizontal surface exerts a kinetic frictional force with magnitude $F_k \ll F$ in the negative x-direction on ***each*** block. Determine:

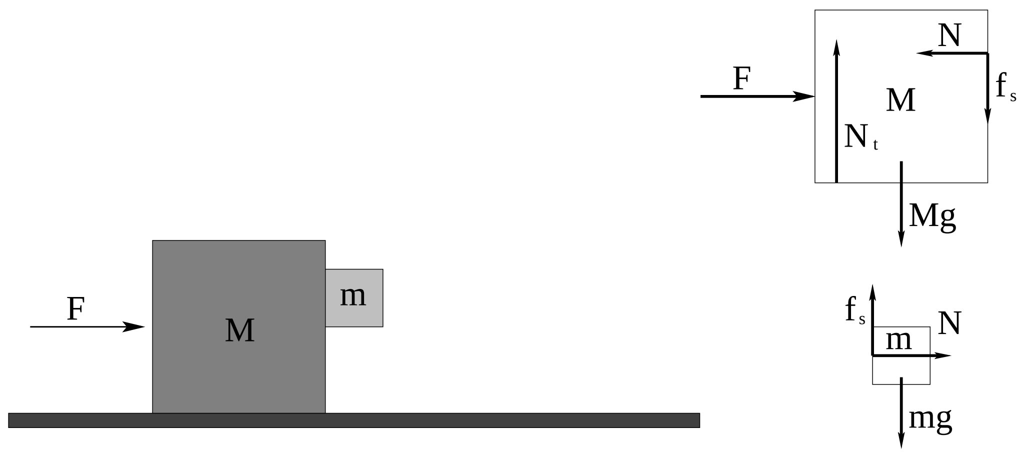
- The acceleration of the system of blocks.
- The normal contact force N_{AB} between block A and block B.
- The normal contact force N_{BC} between block B and block C.
- Evaluate your answers (for this part only) if $F = 100$ N, $M = 10$ kg, and $F_k = 10$ N.

Problem 95. problems-1/force-pr-pushing-vertical-blocks-friction-icp.tex



A force $\vec{F}$ is applied to a large block with mass M, which pushes a smaller block of mass m as shown. The large block is supported by a frictionless table. The coefficient of static friction between the large block and the small block is μ_s. Find the **magnitude** of the minimum force F_{min} such that the small block does not slide down the face of the large one. Draw free body diagrams and show all of your reasoning.

Problem 96. problems-1/force-pr-pushing-vertical-blocks-friction-icp-soln.tex



A force $\vec{F}$ is applied to a large block with mass M, which pushes a smaller block of mass m as shown. The large block is supported by a frictionless table. The coefficient of static friction between the large block and the small block is μ_s. Find the **magnitude** of the minimum force F_{min} such that the small block does not slide down the face of the large one. Draw free body diagrams and show all of your reasoning.

In this case I drew two actual "free body diagrams" distinct from the figure, largely because there are a lot of forces acting on M so that the provided diagram would have gotten a bit crowded.

Note Well the Newton's Third Law Pairs: N and f_s. N_t is the normal force exerted by the table on the bottom of the big block – do not confuse it.

Here are some questions to ask yourself as you (ideally working with your team) try to get the answer below ***on your own*** after looking over these hints and putting them away:

- Static friction is ***always less than*** $\boldsymbol{\mu_s N}$. What must N be in order to *barely* support the weight of mass m?

$$mg = f_s < \mu_s N \quad \Rightarrow \quad N > \frac{mg}{\mu_s}$$

- What must the acceleration of the two masses together be?

$$F - N = Ma \quad \text{and} \quad N = ma \quad \Rightarrow \quad F = (M+m)a$$

- How is that acceleration related to F?

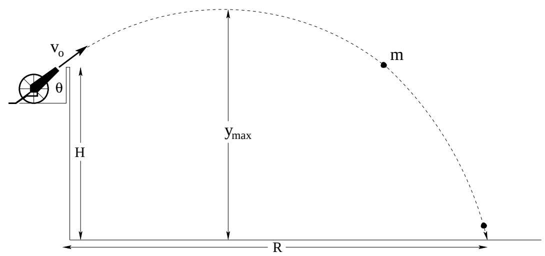
$$N = ma = \frac{m}{M+m}F > \frac{mg}{\mu_s} \quad \text{or} \quad F > \frac{(M+m)g}{\mu_s}$$

for the box *not* to fall. If:

$$F > \boxed{F_{\rm min} = \frac{(M+m)g}{\mu_s}}$$

then the little block will not slide down the front face of the big block!

Problem 97. problems-1/force-pr-range-of-cannon-on-hill.tex



A cannon sits on at the top of a rampart of height (to the mouth of the cannon) H. It fires a cannonball of mass m at speed v_0 at an angle θ relative to the ground. Find:

a) The maximum height y_{max} of the cannonball's trajectory.

b) The time the cannonball is in the air.

c) The range of the cannonball.

Problem 98. problems-1/force-pr-terminal-velocity-tom-and-jerry.tex

The script calls for Tom (cat) to chase Jerry (mouse) across the top of a cartoon skyscraper of height H and off the edge where they both fall straight down (their initial x-velocity is negligible as they fall off) towards a soft pile of dirt that will keep either one from getting hurt by the fall no matter how hard they land (no toon animals were injured in this problem).

Your job is to work out the physics of a "realistic" fall for the animation team. You decide to use the following for the drag force acting on either one:

$$\vec{F}_i = -b_i v^2 \hat{\boldsymbol{v}}$$

where hv is a unit vector in the direction of the velocity and $i = \mathrm{t}, \mathrm{j}$ for Tom or Jerry respectively and where:

$$b_i = CL_i^2$$
$$m_i = DL_i^3$$

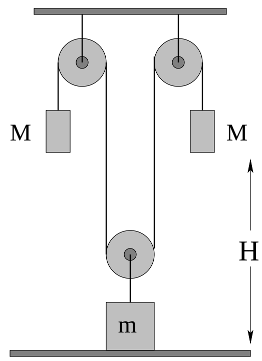
(that is, the drag force is proportional to their cross-sectional area and their mass is proportional to their volume). Their relative size is $L_t = 5L_j$ (Tom is five times the height of Jerry).

a) Draw a on the back of the preceding page showing the building, Tom and Jerry at the instant that Tom runs off of the top. Jerry (who is ahead) should have fallen a short distance d towards the ground.

b) Using the laws of physics, determine the equation of motion (find an expression for the acceleration and write it as a differential equation) algebraically (so the solution applies to Tom or Jerry equally well). Your answer can be given in terms of b_i and m_i.

c) *Without* solving the equation of motion, find an algebraic expression for the terminal velocity of Tom or Jerry as functions of L_i. Explain/show your *reasoning*, don't just write down an answer.

Problem 99. problems-1/force-pr-triple-atwoods-machine.tex



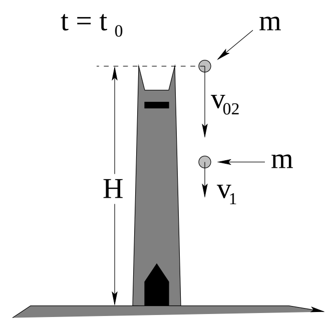
A block and tackle arrangement is built with three massless pulleys and three hanging masses with masses M, m, and M as shown above. The two M masses are a height H off the ground, and m is on the ground. At time $t = 0$ the masses are released from rest from this configuration.

a) Draw a GOOD free body diagram. Clearly label all quantities.

b) Find the acceleration (magnitude and direction) of each block and the tension T in the string as a function of the givens, assuming that $M + M > m$.

c) Find the velocity of each block when the blocks of mass M hit the ground.

Problem 100. problems-1/force-pr-two-balls-1D-2.tex



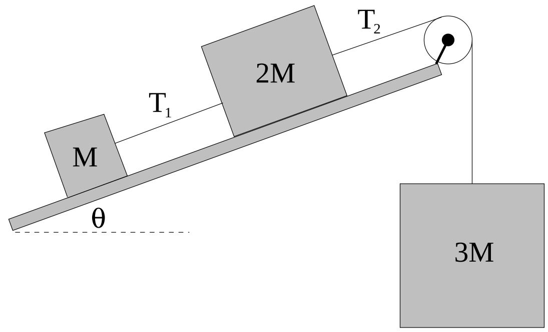
A ball of mass m is dropped at time $t = 0$ from rest ($v_{01} = 0$) from the top of the Duke Chapel (which has height H) to fall freely under the influence of *gravity*. A short time $t = t_0$ later a second ball, also of mass m, is *thrown* down after it at speed v_{02}. Neglect drag.

a) (2 points) Draw a free body diagram for and compute the net force acting on each mass **separately**.

b) (4 points) **From the equation of motion for each mass**, determine their one dimensional trajectory functions, $y_1(t)$ and $y_2(t)$.

c) (3 points) Sketch *qualitatively correct* graphs of $y_1(t)$ and $y_2(t)$ on the same axes in the case where the two collide.

Problem 101. problems-1/force-pr-two-blocks-on-inclined-plane-plus-pulley.tex

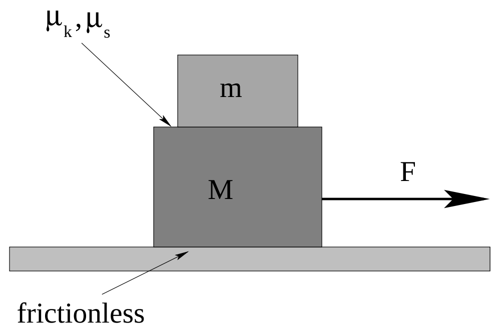


Three blocks of mass M, $2M$ and $3M$ are drawn above. The two smaller blocks sit on a frictionless table tipped at an angle θ with respect to the horizontal as shown. The three blocks are connected by massless unstretchable strings, one of which runs over a massless frictionless pulley to the largest mass. At time $t = 0$ the system is released from rest. Find:

a) The acceleration *vector* in *Cartesian components* of the middle block on the incline. Any correct way of uniquely specifying the Cartesian vector will be accepted, for example $\vec{\boldsymbol{a}} = (a_x, a_y)$, or $\vec{\boldsymbol{a}} = a_x\hat{\boldsymbol{x}} + a_y\hat{\boldsymbol{y}}$.

b) The magnitude of the tensions T_1 and T_2 in the strings as indicated on the diagram.

Problem 102. problems-1/force-pr-two-blocks-with-friction-icp.tex

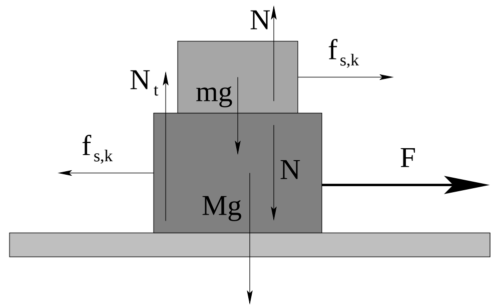


A small block of mass m sits on top of a large block of mass M that sits on a frictionless table. The coefficient of static friction between the two blocks is μ_s and the coefficient of kinetic friction between the two blocks is μ_k. A force $\vec{F} = F\hat{x}$ is directly applied to the *lower* block as shown. All answers should be given in terms of m, M, μ_s, μ_k, and g.

a) What is the largest force $F_{\rm max}$ that can be applied such that the upper block does not slide on the lower block?

b) Suppose that $F = 2F_{\rm max}$ (so that the upper block slips freely). What is the acceleration of each block?

Problem 103. problems-1/force-pr-two-blocks-with-friction-icp-soln.tex



See force diagram above. All arrows *start* on the mass the force acts on. N and $f_{s,k}$ are visible Newton's Third Law pairs where the blocks act on each other. To rigorously solve this, Use the Force (Rubric), Luke!

a) What is the largest force $F_{\rm max}$ that can be applied such that the upper block does not slide on the lower block?

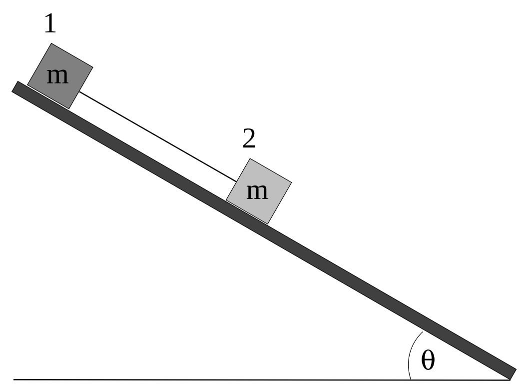
   $a_{\rm max} = \mu_s g$ (why?) so

   $F_{\rm max} = (m + M)\mu_s g$ (also why?)

b) Suppose that $F = 2F_{\rm max}$ (so that the upper block slips freely). What is the acceleration of each block?

   Small block: $a_{\rm max} = \mu_k g$ Large block: $a_{\rm max} = (F - m\mu_k g)/M$

Problem 104. problems-1/force-pr-two-masses-incline-different-friction.tex



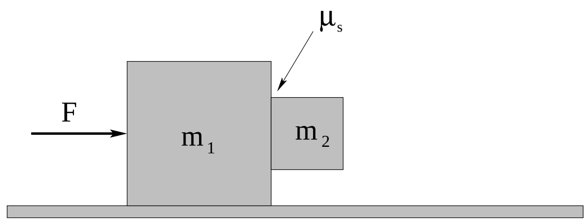
Two blocks, each with the same mass m but ***made of different materials***, sit on a rough plane inclined at an angle θ that is large enough that they will ***definitely slide down***. The first (upper) block has a coefficient of *kinetic* friction of μ_{k1} between block and inclined plane; the second (lower) block has coefficient of kinetic friction μ_{k2}. The two blocks are connected by a massless unstretchable ***string***.

Find the acceleration of the two blocks a_1 and a_2 down the incline:

a) when $\mu_{k2} > \mu_{k1}$.

b) when $\mu_{k1} > \mu_{k2}$;

Problem 105. problems-1/force-pr-vertical-block-friction.tex



A block of mass m_1 is pushed on a frictionless table by a force $\vec{F}$ to the right. A mass m_2 is positioned on the front face as shown. There is a coefficient of static friction μ_s between the big and little block.

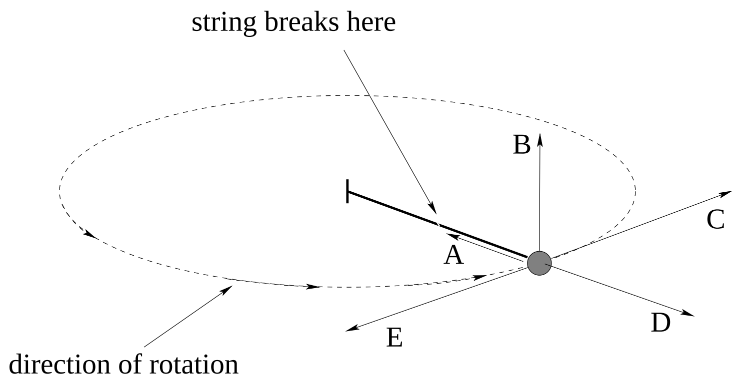
a) What is the horizontal force exerted on block m_2 by the block m_1?

b) Find the ***minimum*** magnitude of force $F_{\rm min}$ that will keep the little block from slipping down the big one.

4.3 Circular Motion

4.3.1 Multiple Choice Problems

Problem 106. problems-1/circular-motion-mc-ball-on-string-breaks.tex



A ball is being whirled on a string. At the instant shown, the string *breaks*. Select the correct trajectory of the ball after it breaks.

a) A

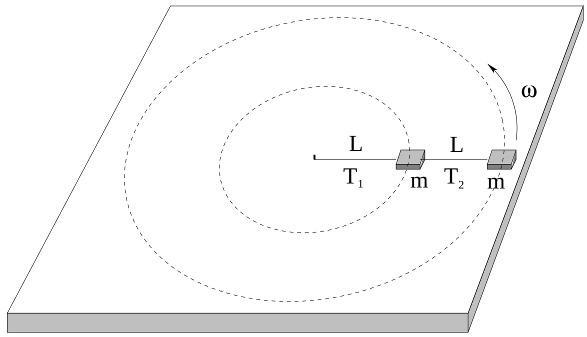
b) B

c) C

d) D

e) E

Problem 107. problems-1/circular-motion-mc-two-masses-on-strings-qual.tex



A block of mass m is tied to a cord of length L that is pivoted at the center of a frictionless table. A second block of mass m is tied to the first block also on a cord of length L, and both are set in motion so that they rotate together at angular speed ω as shown above. The tensions T_1 and T_2 in the cords are:

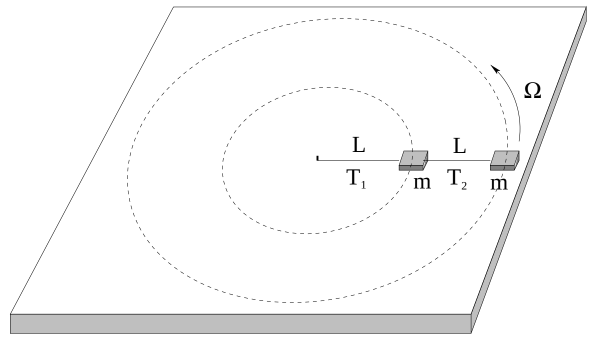
a) $T_1 = T_2$

b) $T_1 > T_2$

c) $T_1 < T_2$

d) $T_1 > T_2$ for $\omega > 0$ and $T_1 < T_2$ for $\omega < 0$

Problem 108. problems-1/circular-motion-mc-two-masses-on-strings-icp.tex



A block of mass m is tied to a cord of length L that is pivoted at the center of a frictionless table. A second block of mass m is tied to the first block also on a cord of length L, and both are set in motion so that they rotate together counterclockwise at angular speed Ω as shown above. The tensions T_1 and T_2 in the cords are:

a) $T_1 = 3m\Omega^2 L$, $T_2 = 2m\Omega^2 L$

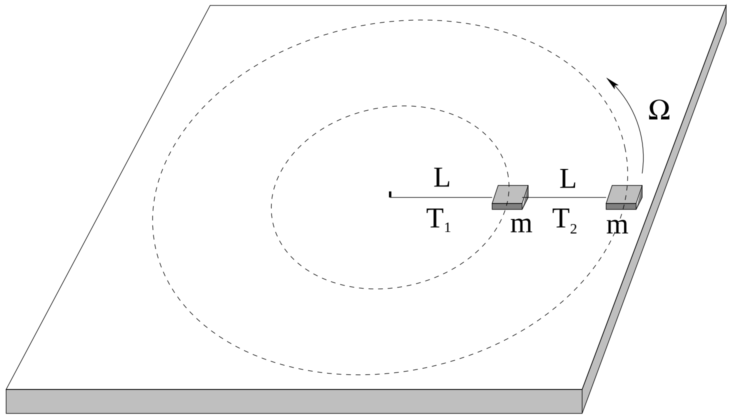
b) $T_1 = m\Omega^2 L$, $T_2 = 2m\Omega^2 L$

c) $T_1 = m\Omega^2 L$, $T_2 = m\Omega^2 L$

d) $T_1 = 2m\Omega^2 L$, $T_2 = 2m\Omega^2 L$

e) $T_1 = m\Omega^2 L$, $T_2 = 4m\Omega^2 L$

Problem 109. problems-1/circular-motion-mc-two-masses-on-strings-icp-soln.tex



Use $F_{\rm tot} = ma_c = mr\Omega^2$ for each mass:

$T_2 = m(2L)\Omega^2$ (toward center)

$T_1 - T_2 = mL\Omega^2$ so $T_1 = m(3L)\Omega^2$ (toward center)

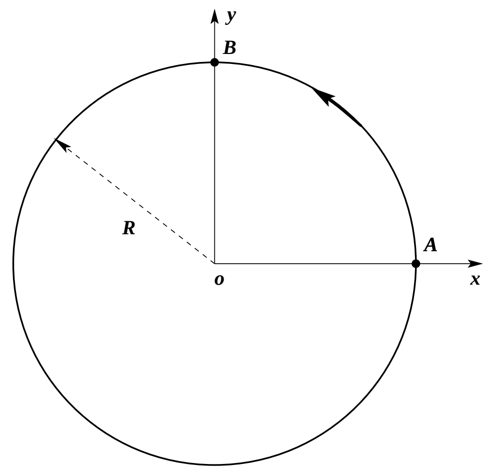
Alternative solution (faster). By inspection, $T_1 > T_2$ – the inner string has to support *both* masses moving in circles where the outer string only has to support one. Only answer a) has $T_1 > T_2$!

Be sure to look for things like this. Short answer questions often have a trick to them, a conceptual solution that doesn't require major computation. In a few cases, one *can't* reasonably "compute" the solution as it is missing the data required to explicitly do so – it is *only* obtainable by a mix of inspection and conceptual reasoning!

4.3.2 Short Answer Problems

Problem 110. problems-1/circular-motion-sa-runner-on-track-icp.tex

Starting from point A, a runner runs at a ***constant speed*** counter-clockwise along a circular race track of radius $R = 20$ m.



a) When the runner has reaches point B, draw and label the runner's velocity $\vec{v}_B$ and acceleration $\vec{a}_B$;

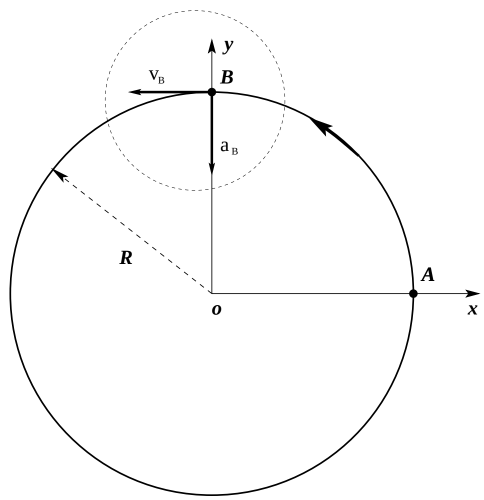
b) If the speed of the runner is 4 m/s, find the angular speed and the magnitude of the acceleration.

$\Omega = \boxed{}$ and $a = \boxed{}$.

c) If the runner runs on the same circular track, but he finishes in one-half of the original time, his new angular speed will be (say) Ω_2, and the new magnitude of his acceleration will be a_2. Find the following ratios:

$\frac{\Omega_2}{\Omega} = \boxed{}$ and $\frac{a_2}{a} = \boxed{}$.

Problem 111. problems-1/circular-motion-sa-runner-on-track-icp-soln.tex



a)

Solution: For a), see above.

For b):

$$\Omega = v/R = 4/20 = \boxed{0.2 \text{ rad/sec}}.$$

The acceleration is *purely centripetal* (the speed of the runner is constant) so

$$a = a_c = v^2/R = \Omega^2 R = 16/20 = \boxed{0.8 \text{ m/sec}^2}.$$

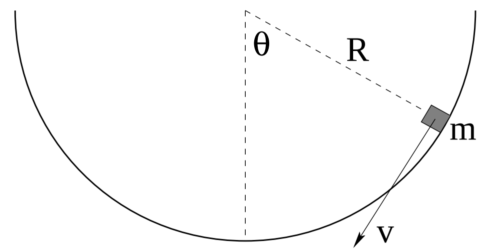
For c), ***half the time is twice the speed***, $\boldsymbol{v \to 2v}$, while ***R doesn't change***. Using the same relations above but now only for *scaling*:

$$\Omega_2/\Omega = \boxed{2}.$$

but

$$a_2/a = \boxed{4}.$$

Problem 112. problems-1/circular-motion-sa-sliding-down-curve.tex
In the figure on the right, a small block slides down a frictionless curved track of circular radius R. When it reaches the angle θ as shown, it has speed v (in a later chapter, we'll learn how to find v from initial conditions).



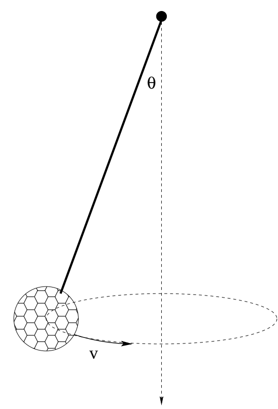
a) Draw a free body diagram for the mass. You may draw it directly on the figure if you wish.

b) Select and draw the best coordinate system to use to analyze its motion. This is tricky!

c) Find the normal force exerted by the track in terms of v and θ.

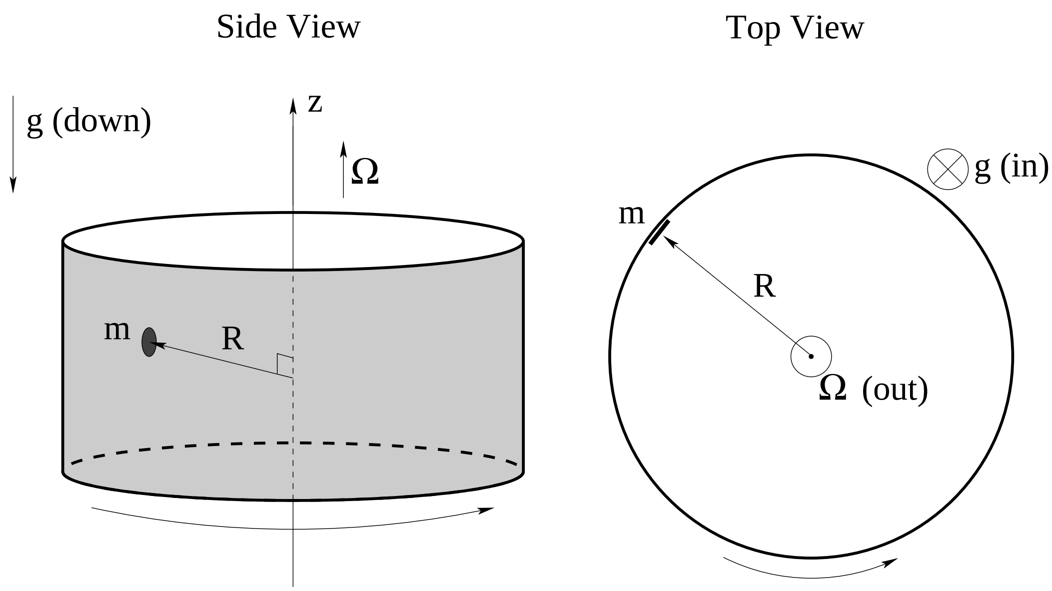
4.3.3 Regular Problems

Problem 113. problems-1/circular-motion-pr-conic-pendulum-tether-ball.tex



A tether ball of mass m is suspended by a rope of length L from the top of a pole. A youngster gives it a whack so that it moves in a circle of radius $r = L\sin(\theta) < L$ around the pole. Find an expression for the speed v of the ball as a function of θ.

Problem 114. problems-1/circular-motion-pr-puck-on-cylinder-friction-review.tex



A hockey puck with mass m is placed against the wall of a hollow cylinder of radius r that is ***rotating at a constant angular speed*** Ω around the z-axis as shown in ***side and top views*** above. The coefficient of static fraction between the puck and the wall of the cylinder is μ_s. Gravity points in the negative $-z$ direction: down in the left hand figure and into the page in the right hand figure.

a) On the diagrams above draw in ***all real forces that act on the mass*** while the cylinder rotates. For forces acting vertically, **use the side view**. For forces acting in the horizontal directions, **use the top view**.

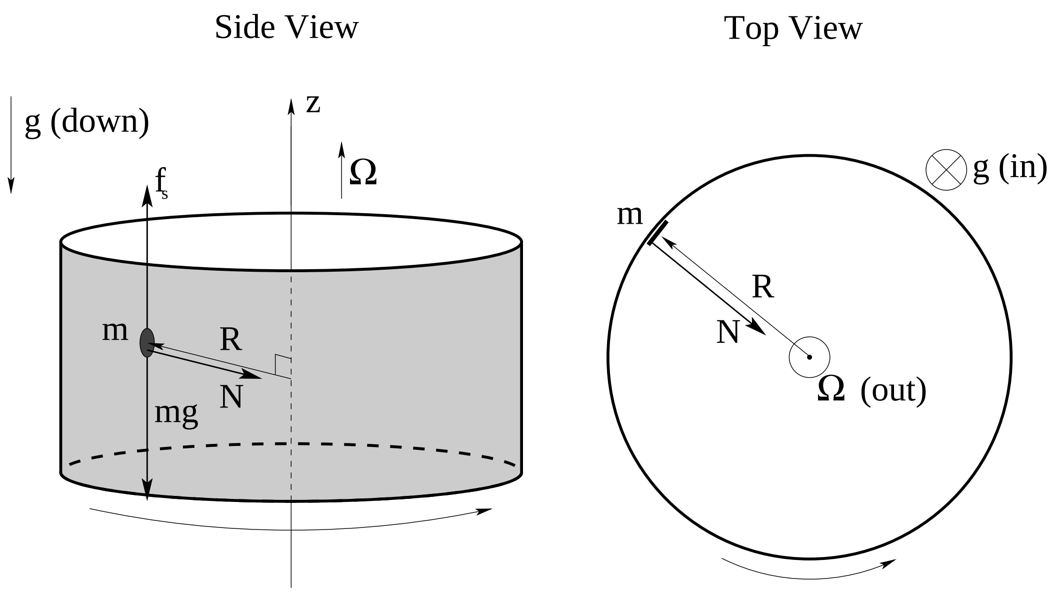
b) If the cylinder is rotating fast enough, the puck does not slide down the wall. What is the value of f_s, the magnitude of static friction, in this case?

$$f_s = \boxed{}$$

c) What is the ***minimum angular speed*** $\boldsymbol{\Omega_{\mathrm{min}}}$ such that the hockey puck does not slide down the wall?

$$\Omega_{\mathrm{min}} = \boxed{}$$

Problem 115. problems-1/circular-motion-pr-puck-on-cylinder-friction-review-soln.tex



A hockey puck with mass m is placed against the wall of a hollow cylinder of radius r that is ***rotating at a constant angular speed*** Ω around the z-axis as shown in ***side and top views*** above. The coefficient of static fraction between the puck and the wall of the cylinder is μ_s. Gravity points in the negative $-z$ direction: down in the left hand figure and into the page in the right hand figure.

a) See above.

b) Comment: f_s is a ***variable*** force and is ***always*** equal to mg if the puck does not slide down. Presumably this means that $\Omega > \Omega_{\min}$, obtained next!

$$f_s = \boxed{mg}$$

c) Here we need to use ***Newton's Second Law and circular motion kinematics*** to find N, use N to find $F_s = \mu_N$ and F_s and $f_s = mg$ to determine $\Omega_{\min}$. That is:

$$F_c = N = ma_c = m\Omega^2 R$$

$$F_s = \mu_s N = \mu_s m\Omega^2 R$$

$$f_s = mg < F_s = \mu_s m\Omega^2 R$$

so (cancelling m and rearranging):

$$\Omega > \sqrt{\frac{g}{\mu_s R}} = \Omega_{\min}$$

or:

$$\Omega_{\min} = \boxed{\sqrt{\frac{g}{\mu_s R}}}$$

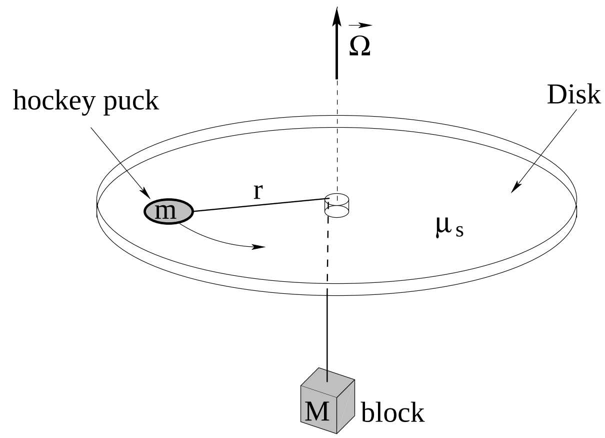
Scoring:

a) +6 total. +2 each for the three forces in the figure. The common mistakes are making f_s tangent to the circle of motion instead of opposite to gravity, and to use $m\Omega^2 R$ outward ("centrifugal force") instead of N inward. These mistakes will likely cost more points later.

b) +5 points straight up, not much room for partial credit. Student loses all five if they write $f_s = \mu_s N$.

c) +9 total. +2 each for each of the three equations above. +3 for the general algebra and final result. Note that they lose at least another 2-3 points if they got b) wrong and hence fail to write $f_s < F_s$ and even if they wrote F_c and F_s correctly.

Problem 116. problems-1/circular-motion-pr-puck-on-wheel-friction.tex



A disk is rotating with a ***constant angular velocity*** $\vec{\Omega}$ (up). A small hockey puck of mass m is placed on the disk at a distance r from the center, and is attached to another block with mass M hanging below by a massless unstretchable string that passes through a tiny (frictionless) hole right in the center of the disk. The static friction coefficient between the hockey puck m and the disk is μ_s.

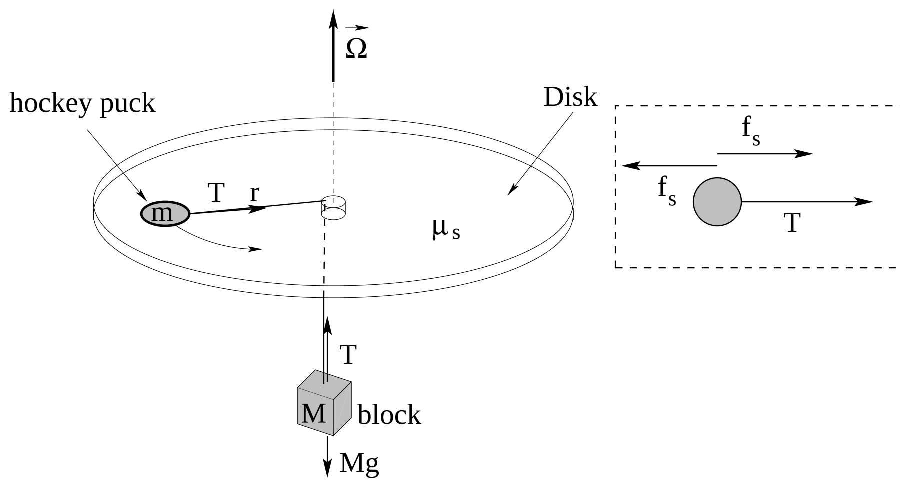
a) Which direction(s) could static friction need to point to keep the puck stationary on the rotating disk (check all that are possible for different/given Ω, r, M, m, μ_s):

☐ In (towards hole) ☐ Out ☐ Tangent to circle of motion

b) Find a formula for the largest M that will ***not move down*** (as the puck slips on the disk), given Ω.

c) Find a formula for the smallest M that will ***not move up*** (as the puck slips on the disk), given Ω.

Problem 117. problems-1/circular-motion-pr-puck-on-wheel-friction-soln.tex



a) Which direction(s) could static friction need to point to keep the puck stationary on the rotating disk (check all that are possible for different/given Ω, r, M, m, μ_s):

   ☒ In (towards hole) ☒ Out ☐ Tangent to circle of motion

   Reason: If it is stationary with respect to the disk, rotating at a constant (angular) velocity, there is no tangential force or resultant torque to speed it up or slow it down. But there ***may be*** frictional forces needed to add, or subtract from, the tension T so that the total force acting on the puck adds up to the kinematically specified ma_c towards the center.

b) Find a formula for the largest M that will not move ***down*** (as the puck slips on the disk), given Ω.

c) Find a formula for the smallest M that will not move ***up*** (as the puck slips on the disk), given Ω.

It's easiest to do both at once. First, for M to be stationary, $T - Mg = 0$ or $T = Mg$. Next, look at the inset in the figures above that shows the forces only on the hocky puck along the radial line towards the hole. If it is rotating slowly enough, and M is large enough, $T = Mg > m\Omega^2 r$ and if there were no friction, the puck would move towards the hole and mass M would move down. In this case, f_s has to point *out*, away from the hole. Then:

$$T - f_s = m\Omega^2 r \Longrightarrow f_s = Mg - m\Omega^2 r < \mu_s N = \mu_s mg$$

This can be rearranged into:

$$Mg < \mu\Omega^2 r + mu_s mg$$

and the largest M can be without slipping down is *just* less than the critical value:

$$\text{b)}\ \boxed{M_c = \frac{m}{g}(\Omega^2 r + \mu_s g)}$$

If it is rotating quickly enough (and/or M is small enough), the tension in the string won't be large enough to keep it moving in a circle and the puck will move *out* (causing M to *rise*) if there is no friction. Friction in this case will have to point *in* towards the hole to help hold the puck in place. Then:

$$T + f_s = m\Omega^2 r \Longrightarrow f_s = m\Omega^2 r - Mg < \mu_s N = \mu_s mg$$

This can be rearranged into:

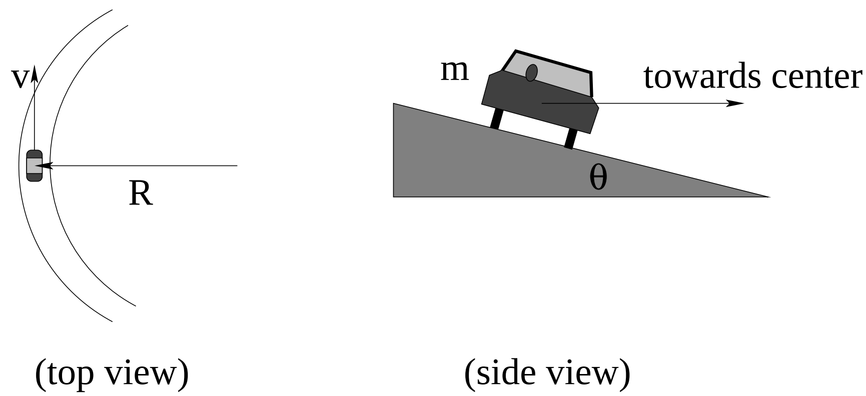
$$m\Omega^2 r - \mu_s mg < Mg$$

not to slip, and the smallest M can be without slipping up is *just* greater than the critical value:

$$\text{c)}\ \boxed{M_c = \frac{m}{g}(\Omega^2 r - \mu_s g)}$$

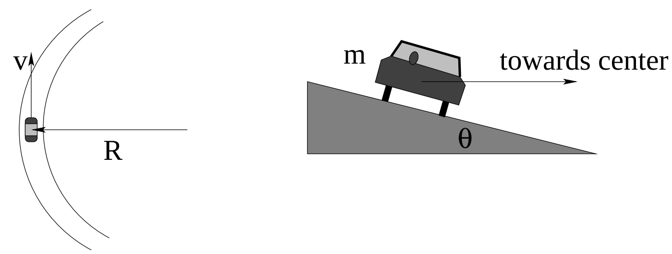
Scoring: Let's do a generous +2 each for getting the directions right, and a -2 for putting down tangential incorrectly, for a maximum total loss of 6 possible if you answer tangential only (and hence never have a chance to solve the problem correctly). That leaves 12 to split among the next two, but really, we're going to put over half of that in just writing some version(s) of: $T = Mg$ (2 points) $F_c = T \pm f_s = m\Omega^2 r$ (4 points) and $f_s < \mu_s mg$ 2 points. This leaves us 4 points for minor algebra errors, direction problems, confusion over the inequality, "getting the wrong answer" with the right general physics and ideas. And yes, if they write $f_s = \mu_s N$ in any direct form, they lose 5 points on the spot.

Problem 118. problems-1/circular-motion-pr-rounding-a-banked-curve-frictionless.tex



A car of mass m is rounding an icy ***frictionless*** banked curve that has radius of curvature R and banking angle θ. What must the speed v of the car be such that it can succeed in making it around the curve without sliding off of the road uphill or down?

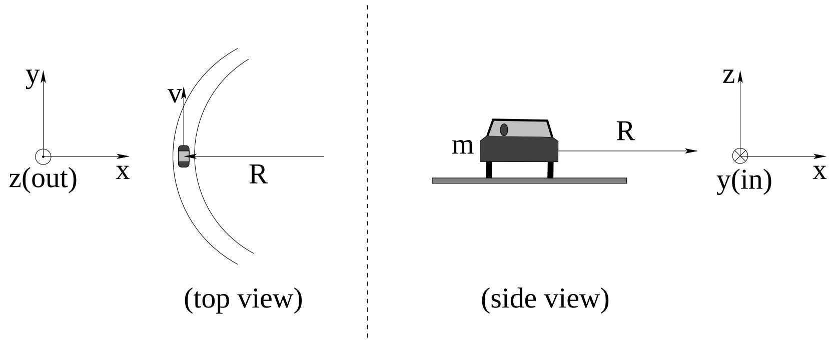
Problem 119. problems-1/circular-motion-pr-rounding-a-banked-curve.tex



(top view) (side view)

A car of mass m is rounding a banked curve that has radius of curvature R and banking angle θ. The coefficient of static friction between the car's tires and the road is μ_s. Find the *range* of speeds v of the car such that it can succeed in making it around the curve without skidding.

Problem 120. problems-1/circular-motion-pr-rounding-a-flat-curve.tex



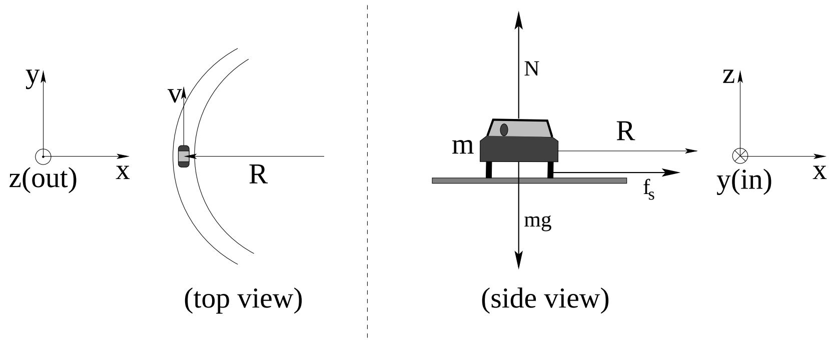
A car of mass m is rounding a flat (unbanked) curve that has radius of curvature R. The coefficient of static friction between the car's tires and the road is μ_s. We will define $v_{\max}$ such that the car makes it around the curve ***without skidding*** as long as:

$$v < v_{\max}$$

a) Suppose that the car is travelling at some constant speed $v < v_{\max}$ (so it does not skid and successfully rounds the curve). Find the total frictional force $\vec{f}_s$ exerted by the tires as it rounds the curve at that speed. Note that force is a vector – be sure to give its magnitude and direction ***using the provided coordinate frame(s)***.

b) Find $v_{\max}$.

Problem 121. problems-1/circular-motion-pr-rounding-a-flat-curve-soln.tex



A car of mass m is rounding a flat (unbanked) curve that has radius of curvature R. The coefficient of static friction between the car's tires and the road is μ_s. We will define $v_{\rm max}$ such that the car makes it around the curve ***without skidding*** as long as:

$$v < v_{\rm max}$$

a) Suppose that the car is travelling at some constant speed $v < v_{\rm max}$ (so it does not skid and successfully rounds the curve). Find the total frictional force $\vec{\boldsymbol{f}}_s$ exerted by the tires as it rounds the curve at that speed. Note that force is a vector – be sure to give its magnitude and direction ***using the provided coordinate frame(s)***.

b) Find $v_{\rm max}$.

Solution: Part a) asks for $\vec{\boldsymbol{f}}_s$. We find it from N2 plus circular motion (centripetal acceleration) as it is the *only* force pushing the car towards the center of the curve:

$$f_s = ma_c = \frac{mv^2}{R} \quad \Rightarrow \quad \boxed{\vec{\boldsymbol{f}}_s = \frac{mv^2}{R}\hat{\boldsymbol{x}}}$$

For part b), we note that:

$$f_s < \mu_s N \quad \text{and} \quad N - mg = ma_y = 0$$

so:

$$f_s = \frac{mv^2}{R} \quad < \quad \mu_s mg$$

or:

$$v^2 < \mu_s g R = v_{\rm max}^2$$

and the car will just start to slide as:

$$v \to \boxed{v_{\rm max} = \sqrt{\mu_s g R}}$$

from below.

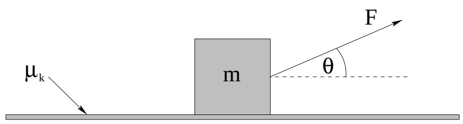
Chapter 5

Work and Energy

5.1 The Work-Kinetic Energy Theorem

5.1.1 Multiple Choice Problems

Problem 122. problems-1/wke-mc-block-and-friction-icp.tex



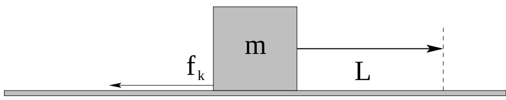
A block of mass m is on a floor. The kinetic friction coefficient between the block and the floor is μ_k. A student pulls a block with a force $\vec{\boldsymbol{F}}$ directed upward at an angle θ with respect to the horizontal as shown. What is the work done ***by friction*** when the block moves a distance L along the floor to the right?

☐ $-\mu_k mgL$ ☐ $-\mu_k(mg + F\sin(\theta))L$

☐ $FL\ \cos(\theta) - \mu_k mgL + \mu_k F\sin(\theta)L$

☐ $FL\ \cos(\theta)$ ☐ $-\mu_k(mg - F\sin(\theta))L$

Problem 123. problems-1/wke-mc-block-and-friction-icp-soln.tex



A block of mass m is on a floor. The kinetic friction coefficient between the block and the floor is μ_k. A student pulls a block with a force $\vec{F}$ directed upward at an angle θ with respect to the horizontal as shown. What is the work done ***by friction*** when the block moves a distance L along the floor to the right?

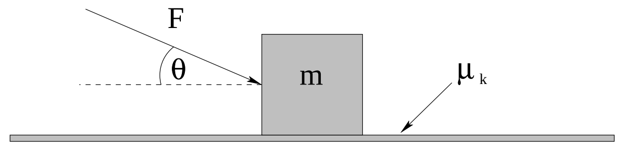
- ☐ $-\mu_k mgL$
- ☐ $-\mu_k(mg + F\sin(\theta))L$
- ☐ $FL\ \cos(\theta) - \mu_k mgL + \mu_k F\sin(\theta)L$
- ☐ $FL\ \cos(\theta)$
- ☒ $-\mu_k(mg - F\sin(\theta))L$

Solution: Force balance in y:

$$F\sin\theta + N - mg = ma_y = 0 \quad \Rightarrow \quad N = mg - F\sin\theta \quad \Rightarrow \quad f_k = \mu_k N = \mu_k(mg - F\sin\theta) \quad \text{(left or } -\hat{\boldsymbol{x}}\text{)}$$

$$W = \int \vec{F} \cdot d\vec{\ell} = \int_0^L -f_k dx = -\mu_k(mg - F\sin\theta)\int_0^L dx = \boxed{-\mu_k(mg - F\sin(\theta))L}$$

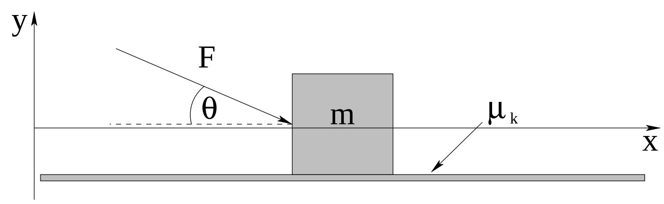
Problem 124. problems-1/wke-mc-block-and-friction.tex



A block of mass m is on a floor. The kinetic friction coefficient between the block and the floor is μ_k. A student pushes a block with a force $\vec{F}$ directed down at an angle θ with respect to the horizontal as shown that makes the block slide ***to the right***. What is the work done ***by the student*** (the force $\vec{F}$) when the block moves a distance L along the floor?

- ☐ $FL\ \cos\theta$
- ☐ $\mu_k(mg + F\sin\theta)L$
- ☐ FL
- ☐ $\mu_k mgL$
- ☐ Cannot tell from the information given.

Problem 125. problems-1/wke-mc-block-and-friction-soln.tex



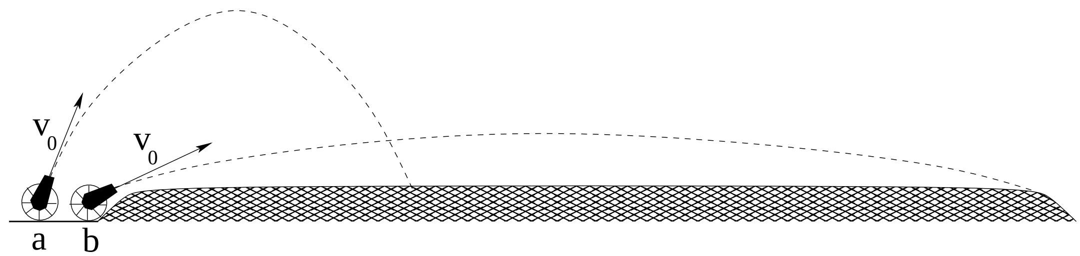
A block of mass m is on a floor. The kinetic friction coefficient between the block and the floor is μ_k. A student pushes a block with a force $\vec{\boldsymbol{F}}$ directed down at an angle θ with respect to the horizontal as shown that makes the block slide ***to the right***. What is the work done ***by the student*** (the force $\vec{\boldsymbol{F}}$) when the block moves a distance L along the floor?

- [x] $FL\ \cos\theta$
- [ ] $\mu_k(mg + F\sin\theta)L$
- [ ] FL
- [ ] $\mu_k mgL$
- [ ] Cannot tell from the information given.

Solution: Note the emphasis on the words 'by the student'. This is a direct hint that one shouldn't try to compute e.g. the work done by friction or the net total work – both are foolers in the list of possibilities. One also *does* need to use the dot product, as part of $\vec{\boldsymbol{F}}$ appears to just push the block harder into the table (increasing kinetic friction) – only the component parallel to the table contributes. Hence:

$W = \vec{\boldsymbol{F}} \cdot L\hat{\boldsymbol{x}} = \boxed{FL\ \cos\theta}$

Problem 126. problems-1/wke-mc-cannonball-energy-speed-trajectories-icp.tex



Two cannons fire cannonballs ***at the same initial speed v_0*** into the air along the trajectories shown. Neglect the drag force of the air.

Which cannonball strikes the ground faster?

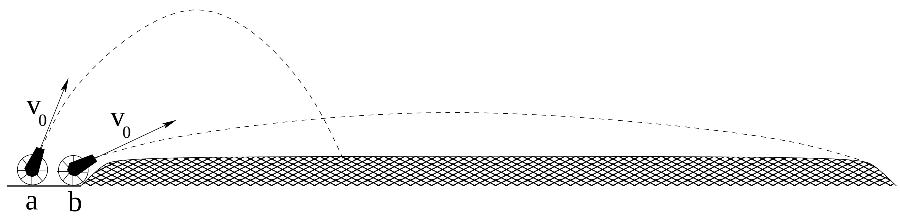
a) Cannonball **a** hits going faster.

b) Cannonball **b** hits going faster.

c) Cannonball **a** and **b** hit at the same speed

d) We cannot tell which hits the ground going faster without more information than is given in the problem and picture.

Problem 127. problems-1/wke-mc-cannonball-energy-speed-trajectories-icp-soln.tex



Which cannonball strikes the ground faster?

Cannonball **a** and **b** hit ***at the same speed*** because total mechanical energy is conserved ***or*** the work done by gravity going up equals the work done going back down to the same height, so there is no change in the kinetic energies of the cannonballs (which are ***initially equal***).

a) Cannonball **a** hits going faster.

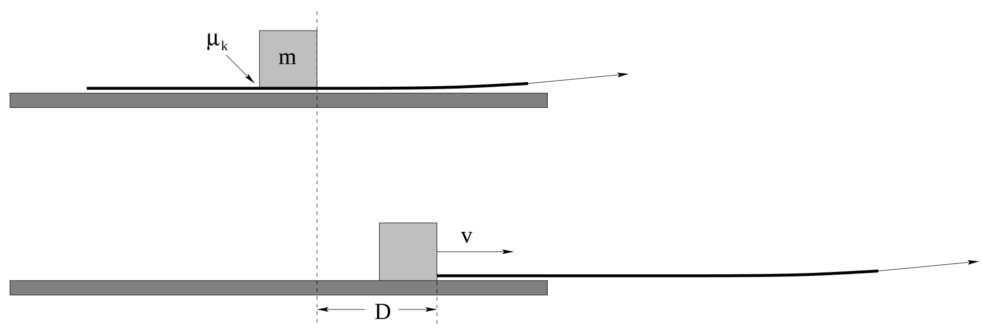
b) Cannonball **b** hits going faster.

c) Cannonball **a** and **b** hit at the same speed

d). We cannot tell which hits the ground going faster without more information than is given in the problem and picture.

5.1.2 Short Answer Problems

Problem 128. problems-1/wke-sa-block-on-paper-icp.tex



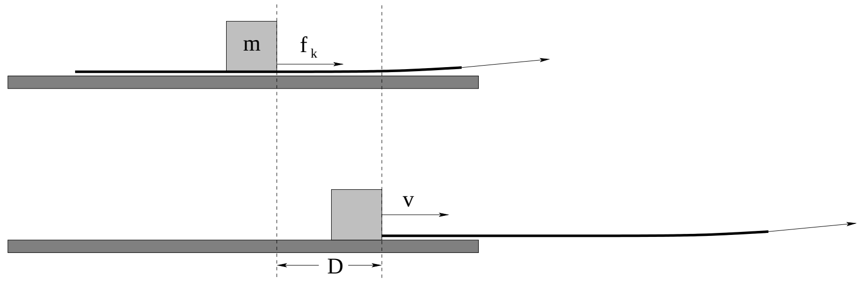
A block of mass m is initially at rest on a long piece of smooth paper on a frictionless table. The block has a coefficient of kinetic friction μ_k with the paper. You pull the paper horizontally out from under the block quickly in the direction indicated by the arrow, such that the block moves a distance D (relative to the ground) while still on the paper.

a) What is the ***magnitude*** of the work done by kinetic friction on the block?

b) Is the work done positive (increasing the kinetic energy of the block) or negative (decreasing the kinetic energy of the block).

c) What is the final velocity of the block when it comes off of the paper and slides along the frictionless table? ***Use +x to the right!***

Problem 129. problems-1/wke-sa-block-on-paper-icp-soln.tex



A block of mass m is initially at rest on a long piece of smooth paper on a frictionless table. The block has a coefficient of kinetic friction μ_k with the paper. You pull the paper horizontally out from under the block quickly in the direction indicated by the arrow, such that the block moves a distance D (relative to the ground) while still on the paper.

a) What is the ***magnitude*** of the work done by kinetic friction on the block?

$$W_{f_k} = \left| \int \vec{\boldsymbol{f}}_k \cdot d\vec{\boldsymbol{\ell}} \right| = \mu_k mgD$$

b) Is the work done positive (increasing the kinetic energy of the block) or negative (decreasing the kinetic energy of the block).

The force of friction acts to the ***right*** and the block moves to the ***right***. So the work done by friction is:

$$\boxed{\textbf{positive}}$$

c) What is the final velocity of the block when it comes off of the paper and slides along the frictionless table? ***Use +x to the right!***

Again, use the work-kinetic energy theorem:

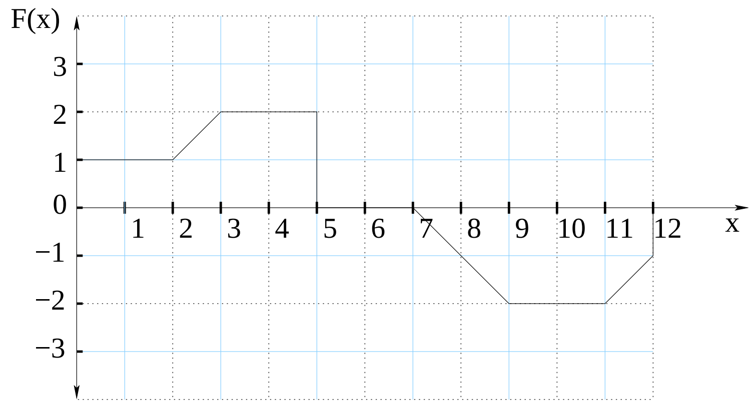
$$W = \mu_k ND = \mu_k mgD = \frac{1}{2}mv^2$$

or:

$$\vec{\boldsymbol{v}} = \sqrt{2\mu_k gD}\hat{\boldsymbol{x}}$$

(which is also $v = \sqrt{2aD}$ as usual, from N2.) Don't forget the direction! Velocity is a ***vector!***

Problem 130. problems-1/wke-sa-graph-work-1.tex



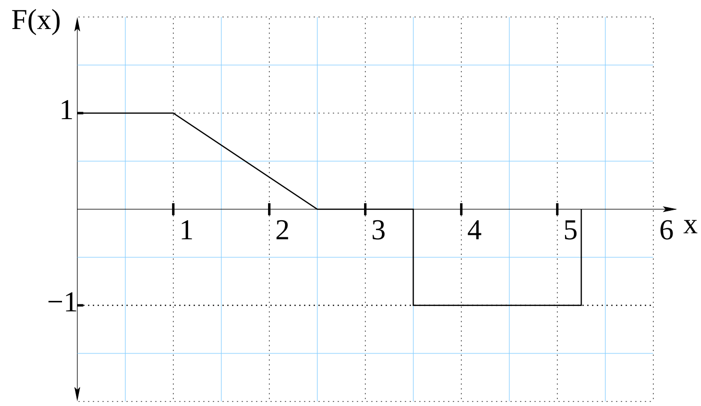
The graph above represents a force in the positive x direction $F(x)$ applied to a mass m as a function of its position. The mass begins at rest at $x = 0$. The force F is given in ***Newtons***, the position x is given in ***meters***.

a) ***How much work*** is done going from $x = 0$ to $x = 6$?

b) ***How much work*** is done going from $x = 6$ to $x = 12$?

c) Assuming $m = 1$ kg, ***what is the final velocity*** of the object at $x = 12$?

Problem 131. problems-1/wke-sa-graph-work-icp.tex



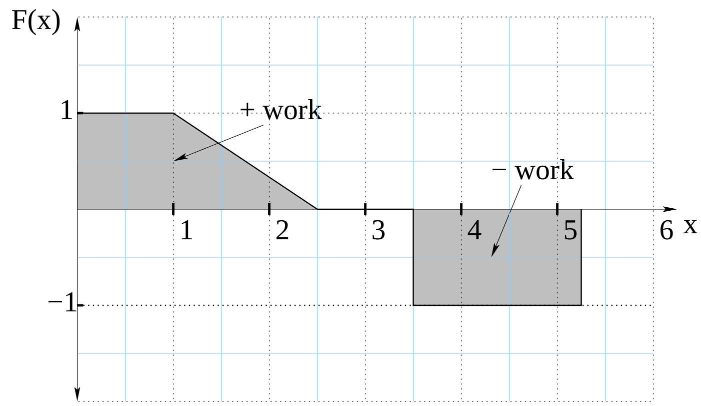
The graph above represents the total one-dimensional force in the x direction $F(x)$ being applied to a mass m as a function of its position. The mass begins at rest at $x = 0$ and moves only along the x axis. The force F is given in Newtons, the position x is given in meters. Answer the following questions (and give the units of your answers):

a) How much work is done going from $x = 0$ to $x = 3$? $W(0 \to 3) =$ $\boxed{}$

b) How much work is done going from $x = 3$ to $x = 6$? $W(3 \to 6) =$ $\boxed{}$

c) Assuming that $m = 1$ kg and that it begins ***at rest*** at the beginning of the motion, what is the speed of the particle at $x = 6$? $v(x = 6) =$ $\boxed{}$

Problem 132. problems-1/wke-sa-graph-work-icp-soln.tex



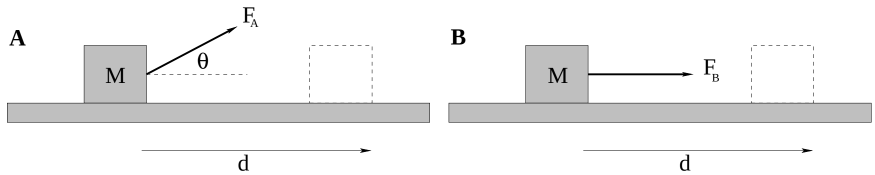
Work is the ***area under the force curve***, positive for positive area, negative for negative area. In this case, count the boxes! Each box represents 0.25 Joules.

a) How much work is done going from $x = 0$ to $x = 3$? $W(0 \to 3) =$ $\boxed{\text{7/4 J.}}$

b) How much work is done going from $x = 3$ to $x = 6$? $W(3 \to 6) =$ $\boxed{\text{-7/4 J.}}$

c) Assuming that $m = 1$ kg and that it begins ***at rest*** at the beginning of the motion, what is the speed of the particle at $x = 6$? $v(x = 6) =$ $\boxed{\text{0.0 m/sec.}}$

Problem 133. problems-1/wke-sa-work-done-by-force.tex



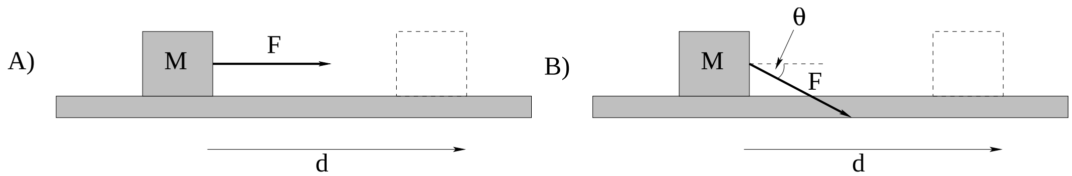
In the figure above a force with a ***constant*** magnitude $F_A = F_B = F$ is applied to a block of mass M resting on a table with a rough surface at two different angles as shown. The coefficient of kinetic friction between the block and table is μ_k. As the block slides to the right a distance d, the work done by $\vec{\boldsymbol{F}}$ is W_F, and the work done by friction is W_{f_k} in the two cases, **A** and **B**.

a) For case **A**, find the work done by $\vec{\boldsymbol{F}}$ and friction, W_F^A and $W_{f_k}^A$, respectively. Your answers should have the correct sign.

b) For case **B**, find the work done by $\vec{\boldsymbol{F}}$ and friction, W_F^B and $W_{f_k}^B$, respectively. Your answers should have the correct sign.

5.1.3 Ranking Problems

Problem 134. problems-1/wke-ra-work-by-force-2.tex



In the figure above a force with a ***constant magnitude*** F is applied to a block of mass M resting on a table with a rough surface at two different angles as shown. The coefficient of kinetic friction between the block and table is μ_k. ***As the block slides to the right a distance d***, the work done by $\vec{F}$ is W_F, and the work done by friction is W_{f_k}.

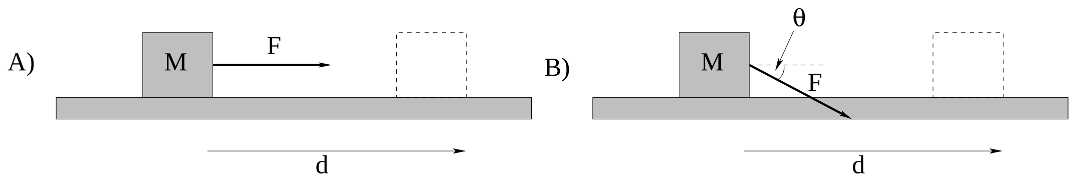
a) Rank the ***magnitude*** of the work done ***by*** $\vec{F}$ in the two cases (put <>= in box):

$$W_F^A \ \square \ W_F^B$$

b) Rank the ***magnitude*** of the work done ***by friction*** in the two cases (put <>= in box):

$$W_{f_k}^A \ \square \ W_{f_k}^B$$

Problem 135. problems-1/wke-ra-work-by-force-2-soln.tex



In the figure above a force with a ***constant magnitude*** F is applied to a block of mass M resting on a table with a rough surface at two different angles as shown. The coefficient of kinetic friction between the block and table is μ_k. ***As the block slides to the right a distance d***, the work done by $\vec{\boldsymbol{F}}$ is W_F, and the work done by friction is W_{f_k}.

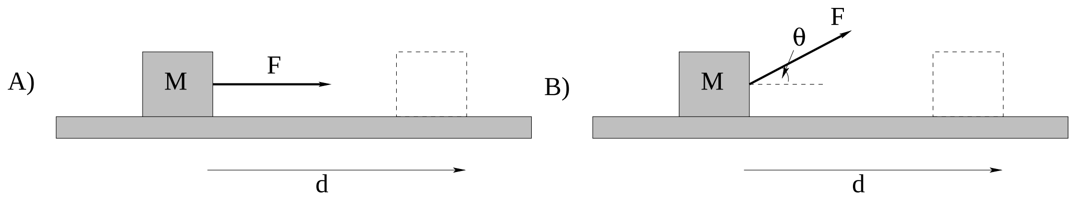
a) Rank the ***magnitude*** of the work done ***by*** $\vec{\boldsymbol{F}}$ in the two cases (put <>= in box):

$$W_F^A \ \boxed{>} \ W_F^B$$

b) Rank the ***magnitude*** of the work done ***by friction*** in the two cases (put <>= in box):

$$W_{f_k}^A \ \boxed{<} \ W_{f_k}^B$$

Problem 136. problems-1/wke-ra-work-by-force-3.tex



In the figure above a force with a ***constant*** magnitude $F < Mg$ is applied to a block of mass M resting on a table with a rough surface at two different angles as shown. The coefficient of kinetic friction between the block and table is μ_k. As the block slides to the right a distance d, the work done by $\vec{\boldsymbol{F}}$ is W_F, and the work done by friction is W_{f_k}.

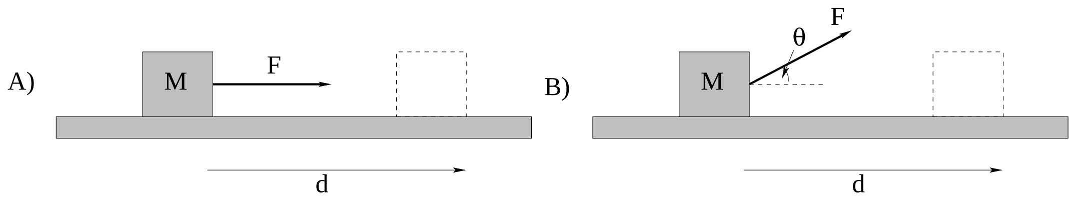
a) Rank the ***magnitude*** of the work done by $\vec{\boldsymbol{F}}$ in the two cases (<>= in box):

$$W_F^A \ \square W_F^B$$

b) Rank the ***magnitude*** of the work done by friction in the two cases (<>= in box):

$$W_{f_k}^A \ \square W_{f_k}^B$$

Problem 137. problems-1/wke-ra-work-by-force-3-soln.tex



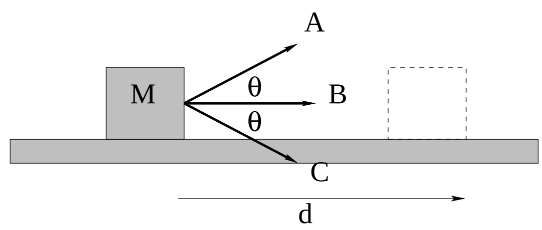
a) Rank the ***magnitude*** of the work done by $\vec{\boldsymbol{F}}$ in the two cases (<>= in box):

$$W_F^A > W_F^B$$

b) Rank the ***magnitude*** of the work done by friction in the two cases (<>= in box):

$$W_{f_k}^A > W_{f_k}^B$$

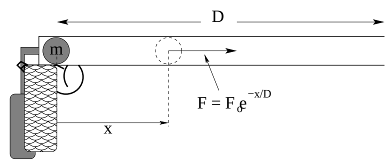
Problem 138. problems-1/wke-ra-work-by-force.tex



In the figure above a force with a *constant* magnitude F is applied to a block of mass M resting on a smooth (frictionless) table at three different angles as shown. Rank the work done by $\vec{\boldsymbol{F}}$ as the block slides to the right a distance d, where equality is allowed. (A possible answer might be $A = B > C$ for example.)

5.1.4 Regular Problems

Problem 139. problems-1/wke-pr-painball-gun-exponential-example.tex



A simple schematic for a paintball gun with a barrel of length D is shown above; when the trigger is pulled carbon dioxide gas under pressure is released into the approximately frictionless barrel behind the paintball (which has mass m). The expanding, cooling gas exerts a force on the ball of magnitude:

$$F = F_0 e^{-\frac{x}{D}}$$

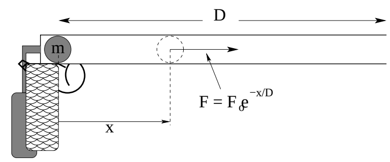
on the ball to the *right*, where x is measured from the paintball's initial position as shown.

a) Find the work done on the paintball by the force as the paintball is accelerated down the barrel.

b) Use the work-kinetic-energy theorem to compute the kinetic energy of the paintball after it has been accelerated.

c) Find the speed with which the paintball emerges from the barrel after the trigger is pulled.

Problem 140. problems-1/wke-pr-painball-gun-exponential-solution.tex



A simple schematic for a paintball gun with a barrel of length D is shown above; when the trigger is pulled carbon dioxide gas under pressure is released into the approximately frictionless barrel behind the paintball (which has mass m). The expanding, cooling gas exerts a force on the ball of magnitude:

$$F = F_0 e^{-\frac{x}{D}}$$

on the ball to the *right*, where x is measured from the paintball's initial position as shown.

a) Find the work done on the paintball by the force as the paintball is accelerated down the barrel.

b) Use the work-kinetic-energy theorem to compute the kinetic energy of the paintball after it has been accelerated.

c) Find the speed with which the paintball emerges from the barrel after the trigger is pulled.

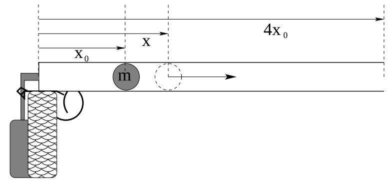
The only force acting on the paintball is the force applied by the prssurized gas (gravity is countered by a normal force from the barrel, and in any case neither does work when the motion is horizontal; we are neglecting friction which may be less realistic). So WKE reads simply

$$\begin{aligned} K_f - K_i &= \int_i^f bfF \cdot d\mathbf{x} \\ &= \int_0^D F_0 e^{-\frac{x}{D}} dx \\ &= F_0 \left(-De^{-d/D}\right)\Big|_0^D \\ &= -FD(e^{-1} - 1) = FD(1 - 1/e) \ . \end{aligned} \tag{5.1}$$

This is the work done by the gas. Since the paintball starts at rest so $K_i = 0$ it is also the kinetic energy the ball has when it leaves the barrel. To find the speed with which it leaves we set

$$\begin{aligned} \frac{mv^2}{2} &= K_f = FD(1 - 1/e) \\ v &= \sqrt{\frac{2FD(1 - 1/e)}{m}} \ . \end{aligned} \tag{5.2}$$

Problem 141. problems-1/wke-pr-paintball-gun-adiabatic.tex



A simple schematic for a paintball gun is shown above; when the trigger is pulled carbon dioxide gas under pressure is released into the approximately frictionless barrel behind the paintball (which has mass m) initially resting at x_0. The gas expands approximately ***adiabatically*** and exerts a force on the ball of magnitude:

$$F = F_0 \frac{x_0^\gamma}{x^\gamma}$$

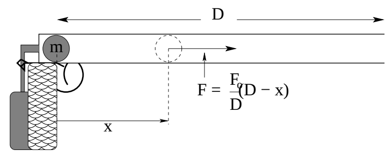
on the ball to the *right*, where F_0 is the initial force exerted at $x = x_0$, and x is measured from the end of the barrel as shown. γ is a constant (equal to 1.4 for carbon dioxide). This force is only exerted up to the end of the barrel at $x = 4x_0$.

a) Find the work done on the paintball by the force as the paintball is accelerated down the barrel.

b) Use the work-kinetic-energy theorem to compute the kinetic energy of the paintball after it has been accelerated.

c) Find the speed with which the paintball emerges from the barrel after the trigger is pulled.

Problem 142. problems-1/wke-pr-paintball-gun-linear.tex



A simple schematic for a paintball gun is shown above; when the trigger is pulled carbon dioxide gas under pressure is released into the approximately frictionless barrel behind the paintball (which has mass m). The gas exerts a force on the ball of magnitude:

$$F = \frac{F_0}{D}(D - x)$$

on the ball to the *right*, where x is measured from the paintball's initial position as shown.

a) Find the work done on the paintball by the force as the paintball is accelerated down the barrel.

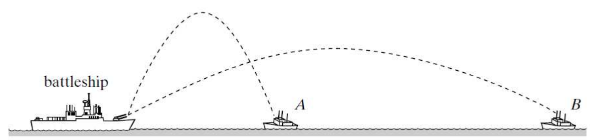
b) Use the work-kinetic-energy theorem to compute the kinetic energy of the paintball after it has been accelerated.

c) Find the speed with which the paintball emerges from the barrel after the trigger is pulled.

5.2 The Non-Conservative Work-Mechanical Energy Theorem

5.2.1 Multiple Choice Problems

Problem 143. problems-1/wme-mc-battleship.tex



A battleship ***simultaneously*** fires two shells at enemy ships along the trajectories shown, such that the shells have the ***same initial speed***. One ship (**A**) is close by; the other ship (**B**) is far away. Ignore drag forces.

a) Which ship is hit first (circle both if they are hit at the same time)?

A **B**

b) Which shell has the greater speed when it hits the ship (circle both if the speeds are equal)?

A **B**

Problem 144. problems-1/wme-mc-zero-of-potential-energy.tex

Tommy is working on a physics problem involving energy. “Look,” he says, “the total energy of this block at rest is zero at the top of this incline of height H and therefore must be zero at the bottom.”

Sally disagrees. “Impossible. The block is at the *top* of the incline. It has total energy mgH at the top and so its total energy must still be mgH at the bottom.”

a) Tommy is right, Sally is wrong.

b) Sally is right, Tommy is wrong.

c) Both Tommy and Sally are right.

d) Both Tommy and Sally are wrong.

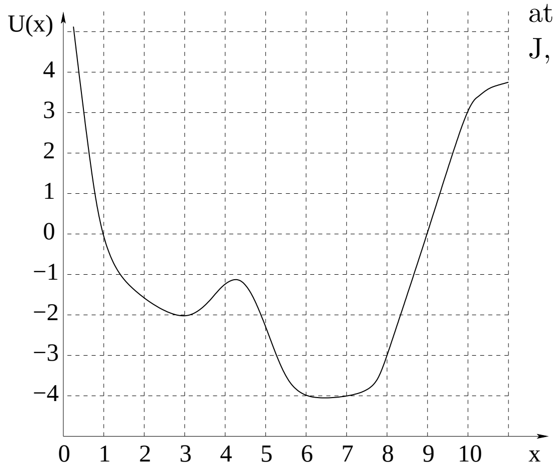
e) There isn’t enough information to tell who is right and who is wrong.

5.2.2 Short Answer Problems

Problem 145. problems-1/wme-sa-ball-to-ground.tex

A ball is thrown with some speed v_0 from the top of a cliff of height H. Show that the speed with which it hits the ground is independent of the direction it is thrown (and determine that speed in terms of g, H, and v_0).

Problem 146. problems-1/wme-sa-potential-energy-graph-2.tex



A conservative one-dimensional force $\vec{F}(x)$ acts on a particle of mass $m = 2$ kg. The potential energy $U(x)$ associated with $\vec{F}(x)$ is shown in the figure to the left, ***where x is in meters and U is in joules***. The particle is at initially located $x_1 = 3$ m with kinetic energy $K_1 = 2$ J, moving to the left (along the negative x direction).

a) Mark all points ***on the x-axis*** where the force on the mass would vanish. Label the points "stable" or "unstable according to the kind of equilibrium point they are.

b) Between what limits of x does the particle move? Mark them as "turning points" on the x-axis of the graph.

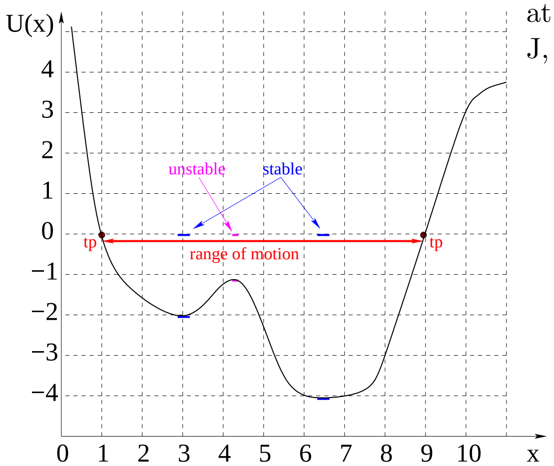
c) What is the particle's *velocity* (magnitude and direction) $\vec{v}(x)$ when it is at ***first*** at $x_2 = 6$ meters after being released as described above?

d) What is the *force* (magnitude and direction) $\vec{F}(x)$ acting on the particle when it arrives at $x_3 = 9$ meters?

$\vec{v}(6) =$ $\boxed{}$

$\vec{F}(9) =$ $\boxed{}$

Problem 147. problems-1/wme-sa-potential-energy-graph-2-soln.tex



A conservative one-dimensional force $\vec{F}(x)$ acts on a particle of mass $m = 2$ kg. The potential energy $U(x)$ associated with $\vec{F}(x)$ is shown in the figure to the left, ***where x is in meters and U is in joules***. The particle is at initially located $x_1 = 3$ m with kinetic energy $K_1 = 2$ J, moving to the left (along the negative x direction).

a) Mark all points ***on the x-axis*** where the force on the mass would vanish. Label the points "stable" or "unstable according to the kind of equilibrium point they are.

b) Between what limits of x does the particle move? Mark them as "turning points" on the x-axis of the graph.

c) What is the particle's *velocity* (magnitude and direction) $\vec{v}(x)$ when it is at ***first*** at $x_2 = 6$ meters after being released as described above?

d) What is the *force* (magnitude and direction) $\vec{F}(x)$ acting on the particle when it arrives at $x_3 = 9$ meters?

$\vec{v}(6) = \boxed{2\hat{x} \quad \text{m/sec}}$ $\qquad$ $\vec{F}(9) = \boxed{-3\hat{x} \quad \text{newtons}}$

Solution: For a), recall that:

$$F_x = -\frac{dU}{dx} \quad \Rightarrow \quad F_x = 0 = -\frac{dU}{dx}$$

Stable equilibria (marked in blue above) are cupped "up"; unstable ones are cupped "down" because the force points towards or away from the equilibrium, respectively, a short distance to either side of the point(s) where the slope of U is zero.

For b), at $x_1 = 3$ m, $K = 2$ J and we can read off $U(3) = -2$ J so that $E_{\rm tot} = 0$ J. At the (red) dots on the graph, $E = U \Rightarrow K = 0$, making these ***turning points of the motion***. Wherever $U > E$, K would have to be negative, which is impossible, so these points are said to be "classically forbidden" and we can never find the particle with *this* total energy to the left of the first or right of the second one. The allowed range of motion is then the red thick arrow region in between.

For c) we read off $U(6) = -4$ J. Then:

$$K(6) = \frac{1}{2}mv^2 = E - U(6)$$

or

$$(1)v^2 = 0 - (-4) \quad \Rightarrow \quad \boxed{v_x = 2 \text{ m/sec}}$$

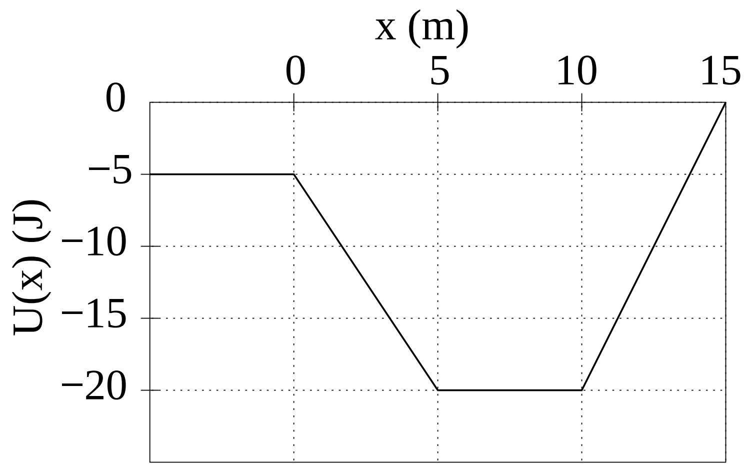
at $x = 6$ m. It is positive (to the right, $+x$) because the particle initially goes *left*, hits the turning point at $x = 1$ m, and returns to the *right*, eventually reaching $x_2 = 6$ m on its way to the RH turning point at $x_3 = 9$ m.

Finally, for d) we just have to read off the slope at 9 meters. At $x = 8$ m, $U = -3$ J. At $x = 10$ m, $U = 3$ J. Then:

$$\boxed{F_x = -\frac{\Delta U}{\Delta x} = -\frac{6 \text{ J}}{2 \text{ m}} = -3 \text{ N}}$$

The direction is to the left, negative x, as positive slope in x implies a force in the $-x$ direction.

Problem 148. problems-1/wme-sa-potential-energy-to-force-icp.tex



A one-dimensional force $F(x)$ acts on a 2 kg particle which moves along the x axis. The potential energy $U(x)$ associated with $F(x)$ is shown in the graph.

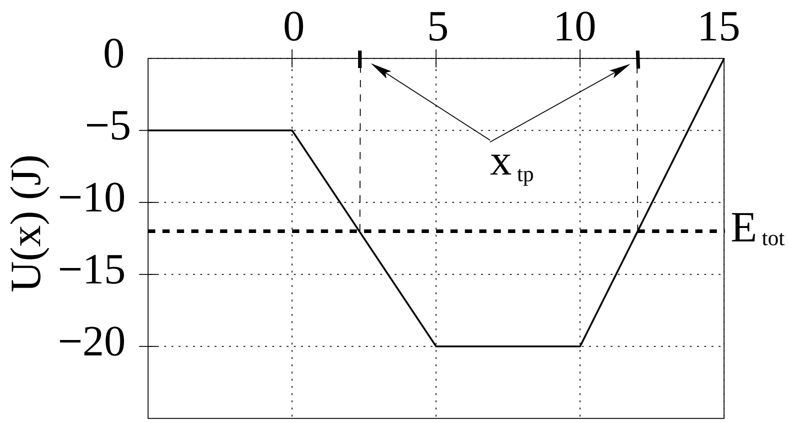
When the particle is at $x = 11$ m, its speed is 2 m/s.

a) What is the magnitude and direction of $F(x)$ at:
$x = -3$ m:
$x = 4$ m:
$x = 6$ m:
$x = 11$ m:

b) Between what limits of x does the particle move?

c) What is its speed at $x = 7$ m?

Problem 149. problems-1/wme-sa-potential-energy-to-force-icp-soln.tex



From the graph, the slope of the graph in the region from 0 to 5 is -3 J/m. In the region from 10 to 15 it is +4 J/m. Recall, $E_{\rm tot} = U + K$ everywhere, so we need to find it using the data. $U(x = 11) = -16$ J. $K = \frac{1}{2}mv^2 = 4$J. Therefore $E_{\rm tot} = -12$ J. Also, $F_x = -\frac{dU}{dx}$, which is the (negative) ***slope*** of the potential energy curve.

a) What is the magnitude and direction of $F(x)$ at:
$x = -3$ m: $F_x = 0$ N
$x = 4$ m: $F_x = 3$ N
$x = 6$ m: $F_x = 0$ N
$x = 11$ m: $F_x = -4$ N

b) These are the values of x for which $E_{\rm tot} = U(x)$. On the left, the turning point is at $x = 7/3$ m, on the right it is at $x = 12$ m.

c) $K = -12 - (-20) = 8$ J, so $v = 2\sqrt{2}$ m/sec.

Problem 150. problems-1/wme-sa-sliding-block-friction.tex

A block of mass m sitting on a horizontal surface is given an initial speed v_0. Travelling in a straight line it comes to rest after sliding a distance d. Find an algebraic expression for the coefficient of kinetic friction in terms of the givens.

$\mu_k = \boxed{}$

Problem 151. problems-1/wme-sa-sliding-block-friction-soln.tex

A block of mass m sitting on a horizontal surface is given an initial speed v_0. Travelling in a straight line it comes to rest after sliding a distance d. Find an algebraic expression for the coefficient of kinetic friction in terms of the givens.

$$\mu_k = \boxed{\frac{v_0^2}{2gd}}$$

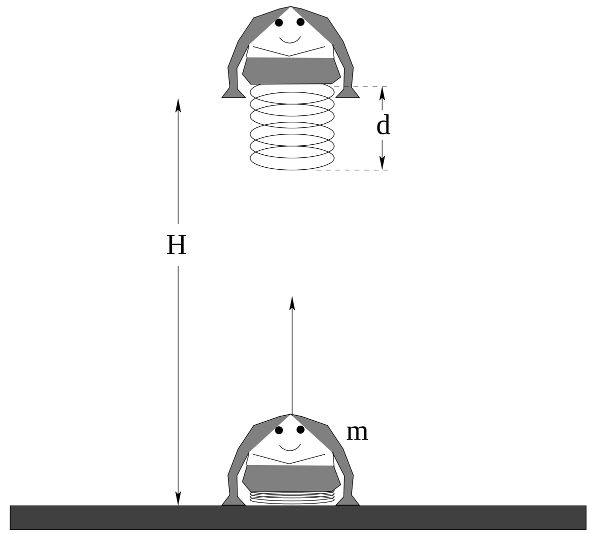
Solution: Use the non-conservative work, mechanical energy theorem:

$$W_{nc} = \int_0^d \{-\mu_k mg\}\, dx = -\mu_k mgd = -\frac{1}{2}mv_0^2 = E_f - E_i = \Delta E$$

or (rearranging):

$$\mu_k = \frac{mv_0^2}{2mgd} = \frac{v_0^2}{2gd}$$

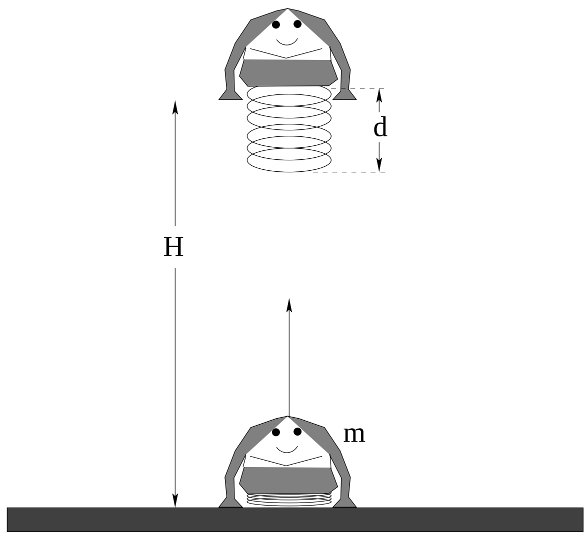
Problem 152. problems-1/wme-sa-spring-jumper-straight-up.tex



A simple child's toy is a jumping frog made up of an approximately massless spring with spring constant k that propels a molded "frog" of mass m. The frog is pressed down onto a table (compressing the spring by a distance d) and at $t = 0$ the spring is released so that the frog leaps high into the air.

Use work and/or mechanical energy to determine how high the frog leaps. Neglect drag forces.

Problem 153. problems-1/wme-sa-spring-jumper-straight-up-soln.tex



A simple child's toy is a jumping frog made up of an approximately massless spring of uncompressed length d and spring constant k that propels a molded "frog" of mass m. The frog is pressed down onto a table (compressing the spring by d) and at $t = 0$ the spring is released so that the frog leaps high into the air.

Use work and/or mechanical energy to determine how high the frog leaps.

It is by far the easiest to use conservation of mechanical energy. Initially, the frog is located, at rest (so its kinetic energy $K = 0$), at $y = 0$ (so $U_g = mgy = 0$) with the spring compressed a distance d (so $U_k = \frac{1}{2}kd^2$). When the frog reaches its maximum height, it is *again* at rest (so $K = 0$), its gravitational potential energy is now $U_g = mgy = mgH$, and the spring is fully expanded so its spring potential energy is $U_k = 0$. Hence:

$$E_i = 0 + 0 + \frac{1}{2}kd^2 = 0 + mgH + 0$$

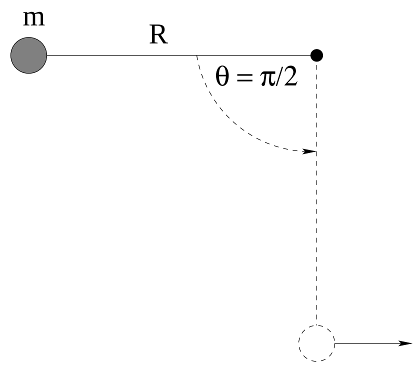
or

$$mgH = \frac{1}{2}kd^2$$

or

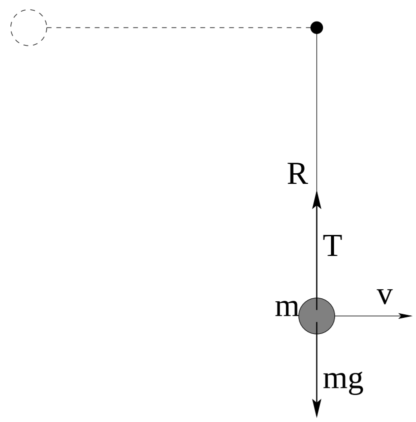
$$H = \frac{kd^2}{2mg}$$

Problem 154. problems-1/wme-sa-tension-pendulum-height-R-icp.tex



In the figure above, a mass m is attached to a massless unstretchable string of length R and held at an initial position at an angle $\theta = \pi/2$ relative to the horizontal as shown. At time $t = 0$ it is released from rest. Find the tension T in the string when it reaches $\theta = 0$.

Problem 155. problems-1/wme-sa-tension-pendulum-height-R-icp-soln.tex



In the figure above, a mass m is attached to a massless unstretchable string of length R and held at an initial position at an angle $\theta = \pi/2$ relative to the horizontal as shown. At time $t = 0$ it is released from rest. Find the tension T in the string when it reaches $\theta = 0$.

This is "like" the loop the loop example. We use ***energy conservation*** to get an expression containing v at the bottom. Note that *experience* tells us (eventually) to solve directly for mv^2 as we look ahead to what we will need to form the right hand side of N2 for circular motion:

$$E_i = mgR = \frac{1}{2}mv^2 = E_f \implies mv^2 = 2mgR$$

At the bottom, N2 in the vertical direction becomes

$$T - mg = ma_{\mathrm{up}} = m\frac{v^2}{R}$$

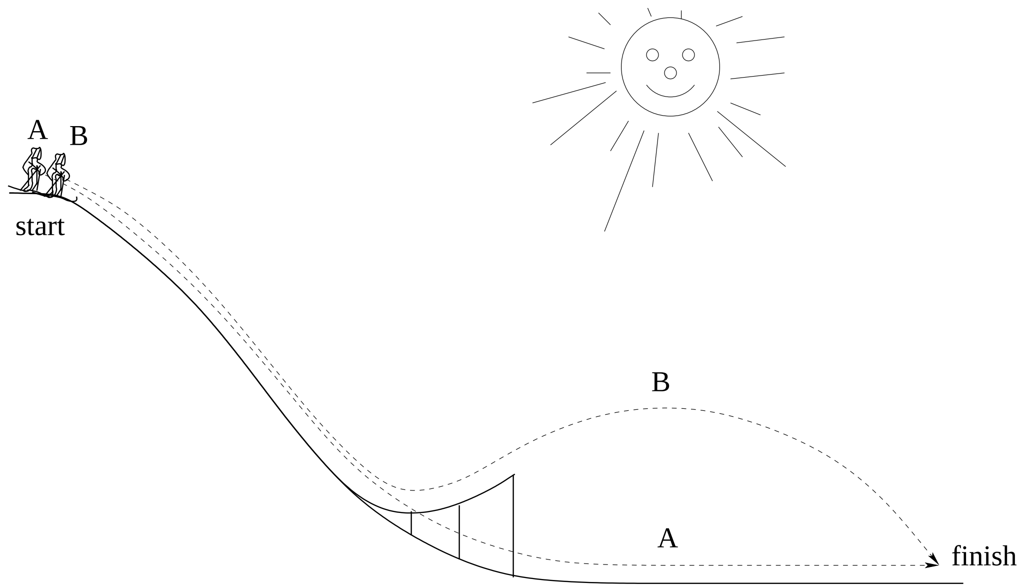
or (using the first result):

$$T = mg + mv^2/R = 3mg$$

Note this is a *substantial* net force. In the accelerating frame of a person swinging on a swing up to the highest point one can go without jerking the chains, it feels like one "weighs" (a bit less than, because one's center of mass isn't at one's bottom) three times as much as normal!

5.2.3 Ranking Problems

Problem 156. problems-1/wme-ra-two-skiers-icp.tex

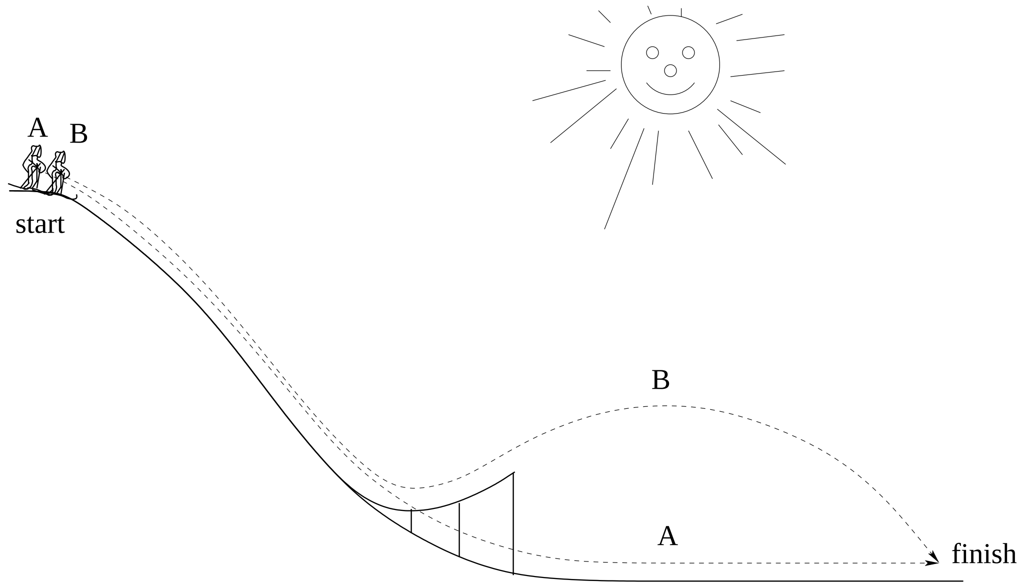


Two skiers start at the same point on a (frictionless) slope. One (A) skis straight down the slope to the finish line. The other (B) passively skis off of a ski jump to arrive at the finish more flamboyantly. ***Rank*** the answers to the following questions, where equality is a possibility, that is, possible answers are $A < B$ or $A = B$. Ignore friction and drag forces and assume that the jumper does not use their leg muscles to “jump”.

a) Rank the ***relative speed*** of the two skiers when they reach the finish line.

b) Rank the ***finish time*** – who arrives at the finish line first (or is it at the same time)?

Problem 157. problems-1/wme-ra-two-skiers-icp-soln.tex

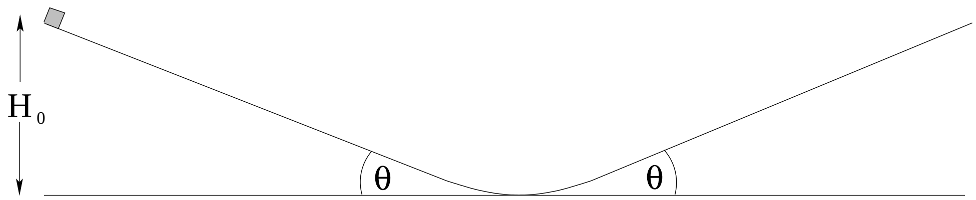


a) Energy conservation says that their final kinetic energies, and hence their speeds, have to be **equal**.

b) Skier A ***both*** goes a shorter distance ***and*** spends a lot of his/her time going faster than skier B, who ***slows down as he/she rises*** and only reaches his/her final speed (equal to A) at the finish. A is at that speed from the bottom of the slope all the way to the finish line! So **A wins**.

5.2.4 Regular Problems

Problem 158. problems-1/wme-pr-double-inclines-with-friction.tex



A block with mass m is released from rest at a height H_0 on an inclined plane that makes an angle θ with the ground. When it reaches the bottom, it smoothly slides up a second incline, also at an angle θ with respect to the ground as shown. The coefficient of static friction between the block and the inclines is μ_s; the coefficient of kinetic friction between the block and the inclines is μ_k.

a) Find the minimum angle θ_{min} such that the block will be able to slide down the incline after being released from rest.

$\theta_{\text{min}} =$ $\boxed{}$

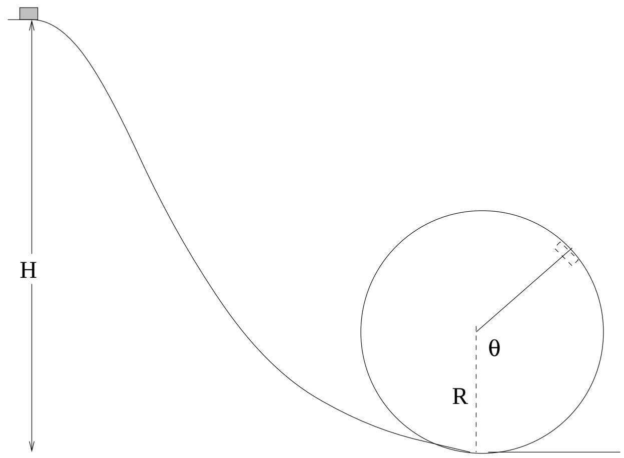
b) Suppose $\theta > \theta_{\text{min}}$. When the block is released from the initial height H_0, what height H_1 will it reach as it slides up the opposite incline before coming momentarily to rest?

$H_1 =$ $\boxed{}$

c) *Bonus question: 5 points*
Suppose that the coefficient of kinetic friction is very small so that it can slide back and forth many times. Approximately how many times will the block slide back and forth before it loses 1/2 of its initial energy?

$N_{\frac{1}{2}} =$ $\boxed{}$

Problem 159. problems-1/wme-pr-loop-the-loop-block.tex

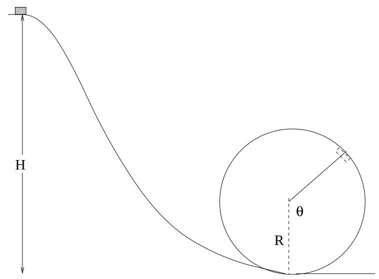


A block of mass M sits at the top of a frictionless loop-the-loop of height H.

a) Find the normal force exerted by the track when the mass is at an angle θ on the loop as shown.

b) Find the minimum height H such that the block loops the loop without coming off of the track.

Problem 160. problems-1/wme-pr-loop-the-loop-classic-example.tex



A block of mass M sits at the top of a frictionless hill of height H leading to a circular loop-the-loop of radius R.

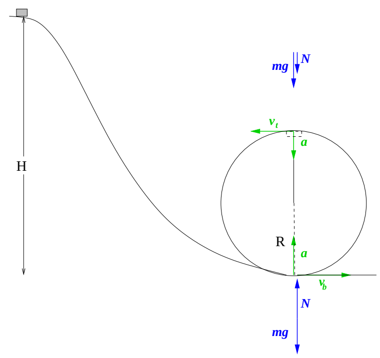
a) Find the minimum height $H_{\rm min}$ for which the block *barely* goes around the loop staying on the track at the top. (Hint: What is the condition on the normal force when it "barely" stays in contact with the track? This condition can be thought of as "free fall" and will help us understand circular orbits later, so don't forget it.).

   Discuss within your recitation group why your answer is a scalar number times R and how this *kind* of result is usually a good sign that your answer is probably right.

b) If the block is started at this position, what is the normal force exerted by the track at the *bottom* of the loop, where it is greatest?

   If you have ever ridden roller coasters with loops, use the fact that your apparent weight *is* the normal force exerted on you by your seat if *you* are looping the loop in a roller coaster and discuss with your recitation group whether or not the results you derive here are in accord with your experiences. If you haven't, consider riding one *aware* of the forces that are acting on you and how they affect your perception of weight and change your direction on your next visit to e.g. Busch Gardens to be, in a bizarre kind of way, a *physics assignment*. (Now c'mon, how many classes have you ever taken that assign riding roller coasters, even as an optional activity?:-)

Problem 161. problems-1/wme-pr-loop-the-loop-classic-solution.tex



A block of mass M sits at the top of a frictionless hill of height H leading to a circular loop-the-loop of radius R.

a) Find the minimum height H_{min} for which the block *barely* goes around the loop staying on the track at the top. (Hint: What is the condition on the normal force when it "barely" stays in contact with the track? This condition can be thought of as "free fall" and will help us understand circular orbits later, so don't forget it.).

   Discuss within your recitation group why your answer is a scalar number times R and how this *kind* of result is usually a good sign that your answer is probably right.

b) If the block is started at this position, what is the normal force exerted by the track at the *bottom* of the loop, where it is greatest?

   If you have ever ridden roller coasters with loops, use the fact that your apparent weight *is* the normal force exerted on you by your seat if *you* are looping the loop in a roller coaster and discuss with your recitation group whether or not the results you derive here are in accord with your experiences. If you haven't, consider riding one *aware* of the forces that are acting on you and how they affect your perception of weight and change your direction on your next visit to e.g. Busch Gardens to be, in a bizarre kind of way, a *physics assignment*. (Now c'mon, how many classes have you ever taken that assign riding roller coasters, even as an optional activity?:-)

Let us follow the hint and think about what is going on here. In this problem the block is *not* bound to the looping track. This means that when it goes over the top of the loop nothing is "holding it up." Like any other object not held up by anything, it must accelerate down with an acceleration g. Yet experience with toy cars, roller coasters, and strings tells us that if it is going fast enough it will not fall off the track. The reason is that going around a circular track does involve an acceleration, towards the center of the circle, of magnitude v^2/R where v is the speed. We will reproduce this in the last problem on this set. If this acceleration is at least g then at the top of the track the block can be in free fall without leaving the track. If the speed is higher, the acceleration required to complete the circle will be higher than g. This means if the track broke, the block would in fact fly off *above* the circular trajectory. This is prevented by a normal force applied by the track. The point of all this is that if the block is moving too slowly around the loop it will leave the track. As the speed is reduced past this minimum, the first failure to stay on the track will occur at the very top of the loop. This is intuitively clear, we will work it out in detail in another problem.

To turn these words into equations, consider applying Newton's second law to the block at the instant when it is at the apex of the looping track, moving (horizontally, to the left) at a speed v_t. The figure indicates forces, velocity, and acceleration at this point, including the initially unknown normal force applied by the track. Newton's second law then reads

$$\mathbf{F} = -mg\hat{\mathbf{y}} - N\hat{\mathbf{y}} = m\mathbf{a} \ . \tag{5.3}$$

In order for the block to continue its circular motion along the track this downward vertical acceleration must be equal to the centripetal acceleration, directed downward towards the center of the circle, i.e. we have $\mathbf{a} = -mv_t^2/R\,\hat{\mathbf{y}}$. This requires

$$N = \frac{mv_t^2}{R} - mg \ . \tag{5.4}$$

Since $N \geq 0$ we see that if the block is moving too slowly it will not stay on the track. The minimum speed needed to just maintain contact with the track at the top is the speed at which $N = 0$, i.e.

$$v_t^2 = gR \ . \tag{5.5}$$

Now our job is to find how high the initial ramp must be in order for the block to reach the top of the look with this speed. Of course, as it goes down the ramp the block accelerates under the influence of gravity, but as it goes up the looping track it slows down under the same influence. Since all forces acting on the block are conservative (gravity) or do no work at all (the normal forces, which are everywhere perpendicular to the direction of motion) the total mechanical energy of the block is conserved throughout its travels along our track. We can thus relate its speed at the top of the loop to the height of the ramp where it was released from rest by equating the total mechanical energy in both configurations, including kinetic and gravitational potential energy. Setting $U_g = 0$ at the base of the loop to determine the irrelevant additive constant we have for these initial and final configurations the following expressions for total energy

$$\begin{aligned} E_i &= mgH \\ E_t &= \frac{mv_t^2}{2} + mg\,2R \ . \end{aligned} \tag{5.6}$$

Setting these equal to each other we find that the speed of the block at the top of the loop is determined by H, R as

$$v_t^2 = g(2H - 4R) \ . \tag{5.7}$$

The minimum H needed to clear the loop will lead to a value for v equal to the minimal value (5.5) i.e.

$$\begin{aligned} g(2H - 4R) &= gR \\ H &= \frac{5}{2}R \ . \end{aligned} \tag{5.8}$$

As predicted, we find a number, determined by various geometric factors, times R. This makes sense, because R is the only parameter in the problem with the right dimensions, length. So to determine a length H related in some way to R we expect to find a result like this. That it makes sense does not make it trivial. Neglecting friction, we predict that to double the height of the loop you must double the height of your ramp. And we could have predicted that just using this kind of dimensional reasoning, without doing any calculations at all!

We now want to find the normal force applied by the track at the bottom of the loop when the block is released from this height. The figure indicates forces, velocity, and acceleration at this instant. The total force on the block is now

$$\mathbf{F} = -mg\hat{\mathbf{y}} + N\hat{\mathbf{y}} \ . \tag{5.9}$$

Once more the net acceleration is vertical. In order to be moving around a circle at speed v_b we must have $\mathbf{a} = v_b^2/R\hat{\mathbf{y}}$ directed upward towards the center of the circle. This requires

$$N = mg + m\frac{v_b^2}{R} \ . \tag{5.10}$$

This makes sense. At the bottom, in addition to holding up the block's weight, the track must apply additional normal force to provide the centripetal acceleration.

To find the value of N we again use energy conservation to find v_b. At the bottom of the loop the gravitational potential energy vanishes but the conserved total energy is equal to its value at the top of the ramp (or at any other time during the block's travels). This means

$$E_b = \frac{mv_b^2}{2} = E_i = mgH \ , \tag{5.11}$$

or

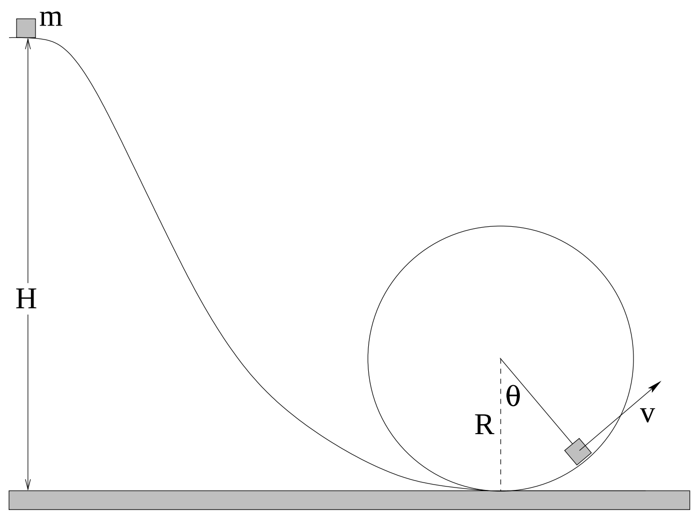
$$v_b^2 = 2gH = 5gR \ , \tag{5.12}$$

where the last equality used (5.8). Inserting this we find

$$N = mg + 5mg = 6mg \ . \tag{5.13}$$

If you are sitting in this block and travelling at the minimal speed needed to traverse the loop, then at the top of the loop (where $N = 0$ you will just barely touch your seat. At the bottom your seat needs to apply six times your weight to your bottom to accelerate you up. Your diaphragm needs to apply six times the force it is accustomed to to hold up the contents of your abdominal cavity, and most importantly your heart must lift your blood out of your feet against an apparent $6g$ of gravity. This is why pilots of WWII planes that first achieved high speeds had trouble with blacking out. Their hearts failed to overcome the increased apparent gravity at the bottom of maneuvers and their oxygen-starved brains lost consciousness. The remedy at the time was inserting wood blocks on the pedals, to raise their feet and put them in a cramped position amenable to tightening their abdominal muscles to restrict blood flow. Modern pressure suits simply squeeze the lower extremities in any configuration so that blood flow is unaffected by acceleration.

Problem 162. problems-1/wme-pr-loop-the-loop-difficult.tex

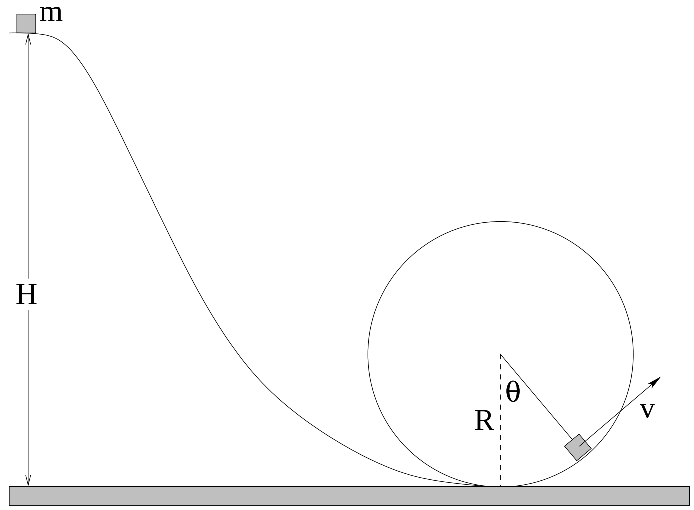


A block of mass m sits at the top of a frictionless hill of height H. It slides down and around a loop-the-loop of radius R to an angle θ as shown.

a) Find the magnitude of the normal force as a function of the angle θ.

b) From this, deduce an expression for the angle θ_0 at which the block will *leave* the track if the block is started at a height $H = 2R$.

Problem 163. problems-1/wme-pr-loop-the-loop-difficult-soln.tex



A block of mass m sits at the top of a frictionless hill of height H. It slides down and around a loop-the-loop of radius R to an angle θ as shown.

a) Find the magnitude of the normal force as a function of the angle θ.

b) From this, deduce an expression for the angle θ_0 at which the block will *leave* the track if the block is started at a height $H = 2R$.

WME theorem:

$$mgH = mgR(1 - \cos\theta) + \frac{1}{2}mv^2$$

or

$$mv^2 = 2mgH - 2mgR + 2mgR\cos\theta$$

Also, in the radial direction:

$$N - mg\cos\theta = \frac{mv^2}{R}$$

or

$$\boxed{N = mg\cos\theta + 2mg\frac{H}{R} - 2mg + 2mg\cos\theta = 3mg\cos\theta + 2mg\frac{H}{R} - 2mg}$$

The block leaves the track when $N \to 0$. When $H = 2R$,

$$3mg\cos\theta_0 + 4mg - 2mg = 3\cancel{mg}\cos\theta_0 + 2\cancel{mg} = 0$$

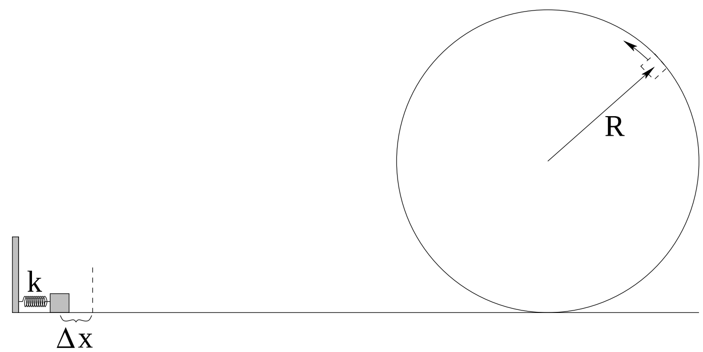
Solving this,

$$\cos\theta_0 = -\frac{2}{3}$$

or

$$\boxed{\theta_0 = \cos^{-1}\left(-\frac{2}{3}\right)}$$

Problem 164. problems-1/wme-pr-loop-the-loop-from-spring.tex



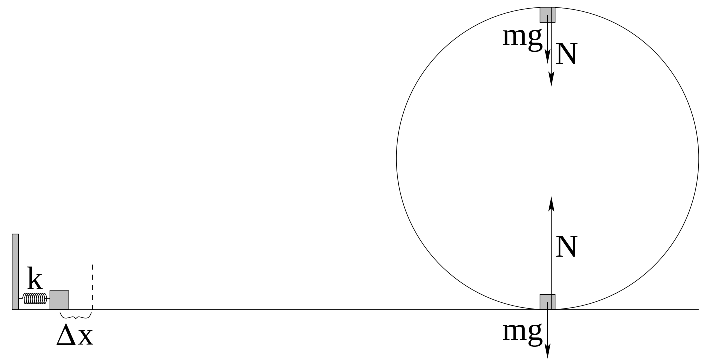
A block of mass M sits in front of a spring with spring constant k compressed by an amount Δx on a frictionless track leading to a circular loop-the-loop of radius R as shown.

a) ***Draw two force diagrams***, one with the block at the top of the loop and one with the block at the bottom of the loop. Clearly label all forces, including ones that you might set to zero or ignore. Use these force diagrams to help answer the following two questions.

b) Find the minimum value of Δx for which the block *barely* goes around the loop staying on the track at the top.

c) If the block is started at this position, what is the normal force exerted by the track at the *bottom* of the loop, where it is greatest?

Problem 165. problems-1/wme-pr-loop-the-loop-from-spring-soln.tex



A block of mass M sits in front of a spring with spring constant k compressed by an amount Δx on a frictionless track leading to a circular loop-the-loop of radius R as shown.

a) ***Draw two force diagrams***, one with the block at the top of the loop and one with the block at the bottom of the loop. Clearly label all forces, including ones that you might set to zero or ignore. Use these force diagrams to help answer the following two questions.

b) Find the minimum value of Δx for which the block *barely* goes around the loop staying on the track at the top.

c) If the block is started at this position, what is the normal force exerted by the track at the *bottom* of the loop, where it is greatest?

Solution: For a) see above. The only two forces at the top and bottom are the weight of the block and the normal force.

For b), we have to use energy conservation to relate Δx to the speed at the top of the loop, write Newton's Second Law at the top of the loop in the limit that $N \to 0$ ("barely" loops the loop on the track) and using circular motion kinematics:

$$E_i = \frac{1}{2}k\Delta x^2 = mg2R + \frac{1}{2}mv_t^2 = E_f \quad \Rightarrow \quad mv_t^2 = k\Delta x^2 - 4mgR$$

N2:

$$mg + N = \frac{mv_t^2}{R} \quad \Rightarrow \quad mg = \frac{k\Delta x^2}{R} - 4mg$$

$$\boxed{\Delta x = \sqrt{\frac{5mgR}{k}}}$$

For c) we can (again) use energy conservation, N2, and circular motion kinematics (plus Δx from part b)):

$$E_i = \frac{1}{2}k\Delta x^2 = \frac{5mgR}{2} = \frac{1}{2}mv_b^2 = E_f \quad \Rightarrow \quad mv_b^2 = 5mgR$$

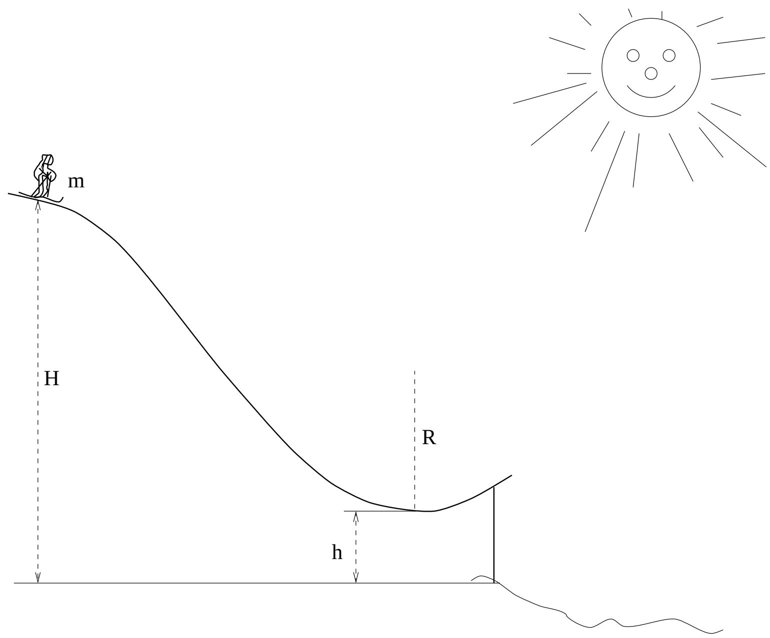
N2:

$$N - mg = \frac{mv_b^2}{R} \quad \Rightarrow \quad N = 5mg + mg = 6mg$$

or

$$\boxed{N = 6mg}$$

Problem 166. problems-1/wme-pr-loop-the-loop-skier-ambitious-amy.tex



Ambitious Amy has a mass m. She skis from initial rest down the (frictionless) ski slope of height H to a ski ramp whose radius of curvature R and whose lowest point is h above the ground (as shown).

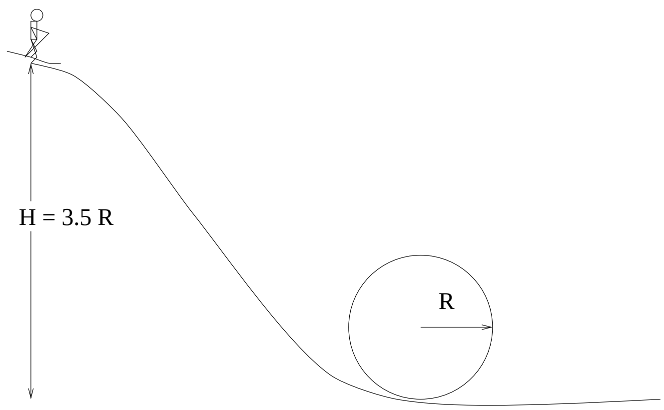
Amy's leg strength must oppose her apparent weight at the bottom of the jump. Is she strong enough? It would be good to know how strong she has to be so that she can work on leg presses if need be before trying the actual jump. So (in terms of the given quantities m, g, R, H, h) :

a) How fast is Amy going when she reaches the lowest point in the curved jump?

b) What is the *total* force that must be directed towards the center of the circle of motion at that point (again, in terms of the given).

c) Using your knowledge of the actual forces acting on her that have to sum to this force, determine her "apparent weight" – the peak force she has to *push down on the ground* with her skis with in order to stay on the circular curve.

Problem 167. problems-1/wme-pr-loop-the-loop-skier.tex

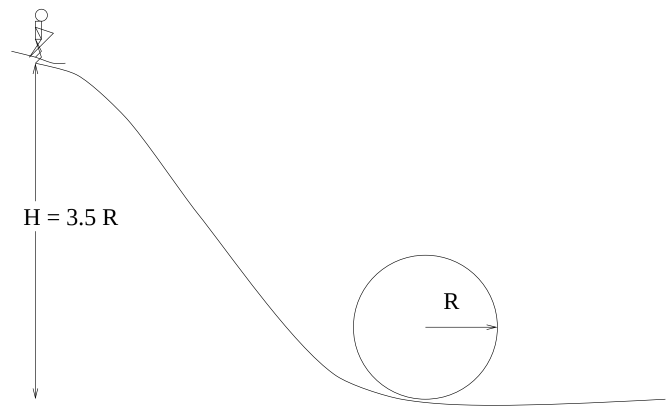


A skier of mass m at an exhibition wants to loop-the-loop on a special (frictionless) ice track of radius R set up as shown. Suppose $H = 3.5R$. All answers should be given in terms of g, m and R. (Note that the skier is really much shorter than R; the picture is not drawn strictly to scale for ease of viewing.)

a) What is her apparent "weight" (the normal force exerted by the track on her skis) when she is upside down at the top of the loop-the-loop?

b) What is her maximum apparent "weight" on the loop-the-loop and where (at what point on the loop-the-loop track) does it occur? Indicate the position on the figure.

Problem 168. problems-1/wme-pr-loop-the-loop-skier-weight.tex

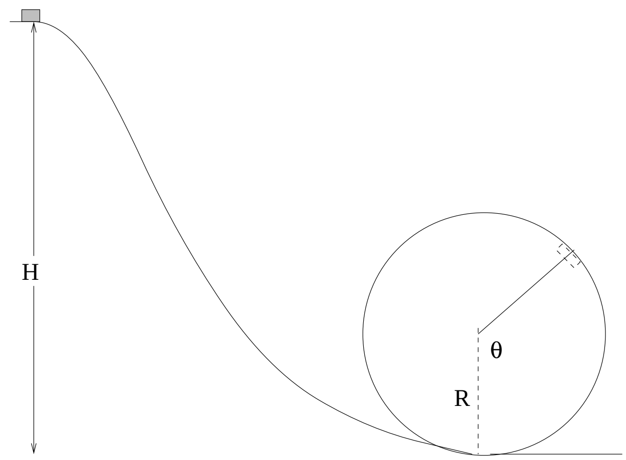


A skier of mass m at an exhibition wants to loop-the-loop on a special (frictionless) ice track of radius R set up as shown. Suppose $H = 3.5R$. All answers should be given in terms of g, m and R. (Note that the picture is not drawn strictly to scale for ease of viewing.)

a) What is her apparent weight (the normal force exerted by the track on her skis) when she is upside down at the top of the loop-the-loop? If she closed her eyes, what direction would she think of as "down"?

b) What is her maximum apparent "weight" on the loop-the-loop and where (at what point on the loop-the-loop track) does it occur? Indicate the position on the figure.

Problem 169. problems-1/wme-pr-loop-the-loop.tex

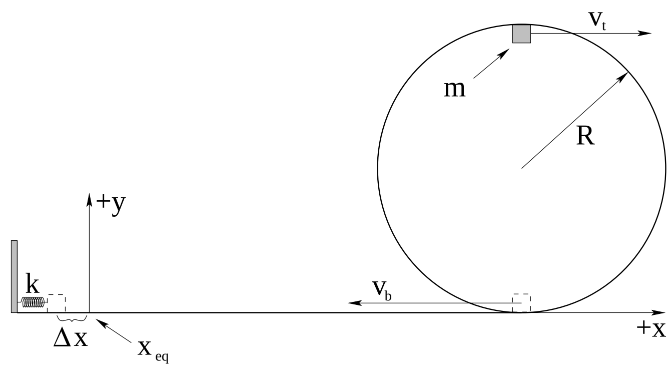


A block of mass M sits at the top of a frictionless hill of height H leading to a circular loop-the-loop of radius R.

a) Find the minimum height $H_{\rm min}$ for which the block *barely* goes around the loop staying on the track at the top.

b) If the block is started at this position, what is the normal force exerted by the track at the *bottom* of the loop, where it is greatest?

Problem 170. problems-1/wme-pr-loop-the-loop-to-spring.tex



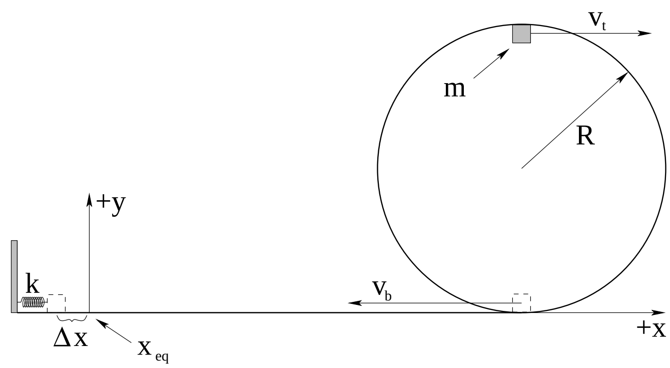
A block of mass m is travelling to the right at the top of a frictionless circular loop-the-loop track of radius R, travelling at speed v_t to the right as shown. v_t is large enough that the mass remains on the track at the top. It then slides around the track to the bottom, slides across the (frictionless) ground, and hits a spring with spring constant k which slows it to rest after the spring has compressed a distance Δx from its initial equilibrium length.

a) What is the speed v_b at the bottom of the circular loop?

b) What is the normal force exerted by the track at the bottom just before/as it leaves the circular loop?

c) By what distance Δx is the spring compressed at the instant the block comes to rest?

Problem 171. problems-1/wme-pr-loop-the-loop-to-spring-soln.tex



A block of mass m is travelling to the right at the top of a frictionless circular loop-the-loop track of radius R, travelling at speed v_t to the right as shown. v_t is large enough that the mass remains on the track at the top. It then slides around the track to the bottom, slides across the (frictionless) ground, and hits a spring with spring constant k which slows it to rest after the spring has compressed a distance Δx from its initial equilibrium length.

a) What is the speed v_b at the bottom of the circular loop?

$$E_i = \frac{1}{2}mv_t^2 + 2mgR = \frac{1}{2}mv_b^2 = E_f$$

so

$$\boxed{v_b = \sqrt{v_t^2 + 4gR}}$$

b) What is the normal force exerted by the track at the bottom just before/as it leaves the circular loop?

$$N - mg = \frac{mv_b^2}{R} \Rightarrow N = mg + \frac{mv_b^2}{R}$$

From the first equation:

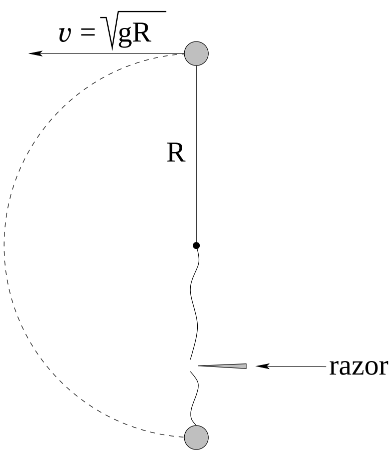
$$mv_b^2 = mv_t^2 + 4mgR \Rightarrow \boxed{N = 5mg + \frac{mv_t^2}{R}}$$

c) By what distance Δx is the spring compressed at the instant the block comes to rest?

Mechanical energy is conserved from the top of the loop to the fully compressed spring:

$$E_i = \frac{1}{2}mv_t^2 + 2mgR = \frac{1}{2}k\Delta x^2 = E_f \Rightarrow \boxed{\Delta x = \sqrt{\frac{m}{k}}\sqrt{v_t^2 + 4gR}}$$

Problem 172. problems-1/wme-pr-razor-cuts-loop-string-1.tex



A small ball of mass m (which can be treated as a point particle for this problem) is attached to a massless, unstretchable string whose other end is attached to a fixed, frictionless pivot. The ball swings in a vertical circle, with gravity acting downward as usual.

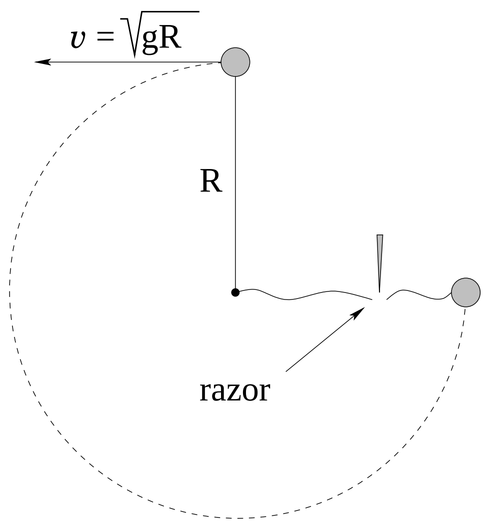
When the ball is at the top of the circle it has velocity $\sqrt{gR}$ to the left as shown, the minimum needed to keep the particle moving in a circle. After the ball has gone half way around the circle and the string is again vertical, a razor blade cuts the string. You can assume that the impulse delivered to the string by the very sharp razor is small enough that it can be ignored.

a) Find the velocity of the ball ***just before*** the string is cut.

b) Find the tension in the string ***just before*** the string is cut.

c) On the diagram, qualitatively show the path of the ball ***just after*** the string is cut.

Problem 173. problems-1/wme-pr-razor-cuts-loop-string.tex



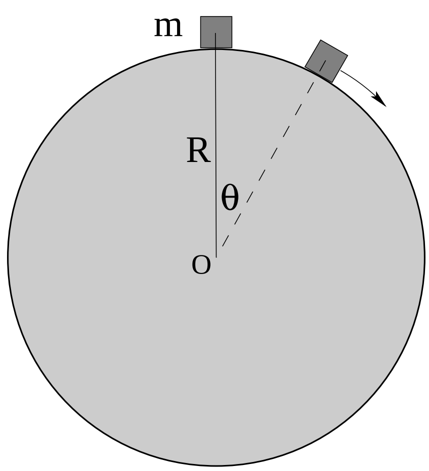
A small ball (which can be treated as a point particle for this problem) is attached to an Acme (massless, unstretchable) string whose other end is attached to a fixed, frictionless pivot. The ball swings in a vertical circle, with gravity acting downward as usual.

When the ball is at the top of the circle it has velocity $\sqrt{gR}$ to the left as shown. After the ball has gone three quarters of the way around and the string is horizontal, a razor blade cuts the string. You can assume that the impulse delivered to the string by the razor is small enough that it can be ignored.

a) Find the tension in the string ***just before*** it is cut.

b) On the diagram, show the path of the ball ***after*** the string is cut. Describe in words any features of the path that you intended to illustrate and be sure to indicate the maximum height you expect the ball to reach relative to the center of the circle of motion.

Problem 174. problems-1/wme-pr-sliding-off-a-cylinder-review.tex



In the figure to the left, a ***small*** (treat as a point mass) block of mass m is on top of a frictionless cylinder so that its center of mass is a distance R from the axis of the cylinder. It is given a nudge so that it slides with negligible initial speed down the side of the cylinder.

a) When its angular position is θ as shown, what is its speed (assuming that it is still on the cylinder)?

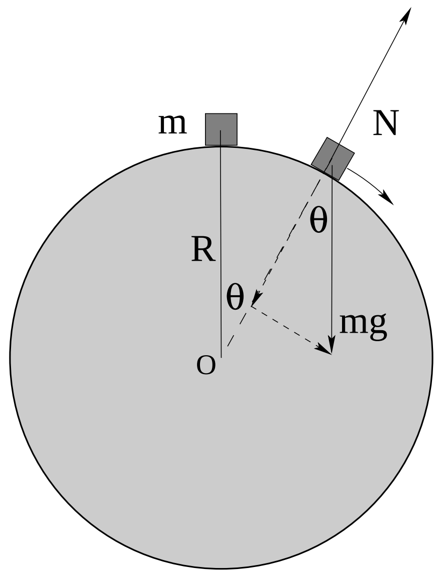
b) What is the magnitude of the normal force exerted on the block by the cylinder at this point?

c) For what value of θ will the block leave the cylinder?

Express your answers in m, R, g and θ.

$v(\theta) = \boxed{}$ $N(\theta) = \boxed{}$ $\theta_{\text{critical}} = \boxed{}$

Problem 175. problems-1/wme-pr-sliding-off-a-cylinder-review-soln.tex



In the figure to the left, a ***small*** (treat as a point mass) block of mass m is on top of a frictionless cylinder so that its center of mass is a distance R from the axis of the cylinder. It is given a nudge so that it slides with negligible initial speed down the side of the cylinder.

a) When its angular position is θ as shown, what is its speed (assuming that it is still on the cylinder)?

b) What is the magnitude of the normal force exerted on the block by the cylinder at this point?

c) For what value of θ will the block leave the cylinder?

Express your answers in m, R, g and θ.

$v(\theta) = \boxed{\sqrt{2gR(1-\cos\theta)}}$ $N(\theta) = \boxed{(3\cos\theta - 2)mg}$ $\theta_{\text{critical}} = \boxed{\cos^{-1}\left(\frac{2}{3}\right)}$

Solution: For a) use ***Energy Conservation***. Let's set $U_g = 0$ at the top of the slope. As it slides, it falls by a vertical height $H = R(1-\cos\theta)$ (why?). Then $E_i = U_i + K_i = 0$ Hence:

$$E_f = \frac{1}{2}mv^2 - mgR(1-\cos\theta) = 0 = E_i \quad \Rightarrow \quad \boxed{v(\theta) = \sqrt{2gR(1-\cos\theta)}}$$

As a quick check, this correctly predicts $v = \sqrt{2gR}$ at $\theta = \pi/2$ and $v = 0$ at $\theta = 0$, and has the right units.

For b) we need to write Newton's Second Law for the component of $\vec{F}$ or $\vec{a}$ ***towards the center of the circle***. As long as one uses this component only, any coordinate frame above can be made to work. That is:

$$F_c = mg\cos\theta - N = ma_c = \frac{mv^2}{R} \quad \text{and} \quad mv^2 = 2mgR(1-\cos\theta) \quad \Rightarrow$$

$$\boxed{N(\theta) = mg\cos\theta - 2mg(1-\cos\theta) = (3\cos\theta - 2)mg}$$

For c), note that $N \to 0$ at the *specific* angle θ_{critical} where it comes off of the cylinder. From $N(\theta)$ in b), θ_{critical} is then easily found:

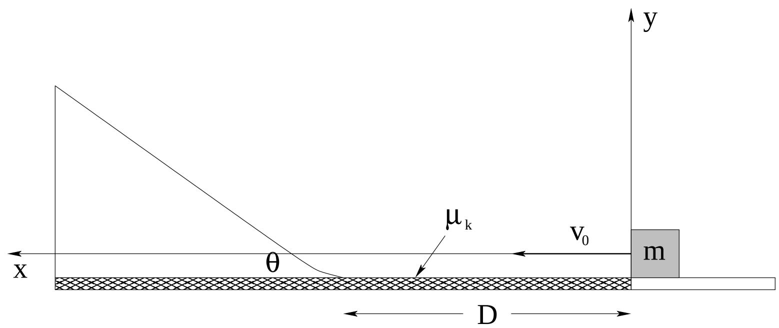
$$3mg\cos\theta_a = 2mg$$

or:

$$\boxed{\theta_{\text{critical}} = \cos^{-1}\left(\frac{2}{3}\right) = 48.2^\circ}$$

Note that because the use of calculators is discouraged, this angle is perfectly well and uniquely (enough) expressed as an inverse cosine, but I put down the angle in degrees just for fun.

Problem 176. problems-1/wme-pr-smooth-inclined-plane-friction-table-2.tex

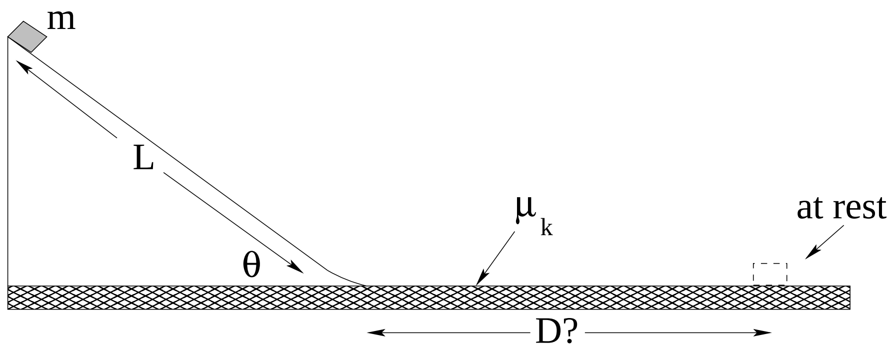


A block of mass m is given a push so that it begins sliding at a speed v_0 from the right to the left over a rough surface of length D leading up to a smooth (frictionless) incline. The incline makes an angle θ with the horizontal as shown. The coefficient of friction between the block and the rough surface is μ_k.

a) What is the minimum speed $v_{0,\mathrm{min}}$ the block must have at the right-hand end of the rough surface such that the block will reach the bottom of the incline a distance D away?

b) Assuming that the block is travelling at some $v_0 > v_{0,\mathrm{min}}$ when it starts at the right-hand end of the rough patch as drawn, how high (to what maximum height y_{max}) will the block slide up the incline (use the coordinate system given)?

Problem 177. problems-1/wme-pr-smooth-inclined-plane-friction-table.tex

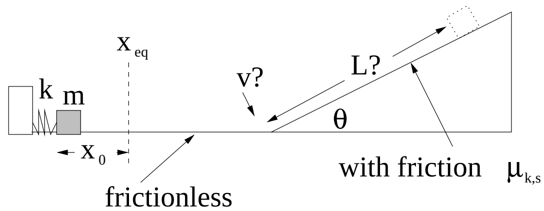


A block of mass m slides down a ***smooth*** (frictionless) incline of length L that makes an angle θ with the horizontal as shown. It then reaches a ***rough*** surface with a coefficient of kinetic friction μ_k.

a) How fast is the block going as it reaches the bottom of the incline?

b) What distance D does the block slide across the rough surface before coming to rest?

Problem 178. problems-1/wme-pr-spring-to-inclined-plane-friction-numbers.tex

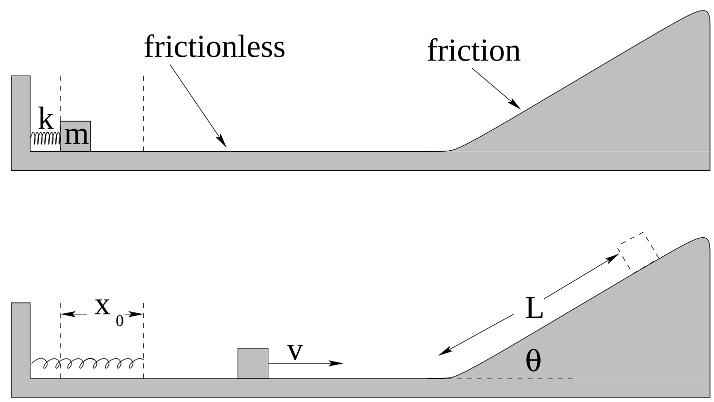


A block of mass $M = 1$ kg is propelled by a spring with spring constant $k = 10$ N/m onto a smooth (frictionless) track. The spring is initially compressed a distance of 0.5m from its equilibrium configuration ($x_i - x_0 = 0.5$ m). At the end of the track there is a rough inclined plane at an angle of 45∘ with respect to the horizontal and with a coefficient of kinetic friction $\mu_k = 0.5$.

a) How far up the incline will the block slide before coming to rest (find H_f)?

b) The coefficient of static friction is $\mu_s = 0.7$. Will the block remain at rest on the incline? If not, how fast will it be going when it reaches the bottom again?

Problem 179. problems-1/wme-pr-spring-to-inclined-plane-friction-icp.tex

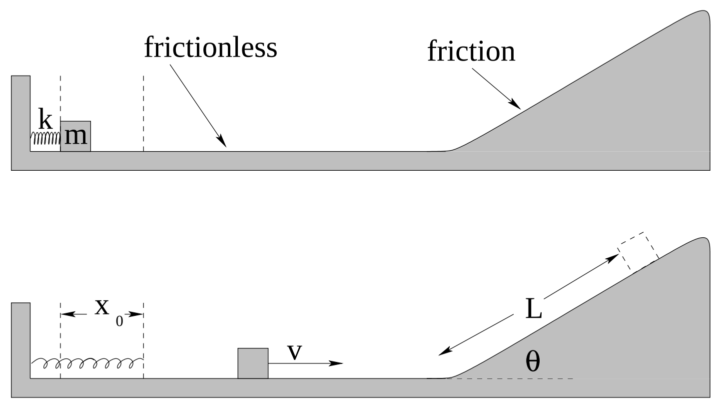


A spring with spring constant k is initially compressed a distance of x_0 from its equilibrium configuration as shown in the top diagram above. A block of mass m is placed against it and the spring is released, propelling it forward on a smooth (frictionless) track. At the end of the track there is a rough inclined plane at an angle of θ with respect to the horizontal and with a coefficient of kinetic friction μ_k.

a) How far up the incline will the block slide before coming momentarily to rest (find L_{max})?

b) Suppose the coefficient of static friction is μ_s. Find the maximum angle θ_{max} such that the block will remain at rest at the top of the incline instead of sliding back down.

Problem 180. problems-1/wme-pr-spring-to-inclined-plane-friction-icp-soln.tex



A spring with spring constant k is initially compressed a distance of x_0 from its equilibrium configuration as shown in the top diagram above. A block of mass m is placed against it and the spring is released, propelling it forward on a smooth (frictionless) track. At the end of the track there is a rough inclined plane at an angle of θ with respect to the horizontal and with a coefficient of kinetic friction μ_k.

a) How far up the incline will the block slide before coming momentarily to rest (find $L_{\max}$)?

The easiest way to answer this is to use the non-conservative work, mechanical energy theorem:

$$W_{nc} = \Delta E$$

The (negative!) non-conservative work is done by kinetic friction on the incline. Hence:

$$W_{nc} = -\mu_k mg\cos(\theta)L_{\max} = mgL_{\max}\sin(\theta) - \frac{1}{2}kx_0^2 = \Delta E_{\text{mech}}$$

One can then solve for $L_{\max}$. First we rearrange:

$$\frac{1}{2}kx_0^2 == mg\left(\sin(\theta) + \mu_k\cos(\theta)\right)L_{\max}$$

and then we get:

$$\boxed{L_{\max} = \frac{kx_0^2}{2mg\left(\sin(\theta) + \mu_k\cos(\theta)\right)}}$$

b) Suppose the coefficient of static friction is μ_s. Find the maximum angle $\theta_{\max}$ such that the block will remain at rest at the top of the incline instead of sliding back down.

We use force balance, assuming that static friction is sufficient to prevent the slide:

$$mg\sin(\theta) - f_s = 0$$

This means that:

$$f_s = mg\sin(\theta)$$

We know that:

$$f_s < F_s = \mu_s N$$

(where F_s is the force at which static friction *fails*). $N = mg\cos(\theta)$, so:

$$f_s = mg\sin(\theta) < \mu_s mg\cos(\theta)$$

or

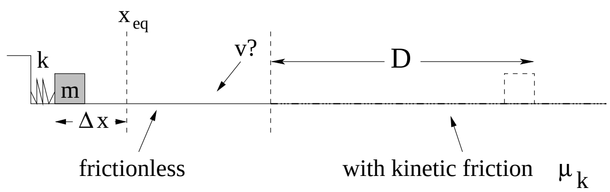
$$\mu_s > \tan(\theta)$$

Take the inverse tangent of both sides to get:

$$\boxed{\theta_{\text{max}} = \tan^{-1}(\mu_s) > \theta}$$

(where θ_{max} is the angle at which it just starts to slide).

Problem 181. problems-1/wme-pr-spring-to-plane-friction.tex



A block of mass m sits against a spring with spring constant k that is initially compressed a distance Δx. At some time the block is released and slides across a frictionless surface until it reaches a *rough* surface with a coefficient of kinetic friction μ_k as shown.

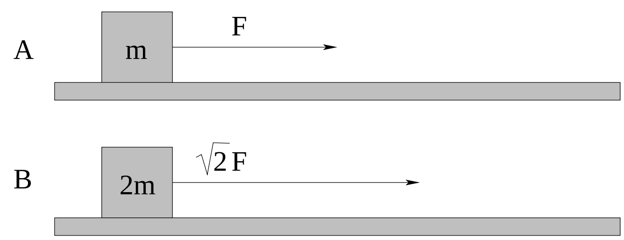
a) How fast is the block going as it leaves the spring at $x_{\rm eq}$?

b) What distance D down the rough surface does the block slide before coming to rest?

5.3 Power

5.3.1 Multiple Choice Problems

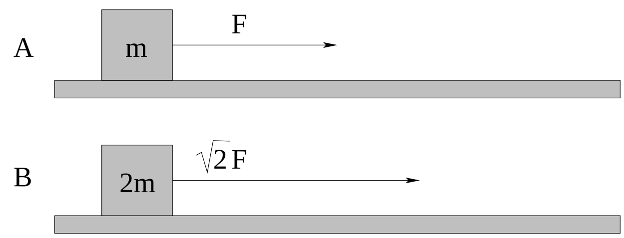
Problem 182. problems-1/power-mc-two-blocks-hard.tex



In the figures **A** and **B** above a force of magnitude F is applied to mass m and a force of magnitude $\sqrt{2}F$ is applied to mass 2m as shown, where both masses are sitting ***initially at rest*** on a frictionless table. For all values of $t > 0$, identify the true statement in the list below:

- ☐ The power provided to block m in **A** is larger than that provided to block $2m$ in **B** at any time $t > 0$.
- ☐ The power provided to the block $2m$ in **B** is larger than that provided to block m in **A** at any time $t > 0$.
- ☐ The power provided to both blocks is identical at any time $t > 0$.
- ☐ It is impossible to tell from the information given which block receives more power from the forces.

Problem 183. problems-1/power-mc-two-blocks-hard-soln.tex



In the figures **A** and **B** above a force of magnitude F is applied to mass m and a force of magnitude $\sqrt{2}F$ is applied to mass 2m as shown, where both masses are sitting ***initially at rest*** on a frictionless table. For all values of $t > 0$, identify the true statement in the list below:

- ☐ The power provided to block m in **A** is larger than that provided to block $2m$ in **B** at any time $t > 0$.
- ☐ The power provided to the block $2m$ in **B** is larger than that provided to block m in **A** at any time $t > 0$.
- ☒ The power provided to both blocks is identical at any time $t > 0$.
- ☐ It is impossible to tell from the information given which block receives more power from the forces.

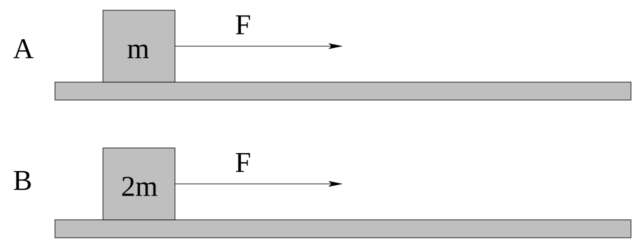
Solution: The fundamental **(scaling)** principle is (at any given time, for constant force in the direction of $\vec{\boldsymbol{v}}$):

$$P = \frac{dW}{dt} = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} = F_i v_i \text{ where } v_i = a_i t = \frac{F_i}{m} t \quad \Rightarrow \quad P_i = \frac{F_i^2}{m_i} t \quad \text{(i = A,B)}$$

Thus:

$$P_A = \frac{F^2}{m} t \quad \text{versus} \quad \boxed{P_B = \frac{(\sqrt{2}F)^2}{2m} t = \frac{F^2}{m} t = P_A} \ \textbf{(!)}$$

Problem 184. problems-1/power-mc-two-blocks.tex



In the figures **A** and **B** above an identical magnitude of force is applied to two masses m and $2m$ respectively, sitting ***initially at rest*** on a frictionless table. For all values of $t > 0$ identify the true statement in the list below:

a) The power provided to both blocks is identical throughout this time, and they end up with the same final kinetic energy at time t.

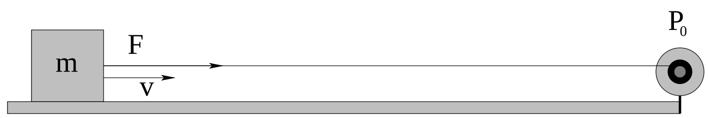
b) The power provided to block m in **A** is larger than that provided to block $2m$ in **B**, and mass m in **A** ends up with more kinetic energy at time t.

c) The power provided to the block $2m$ in **B** is larger than that provided to block m in **A**, but block m in **A** travels further in time t.

d) It is impossible to tell from the information given which block receives more power from the forces.

5.3.2 Regular Problems

Problem 185. problems-1/power-pr-constant-power-v-of-t.tex



In the figure above, a mass m is pulled along on a frictionless table by a motor with ***constant power*** $\boldsymbol{P_0}$. At the instant shown, the mass has been previously accelerated to a speed v towards the motor.

a) Find F as a function of P_0 and v (in the direction of the motor).

b) Write Newton's second law for the mass m in terms of your answer to a), using $a = dv/dt$ for the acceleration.

c) Solve the equation of motion you get for $v(t)$, assuming that $v(0) = 0$.

d) Qualitatively sketch what you expect to get for $v(t)$ (or what you did get in the previous section). Note that you can do this one even if you fail to do the integral correctly, if you think about what happens to the force as the speed gets bigger and bigger.

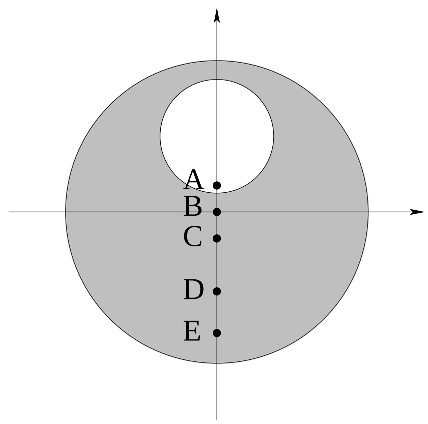
Chapter 6

Many Particle Systems

6.1 The Center of Mass

6.1.1 Multiple Choice Problems

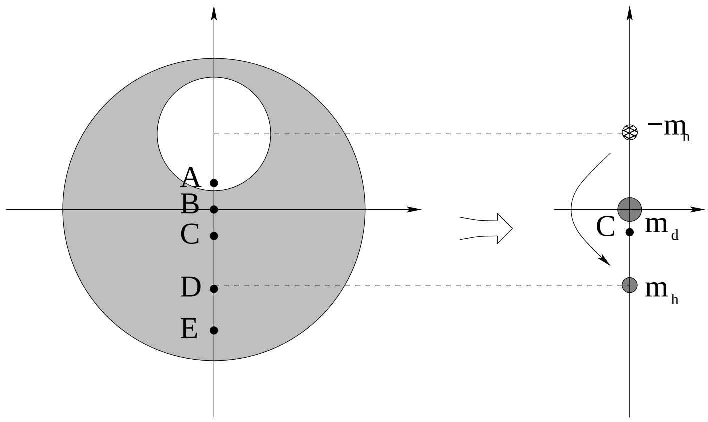
Problem 186. problems-1/center-of-mass-mc-disk-with-hole-icp.tex



A uniform circular disk has a circular hole cut out of it as shown above. Which letter represents the best estimate for the position of its new center of mass?

A **B** **C** **D** **E**

Problem 187. problems-1/center-of-mass-mc-disk-with-hole-icp-soln.tex



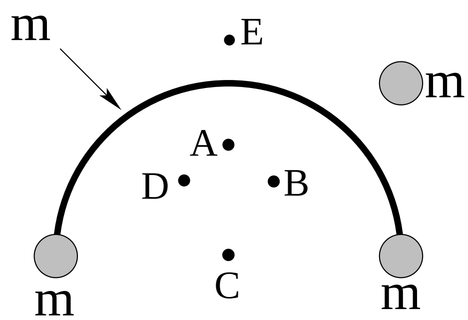
A uniform circular disk has a circular hole cut out of it as shown above. Which letter represents the best estimate for the position of its new center of mass?

Solution: Recall that the center of mass of a collection of objects can be evaluated using the mass of the objects and locating the objects "at" their own centers of mass as if they were point masses. In this case, we can treat the hole as the *removal* of a mass centered at the middle of the hole, or the "addition" of a negative mass at that position to add to the mass that was already there and produce a (mathematical) "hole".

To put it another way, the problem above is equivalent to the problem given on the right, where m_d is the mass of the disk, m_h is the mass removed to make the hole, and $m_d > m_h$ (by inspection). We can handle the idea of "negative mass" at a positive y-position by changing it mentally to a "positive mass" at the same negative y-position as indicated by the curved arrow. The center of mass ***must*** be lower than the center of disk because of the hole, but not TOO low. D is at the (reflected) position of the center of the hole, which is not possible because the mass removed creating the hole m_h is less than the total mass m_d. E is out of the question. C is obviously the best (and only possible) choice.

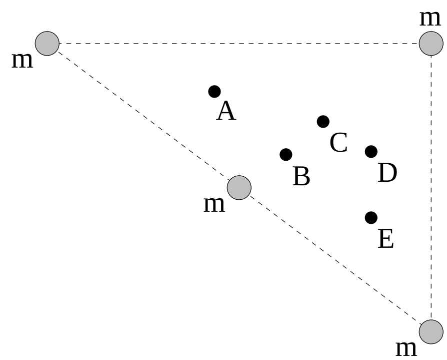
A **B** **C** **D** **E**

Problem 188. problems-1/center-of-mass-mc-particles-1.tex



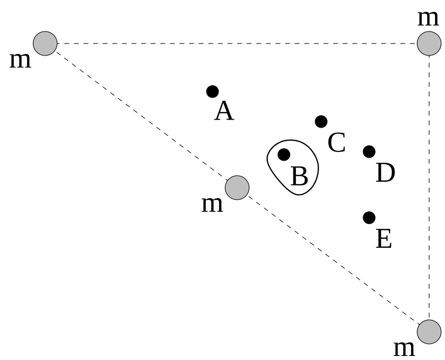
A collection of four equal masses m (including the uniform half-circle of wire) is shown above. Which of the points A-E is a plausible location of the center of mass?

Problem 189. problems-1/center-of-mass-mc-particles-2.tex



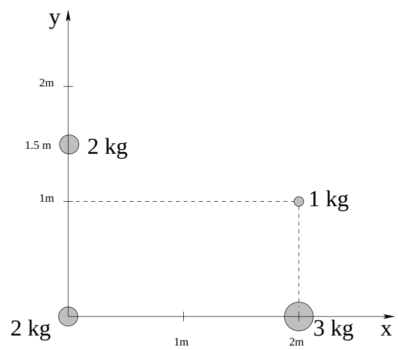
Pick (circle) the point A-E ***closest*** to the center of mass of the system above, given four equal masses m arranged as shown. Note that the dashed lines are drawn simply as a guide to the eye.

Problem 190. problems-1/center-of-mass-mc-particles-2-soln.tex



Rationale: Since the "extra" mass in the triangle of masses is on the midpoint of the line connecting the hypotenuse, and the center of mass of the triangle itself must (from symmetry) also lie on this line, the center of mass must lie on the line between the upper right corner and the mass on the hypoteneuse. This eliminates A, D and E. C is in the *center* of this line, but there is clearly *more mass* along the hypoteneuse than there is on the corner, so the center of mass has to be closer to the central mass on the hypoteneuse than the middle of the line. This leaves B. In actual fact, one expects it to be $2/3 + 1/9 = 7/9$ of the way along the line, starting from the corner, which is very close to where B is drawn.

Problem 191. problems-1/center-of-mass-mc-particles-3.tex



In the figure above, various given masses (in kilograms) are located at the positions shown shown. The center of mass of this system is at:

a) $x = 5/4m$, $y = 1/2m$

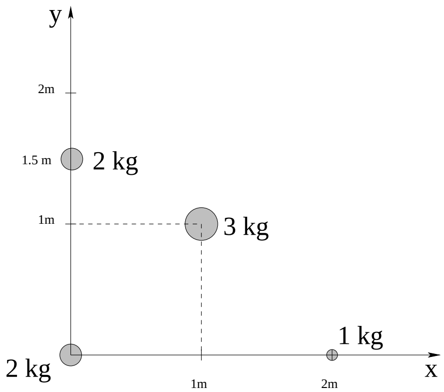
b) $x = 1m$, $y = 1/2m$

c) $x = 1/2m$, $y = 1/4m$

d) $x = 3/4m$, $y = 1m$

e) $x = 1/2m$, $y = 1m$

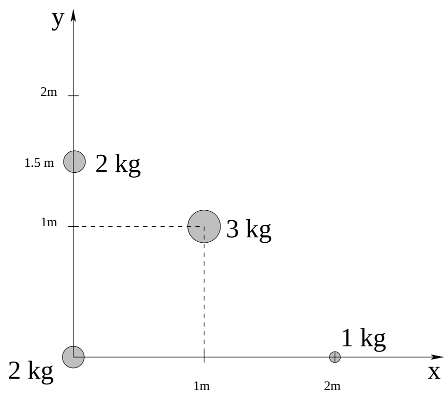
Problem 192. problems-1/center-of-mass-mc-particles-4.tex



In the figure above, various given masses (in kilograms) are located at the positions shown shown. The center of mass of this system is at:

- ☐ $x = 5/4$ m, $y = 1/2$ m
- ☐ $x = 1$ m, $y = 1/2$ m
- ☐ $x = 3/4$ m, $y = 5/8$ m
- ☐ $x = 5/8$ m, $y = 3/4$ m
- ☐ $x = 1/2$ m, $y = 1/4$ m

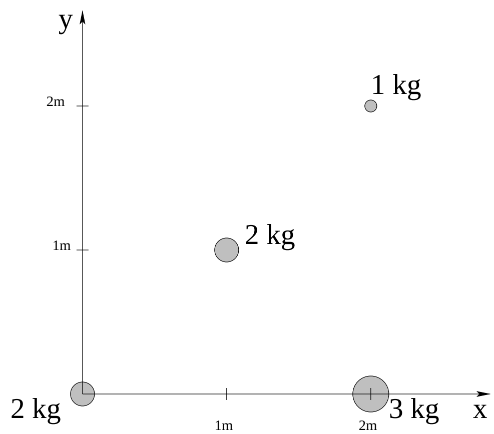
Problem 193. problems-1/center-of-mass-mc-particles-4-soln.tex



In the figure above, various given masses (in kilograms) are located at the positions shown shown. The center of mass of this system is at:

- ☐ $x = 5/4$ m, $y = 1/2$ m
- ☐ $x = 1$ m, $y = 1/2$ m
- ☐ $x = 3/4$ m, $y = 5/8$ m
- ☒ $x = 5/8$ m, $y = 3/4$ m
- ☐ $x = 1/2$ m, $y = 1/4$ m

Problem 194. problems-1/center-of-mass-mc-particles-icp.tex

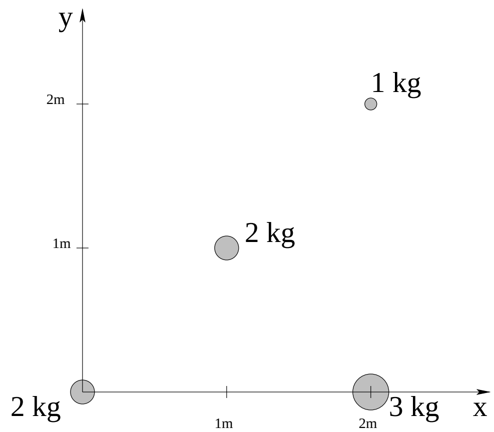


In the figure above, various given masses (in kilograms) are located the positions shown in the coordinate frame (in meters) above. The center of mass of this system is at:

a) $x = 5/4m$, $y = 1/2m$ b) $x = 1m$, $y = 1/2m$ c) $x = 3/2m$, $y = 3/4m$

d) $x = 5/4m$, $y = 3/4m$ e) $x = 3/2m$, $y = 1/2m$

Problem 195. problems-1/center-of-mass-mc-particles-icp-soln.tex



This is straightforward. We use:

$$x_{\rm cm} = \frac{1}{M_{\rm tot}} \sum_i m_i x_i \quad \text{and} \quad y_{\rm cm} = \frac{1}{M_{\rm tot}} \sum_i m_i y_i$$

independently. Obviously:

$$M_{\rm tot} = \sum_i m_i = 2 + 2 + 3 + 1 = 8 \text{ kg}$$

Then, using $m_i \cdot x_i$ order:

$$x_{\rm cm} = \frac{1}{8 \text{ kg}}(2 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 + 1 \cdot 2 = 10 \text{ kg-m}) = \frac{5}{4}\text{m}$$

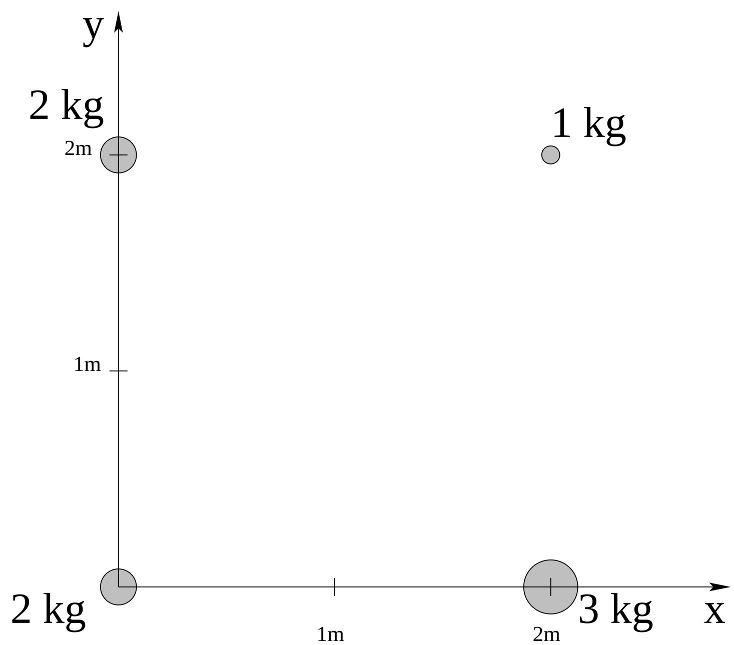
$$y_{\rm cm} = \frac{1}{8 \text{ kg}}(2 \cdot 0 + 2 \cdot 1 + 3 \cdot 0 + 1 \cdot 2 = 4 \text{ kg-m}) = \frac{1}{2}\text{m}$$

Thus the center of mass of this system is at:

a) $x = 5/4m$, $y = 1/2m$ b). $x = 1m$, $y = 1/2m$ c). $x = 3/2m$, $y = 3/4m$

d). $x = 5/4m$, $y = 3/4m$ e). $x = 3/2m$, $y = 1/2m$

Problem 196. problems-1/center-of-mass-mc-particles-6.tex



In the figure above, various given masses (in kilograms) are located at the positions shown above. The center of mass of this system is at:

a) $x = 5/4\ m$, $y = 3/4\ m$

b) $x = 1\ m$, $y = 3/4\ m$

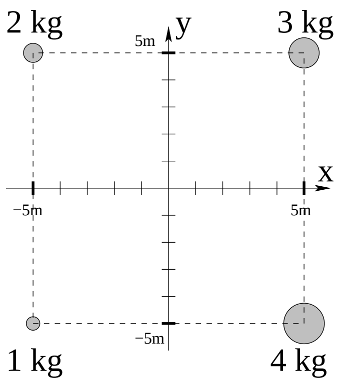
c) $x = 1\ m$, $y = 1\ m$

d) $x = 5/3\ m$, $y = 5/4\ m$

e) $x = 3/4\ m$, $y = 1\ m$

6.1.2 Short Answer Problems

Problem 197. problems-1/center-of-mass-sa-particles.tex

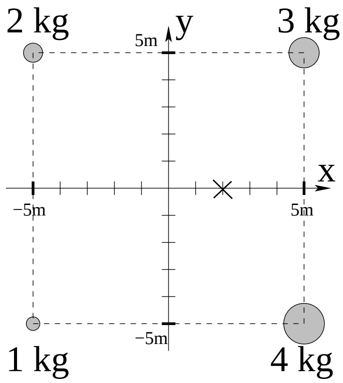


In the figure shown to the left, various given masses (in kilograms) are located at the corners of a square with sides of length 10 meters as shown. ***Using the provided coordinate frame,*** find the coordinates of the center of mass of this system and enter them in the boxes below ***and*** place an "x" on the the graph at its location.

$x_{\mathrm{CoM}} =$ $\boxed{}$

$y_{\mathrm{CoM}} =$ $\boxed{}$

Problem 198. problems-1/center-of-mass-sa-particles-soln.tex



In the figure shown to the left, various given masses (in kilograms) are located at the corners of a square with sides of length 10 meters as shown. ***Using the provided coordinate frame,*** find the coordinates of the center of mass of this system and enter them in the boxes below ***and*** place an "x" on the the graph at its location.

$\boldsymbol{x_{\mathrm{cm}}} =$ **2** meters

$\boldsymbol{y_{\mathrm{cm}}} =$ **0** meters

Solution: We simply use the formula for (mass-weighted average) for the center of mass:

$$\vec{\boldsymbol{x}}_{\mathrm{cm}} = \frac{1}{M_{\mathrm{tot}}} \sum_i m_i \vec{\boldsymbol{x}}_i$$

per coordinate. First find the total mass:

$$M_{\mathrm{tot}} = 1 + 4 + 2 + 3 = 10 \text{ kg}$$

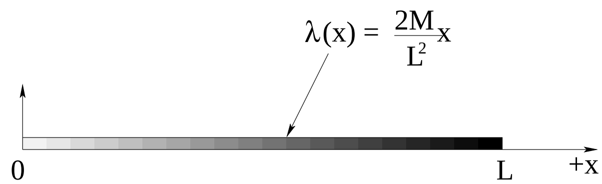
(that was pretty easy!) Next:

$$x_{\mathrm{cm}} = \frac{1}{10}\left(1 * (-5) + 2 * (-5) + 3 * 5 + 4 * 5\right) = \frac{20}{10} = 2 \text{ meters}$$

$$y_{\mathrm{cm}} = \frac{1}{10}\left(1 * (-5) + 4 * (-5) + 2 * 5 + 4 * 5\right) = \frac{0}{10} = 0 \text{ meters}$$

6.1.3 Regular Problems

Problem 199. problems-1/center-of-mass-pr-rod-variable-lambda-icp.tex



In the figure above a rod of total mass M and length L is portrayed (with shading that increases with mass density) that has been machined so that it has a mass per unit length that increases *linearly* along the length of the rod:

$$\lambda(x) = \frac{2M}{L^2}x$$

This might be viewed as a very crude model for the way mass is distributed in something like a human leg or a baseball bat. The rod is so thin that $y_{\rm cm} = z_{\rm cm} \approx 0$ by inspection.

a) verify that the total mass of the rod is indeed M for this mass distribution;

b) find $x_{\rm cm}$, the x-coordinate of the center of mass of the rod.

Problem 200. problems-1/center-of-mass-pr-rod-variable-lambda-icp-soln.tex

Solution: Draw a small chunk dx at general position x on the rod. Start with the litany: "The charge of the chunk is the charge per unit length times the length of the chunk", or:

$$\lambda(x) = \frac{dm}{dx} = \frac{2M}{L^2}x \implies dm = \lambda\, dx = \frac{2M}{L^2}\, x\, dx$$

Then:

a)

$$\int dm = \frac{2M}{L^2}\int_0^L x dx = M$$

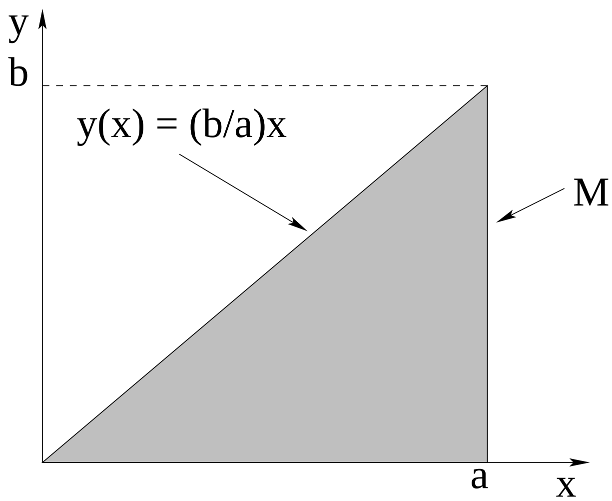
which indeed checks out. We can now evaluate:

b)

$$x_{\rm cm} = \frac{1}{M}\int_0^L x\, dm = \frac{2}{L^2}\int_0^L x^2\, dx = \frac{2}{L^2}\,\frac{x^3}{3}\bigg|_0^L = \frac{2}{3}L$$

This last result is basically the same result you would get evaluating the position of the center of mass along any leg of a uniform sheet of mass in the shape of a right triangle, where the height of the triangle increases linearly with (for example) x.

Problem 201. problems-1/center-of-mass-pr-right-triangle.tex



In the figure to the left, a uniformly thick sheet of plastic is cut into the $a \times b$ right triangle with mass M shown. Find the center of mass of the triangle in the coordinate system given, using integration to find the $x_{\rm cm}$ and $y_{\rm cm}$ components separately.

Suggested solution strategy:

a) Form $\sigma = M/A$ where A is the area of the triangle.

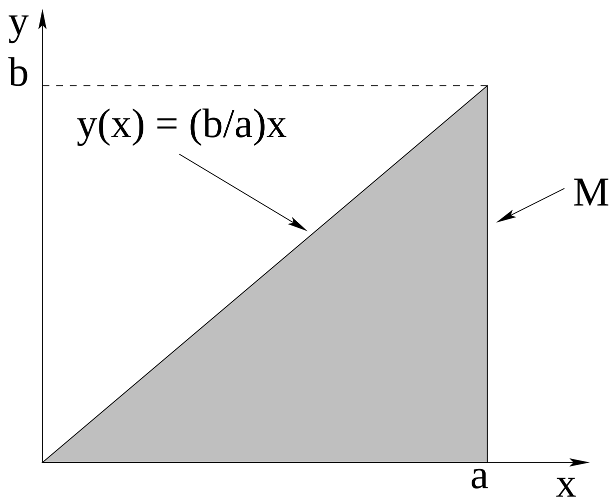
b) Form $dm = \sigma\ dA$ where $dA = dx\ dy$.

c) Do the integrals of e.g. $\int x\ dm$, using the provided functional form of the hypotenuse to set the y-limit of integration.

d) Use symmetry to determine $y_{\rm cm}$!

$x_{\rm cm} = \boxed{}$ $y_c m = \boxed{}$

Problem 202. problems-1/center-of-mass-pr-right-triangle-soln.tex



In the figure to the left, a uniformly thick sheet of plastic is cut into the $a \times b$ right triangle with mass M shown. Find the center of mass of the triangle in the coordinate system given, using integration to find the $x_{\rm cm}$ and $y_{\rm cm}$ components separately.

Suggested solution strategy:

a) Form $\sigma = M/A$ where A is the area of the triangle.

b) Form $dm = \sigma\ dA$ where $dA = dx\ dy$.

c) Do the integrals of e.g. $\int x\ dm$, using the provided functional form of the hypotenuse to set the y-limit of integration.

d) Use symmetry to determine $y_{\rm cm}$!

$$x_{\rm cm} = \boxed{\frac{2}{3}a} \qquad y_{\rm cm} = \boxed{\frac{1}{3}b}$$

Solution: Not much to say here – let's just go through the steps listed above. The area of the triangle is (recall) $A = \frac{1}{2}ab$ so:

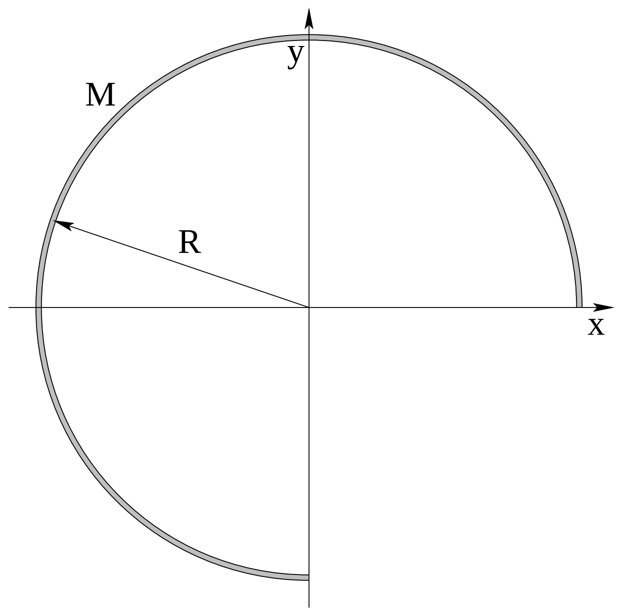
$$\sigma = \frac{M}{A} = \frac{2M}{ab} \qquad dm = \sigma dA = \frac{2M}{ab} dx\, dy$$

$$\begin{aligned} x_{\rm cm} &= \frac{1}{M}\int x\, dm \\ &= \frac{1}{M}\int_0^a \int_0^{y(x)} x \frac{2M}{ab} dx\, dy \\ &= \frac{2}{ab}\int_0^a x dx \left(\int_0^{y(x)} dy = y(x) \right) \\ &= \frac{2}{ab} \times \frac{b}{a}\int_0^a x^2 dx = \frac{2}{a^2} \times \frac{a^3}{3} = \frac{2}{3}a \end{aligned}$$

You can see that $x_{\rm cm}$ is one-third of the distance from the *wide* end of the triangle. Applying the same rule to $y_{\rm cm}$:

$$\boxed{y_{\rm cm} = \frac{b}{3}}$$

Problem 203. problems-1/center-of-mass-pr-arc-270.tex

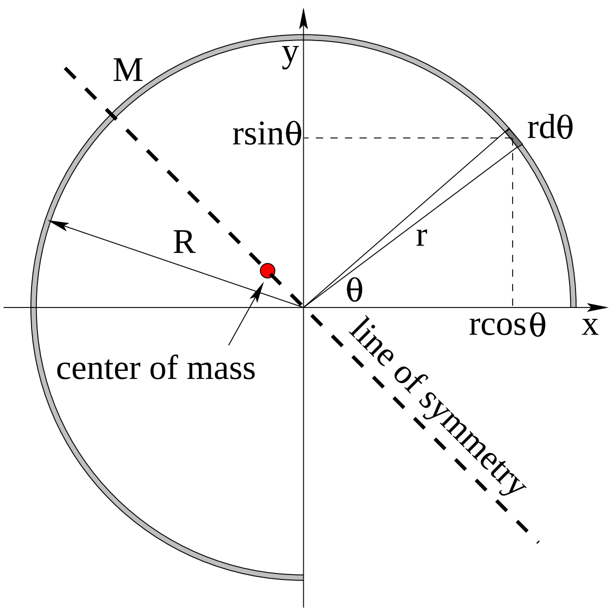


In the figure to the left, a uniformly thick piece of wire with mass M and radius R is bent into 3/4 of a circular arc as shown. Find the center of mass of the wire in the coordinate frame given.

$x_{\text{cm}} =$ $\boxed{}$

$y_{\text{cm}} =$ $\boxed{}$

Problem 204. problems-1/center-of-mass-pr-arc-270-soln.tex



In the figure to the left, a uniformly thick piece of wire with mass M and radius R is bent into 3/4 of a circular arc as shown. Find the center of mass of the wire in the coordinate frame given.

$x_{\rm cm} = \boxed{\dfrac{2R}{3\pi}}$

$y_{\rm cm} = \boxed{\dfrac{2R}{3\pi}}$

Solution: We can't sensibly integrate the center of mass formula in the problem in cartesian coordinates, but it is pretty easy in polar coordinates. So we ***find μ (mass per unit length), form dm for a differential chunk of the wire at angle θ, express e.g. $x = R\cos\theta$ and then integrate to find e.g. $x_{\rm cm}$***.

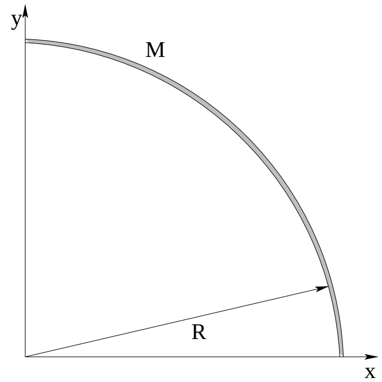
Give this a try before looking further below.

$$\mu = \frac{M}{3\pi R/2} = \frac{2M}{3\pi R} \qquad ds = R\,d\theta \qquad dm = \mu\,ds = \frac{2M}{3\pi}d\theta$$

$$\begin{aligned} x_{\rm cm} &= \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^{3\pi/2} R\cos\theta\frac{2M}{3\pi}d\theta \\ &= \frac{2R}{3\pi}\int_0^{3\pi/2}\cos\theta\,d\theta = \frac{2R}{3\pi}\left(\sin\frac{3\pi}{2} - \sin(0)\right) = -\frac{2R}{3\pi} \end{aligned}$$

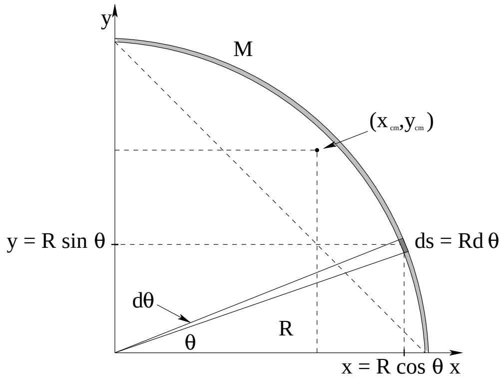
At this point one can *either* set up and do the second integral for $y_{\rm cm}$ – not a bad idea if you are practicing setting up integrals and doing them! – or (smarter!) ***invoke symmetry***. It's pretty obvious that the center of mass has to be on the dashed line bisecting the figure (so it has reflection symmetry about this line), so $y_{\rm cm} = x_{\rm cm}$ without any additional work...

Problem 205. problems-1/center-of-mass-pr-arc-90.tex



In the figure above, a uniformly thick piece of wire is bent into 1/4 of a circular arc as shown. **Find the center of mass of the wire in the coordinate system given**, using integration to find the $x_{\rm cm}$ and $y_{\rm cm}$ components separately.

Problem 206. problems-1/center-of-mass-pr-arc-90-soln.tex



In the figure above, a uniformly thick piece of wire is bent into 1/4 of a circular arc as shown. **Find the center of mass of the wire in the coordinate system given**, using integration to find the $x_{\rm cm}$ and $y_{\rm cm}$ components separately.

Solution: First, select a differential chunk of the wire in polar coordinates at the angle θ and with angular width $d\theta$ as drawn above. This chunk has length:

$$ds = R\,d\theta$$

Next, find the ***mass per unit length of the entire wire:***

$$\mu = \frac{M}{L} = \frac{M}{(\pi/2)R} = \frac{2M}{\pi R}$$

Use this to find the mass of the differential chunk ds:

$$dm = \mu ds = \frac{2M}{\pi R} R\,d\theta = \frac{2M}{\pi} d\theta$$

Next, express its x and y coordinates in polar form:

$$x = R\cos\theta \qquad\qquad y = R\sin\theta$$

Finally, evaluate the center of mass coordinates by integrating $\theta : 0 \to \pi/2$:

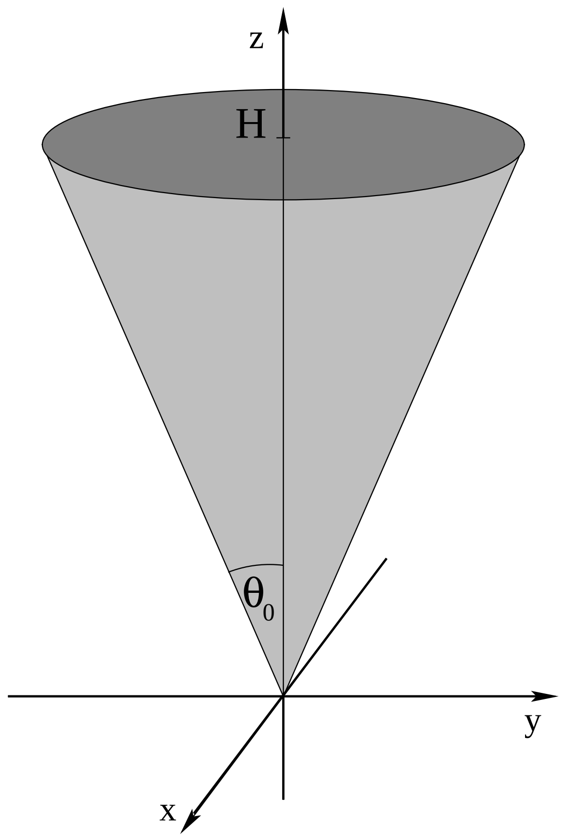
$$x_{\rm cm} = \frac{1}{M}\int x\,dm = \frac{1}{\cancel{M}}\frac{2\cancel{M}}{\pi}\int_0^{\pi/2} R\cos\theta\,d\theta = \frac{2R}{\pi}\sin\frac{\pi}{2} = \boxed{\frac{2}{\pi}R}$$

$$y_{\rm cm} = \frac{1}{M}\int y\,dm = \frac{1}{\cancel{M}}\frac{2\cancel{M}}{\pi}\int_0^{\pi/2} R\sin\theta\,d\theta = \frac{2R}{\pi}\cos 0 = \boxed{\frac{2}{\pi}R}$$

If a student only evaluates *one* of these explicitly and invokes ***symmetry*** to assert that $x_{\rm cm} = y_{\rm cm}$, that's ***perfectly all right!*** It is *good* to use symmetry and conceptual reasoning, when possible, to shorten and simplify a problem!

Note that the solution point is marked in on the figure above, and does lie within the "envelope" of the arc itself, as it must. It's pretty close to where one might *guess* that it is located!

Problem 207. problems-1/center-of-mass-pr-circular-cone.tex



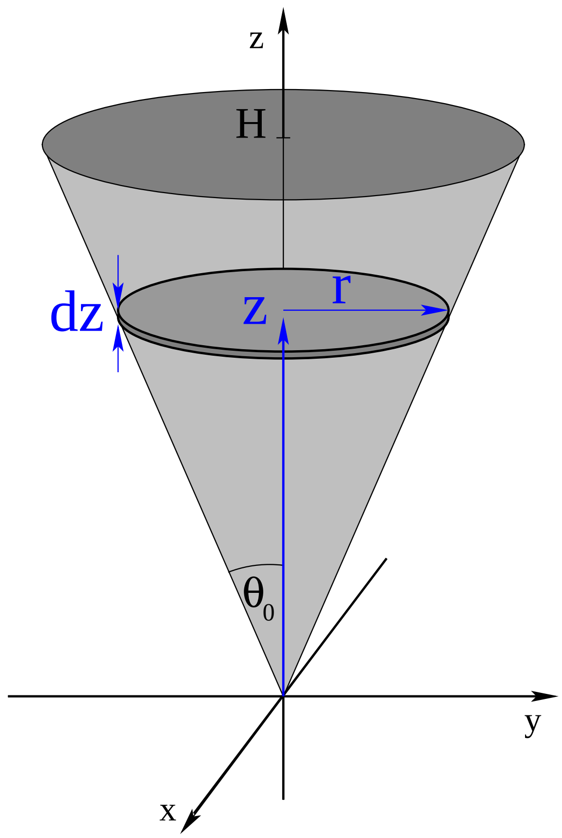
To the left is drawn a solid circular cone with uniform mass density ρ. The cone side makes an angle θ_0 with the positive z axis. The cone height is H.
Find the center of mass of the cone in terms of the quantities given above. Hint: Consider circular slabs of thickness dz a height z above the origin and use symmetry where possible.

$x_{\rm cm} = \boxed{}$

$y_{\rm cm} = \boxed{}$

$z_{\rm cm} = \boxed{}$

Problem 208. problems-1/center-of-mass-pr-circular-cone-soln.tex



To the left is drawn a solid circular cone with uniform mass density ρ. The cone side makes an angle θ_0 with the positive z axis. The cone height is H.
Find the center of mass of the cone in terms of the quantities given above. Hint: Consider circular slabs of thickness dz a height z above the origin and use symmetry where possible.

$$x_{\rm cm} = \boxed{0}$$

$$y_{\rm cm} = \boxed{0}$$

$$z_{\rm cm} = \boxed{\frac{3}{4}R}$$

Solution: First, note that the cone has full ***rotational symmetry around the z-axis!*** That let's immediately fill in $x_{\rm cm} = y_{\rm cm} = 0$ above.

Next, we need to find the "volume element" of the cone in terms of z. This is the disk-shaped slab illustrated above at a height z above the origin. It has a radius $r(z) = z\tan\theta_0$ (why?), cross-sectional area $A = \pi r(z)^2 = \pi z^2 \tan^2\theta_0$, and thickness dz. From this we can evaluate several important quantities:

$$dV = A dz = \pi \tan^2\theta_0 z^2\, dz \qquad\qquad dm = \rho\, dV = \rho\pi\tan^2\theta_0 z^2\, dz$$

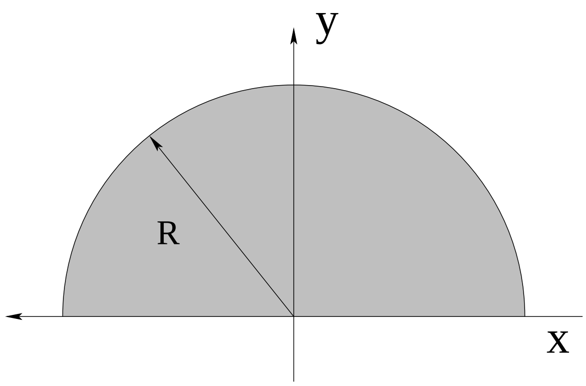
$$M = \int dm = \rho\pi\tan^2\theta_0 \int_0^H z^2 dz = \rho\pi\tan^2\theta_0 \frac{H^3}{3}$$

and finally:

$$z_{\rm cm} = \frac{1}{M}\int z\, dm = \frac{3}{\cancel{\rho\pi\tan^2\theta_0} H^3}\int_0^H \cancel{\rho\pi\tan^2\theta_0} z^3\, dz = \frac{3}{4}H$$

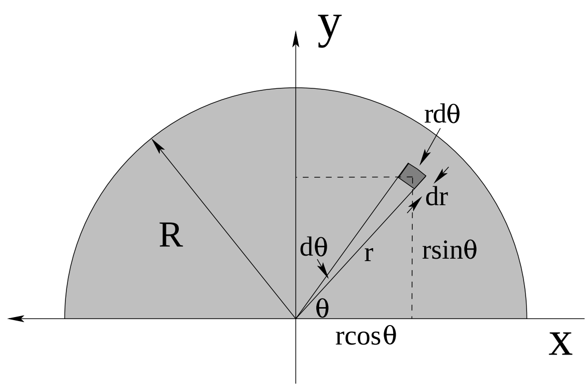
Note that (interestingly!) the center of mass *doesn't depend on the angle of the cone θ_0!* It is a fixed fraction of the height of the cone for all right circular cones!

Problem 209. problems-1/center-of-mass-pr-semicircular-sheet.tex



Find the center of mass of the two-dimensional semicircular sheet drawn above. It has a uniform mass per unit area σ and radius R. You may invoke symmetry for one of the two vector components of the center of mass location.

Problem 210. problems-1/center-of-mass-pr-semicircular-sheet-soln.tex



Find the center of mass of the two-dimensional semicircular sheet drawn above. It has a uniform mass per unit area σ and radius R. You may invoke symmetry for one of the two vector components of the center of mass location.

Solution: We have to use the definition of (vector) center of mass position in integral form, but we have to choose good coordinates to integrate over. In the case of a semicircular domain, polar coordinates are "obviously" the right choice. Things we will need to describe "a chunk of mass at position (r, θ)" include: include:

$$x = r\cos\theta \quad y = r\sin\theta \quad dA = r\,dr\,d\theta \quad dm = \sigma\,dA$$

Then (to get answers in terms of the ***given*** σ):

$$M = \int \sigma dA = \int_0^R \int_0^\pi \sigma r\,dr\,d\theta = \frac{1}{2}\sigma\pi R^2$$

$$x_{\text{cm}} = \frac{1}{M}\int xdm = \frac{2}{\sigma\pi R^2}\int_0^R\int_0^\pi \sigma r\cos\theta\,r\,dr\,d\theta = \frac{2}{\pi R^2}\frac{R^3}{3}(\sin\pi - \sin 0) = 0$$

(which is ***obvious from symmetry***) and:

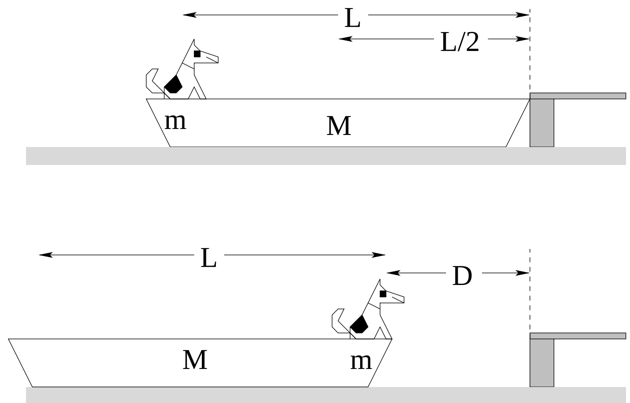
$$y_{\text{cm}} = \frac{1}{M}\int ydm = \frac{2}{\sigma\pi R^2}\int_0^R\int_0^\pi \sigma r\sin\theta\,r\,dr\,d\theta = -\frac{2}{\pi R^2}\frac{R^3}{3}(\cos\pi - \cos 0) = \frac{4R}{3\pi}$$

Hence:

$$\boxed{\vec{\boldsymbol{x}}_{\text{cm}} = 0\hat{\boldsymbol{x}} + \frac{4}{3\pi}R\hat{\boldsymbol{y}}}$$

or any other (equivalent) way of correctly specifying two vector coordinates.

Problem 211. problems-1/center-of-mass-pr-dog-in-a-boat.tex

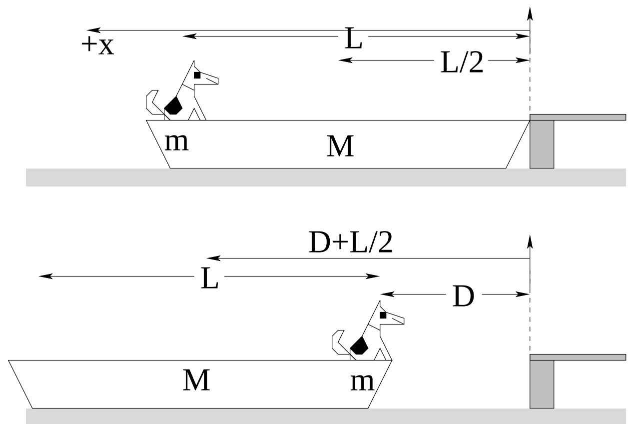


A dog of mass m is sitting at one end of a boat of mass M and length L that is sitting next to a dock as shown. The dog decides he wants some tasty dog chunks that are waiting for him at home and walks to the other end of the boat, expecting to step out onto the dock. Sadly, when he gets there he finds himself a distance D away from the dock.

a) What is D in terms of m, M, and L. You may assume that the boat is symmetric, so that its center of mass is at $L/2$, although this is not strictly necessary to get the answer.

b) The dog can successfully jump to the dock from the boat if $D < L/2$, but otherwise he'll have to swim. Find the ratio m/M for which the dog (first, barely) can't make the leap and has to take a bath to get to the chunks.

Problem 212. problems-1/center-of-mass-pr-dog-in-a-boat-soln.tex



a) The idea is that the water is "frictionless" in the direction of the dock so that there is no net force exerted on the dog-boat system while the dog walks. That means that the center of mass, initially at rest, remains at rest while the dog walks!

So we equate the center of mass before the dog moves to the center of mass after:

$$x_{\rm cm} = \frac{mL + ML/2}{m+M} = \frac{mD + M(D+L/2)}{m+M}$$

cancel $m+M$ and factor a bit:

$$mL + ML/2 = (m+M)D + ML/2$$

cancel the $ML/2$ and divide to get:

$$D = \boxed{\frac{mL}{m+M}}$$

b) This is now easy. In algebra-speak, we require $D \leq L/2$. The marginal case is then:

$$D = \frac{mL}{m+M} = L/2$$

Even if you don't immediately see that $m = M$ works, you can multiply out the $m+M$ on both sides and also multiply both sides by 2:

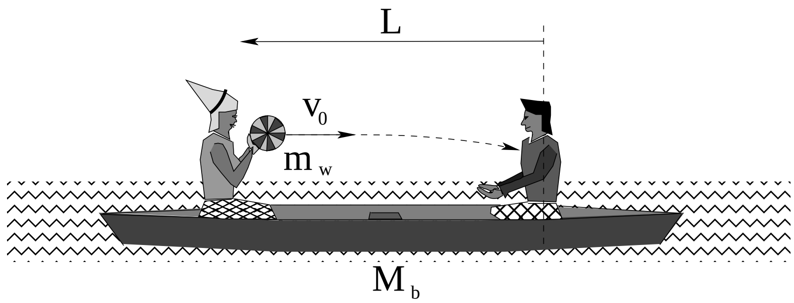
$$2mL = (m+M)L = mL + ML$$

Cancel the L from both sides and simplify to get:

$$m = M \Longrightarrow \frac{m}{M} = \boxed{1}$$

Scoring: +7 points for recognizing that it is a center of mass problem, +3 for writing a correct expression for the CoM before and after, +2 for the algebra to get the correct answer (give or take a point or two for details). Also +5 for recognizing that you set D from a) equal to $L/2$, +3 for getting the answer from this (give or take a point ditto).

Problem 213. problems-1/center-of-mass-pr-romeo-and-juliet.tex



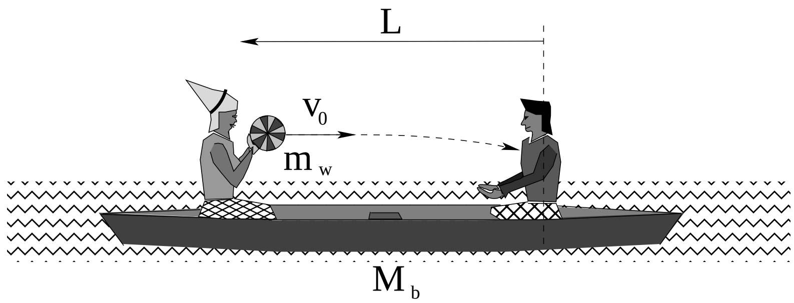
Romeo and Juliet are out in their damn boat again, this time for a picnic on the lake. The boat is ***initially at rest***. Juliet decides she wants a piece of tasty watermelon, and throws the watermelon at horizontal speed v_0 to Romeo at the other end of the boat a distance L away so he can cut her a piece with his ever-handy bodkin (dagger). The combined mass of Romeo, Juliet and the boat is M_b; the mass of the watermelon is m_w. Assume that the boat can move horizontally on the water without drag or friction.

a) What is the horizontal speed of the boat while the watermelon is in the air (neglect its vertical motion – assume that Juliet has thrown it on a flat trajectory as shown).

b) What is the horizontal speed of the boat after Romeo catches the watermelon?

c) How long is the watermelon in the air?

Problem 214. problems-1/center-of-mass-pr-romeo-and-juliet-soln.tex



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a) What is the horizontal speed of the boat while the watermelon is in the air (neglect its vertical motion – assume that Juliet has thrown it on a flat trajectory as shown).

b) What is the horizontal speed of the boat after Romeo catches the watermelon?

c) How long is the watermelon in the air?

Solution: For a), use conservation of (1D) momentum. Everything begins at rest ($p_i = 0$ in the $\hat{\boldsymbol{x}}$ direction) so:

$$p_i = 0 = m_w v_0 + M_b v_b = p_f$$

where p_f is the momentum while the watermelon is in the air. Hence:

$$\boxed{v_b = -\frac{m_w}{M_b} v_0}$$

That is, the boat recoils backwards at a speed that is a smallish fraction of the speed that the watermelon goes forward, assuming reasonably that $M_b \gg m_w$.

b) To find the horizontal speed of the boat after Romeo catches the watermelon, use conservation of momentum again, but now consider p_f to be when Romeo has caught the watermelon and everything is moving "as one" again:

$$p_i = (m_w + M_b)0 = 0 = (m_w + M_b)v_f = p_f$$

or

$$\boxed{v_f = 0}$$

The boat and everything in it *are at rest again* as soon as he catches the watermelon.

c) The easiest way to determine how long the watermelon is in the air is to consider the *relative speed* of the watermelon and the boat. This is:

$$v_{\rm rel} = v_0 - (-\frac{m_w}{M_b} v_0)$$

This is the speed with which the watermelon in the air, and Romeo sitting in the boat, come together, each covering its part of the distance L in between. That is:

$$\boxed{\Delta t = \frac{L}{v_{\rm rel}} = \frac{L M_b}{(m_w + M_b) v_0}}$$

This makes sense – if the boat were very massive compared to the watermelon, the time would just be L/v_0 and that's what this goes to in that limit. In the opposite limit, with a very heavy watermelon and a very light boat, by the time the watermelon is travelling at any decent speed v_0 the boat is rocketing the other way many times faster so the time is very small, much less than L/v_0.

One can get the same result by saying that the watermelon travels a distance D to the right, while Romeo/boat travel a distance $L - D$ to the left, each at its own speed, to meet at time Δt:

$$v_w \Delta t = D$$

$$v_b \Delta t = L - D$$

Add these two equations, and get:

$$(v_w + v_b)\Delta t = L\Delta t \quad \Rightarrow \quad \Delta t = \frac{L}{v_w + v_b} = \frac{L}{v_0 + \frac{m_w}{M_b} v_0} \quad \Rightarrow \quad \boxed{\Delta t = \frac{L M_b}{(m_w + M_b) v_0}}$$

as before.

6.2 Momentum

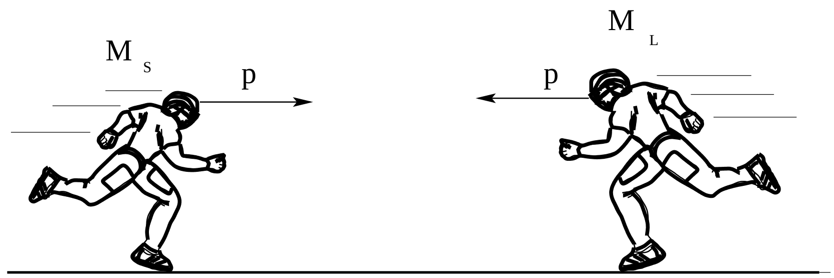
6.2.1 Multiple Choice Problems

Problem 215. problems-1/momentum-mc-also-energy-football-icp.tex

Two football players, one large (L – bigger mass) and one small (S – smaller mass) are running in a straight line directly at one another. They have the ***same magnitude of momentum***. Rank their mechanical energies and speeds right ***before*** they collide.

a) $E_S < E_L$, $v_S < v_L$ b) $E_S < E_L$, $v_S > v_L$ c) $E_S > E_L$, $v_S > v_L$ d) $E_S > E_L$, $v_S < v_L$

Problem 216. problems-1/momentum-mc-also-energy-football-icp-soln.tex



This problem is intended to help you internalize the relationship between the ***scalar magnitude*** of the momentum $\vec{\boldsymbol{p}}$ of an object and K, its kinetic energy:

$$p = |\vec{\boldsymbol{p}}| = mv \qquad \Longrightarrow \qquad K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

To answer these two questions, note that $E = K$ (we might as well set their unchanging potential energy to zero as it is irrelevant). Then ask yourself: "How does (K or v) vary with mass, given equal ***magnitudes*** of ***momentum*** for the two players?" Now it is easy:

$$p_S = M_S v_S = M_L v_L = p_L = p$$

so

$$M_S < M_L \implies v_S > v_L$$

Similarly:

$$K_S = \frac{p^2}{2M_S} \text{ and } K_L = \frac{p^2}{2M_L}$$

which go like the *inverse* of the masses, so:

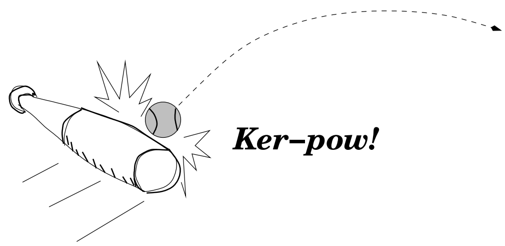
$$E_S = K_S > K_L = E_L$$

Hence:

a) $E_S < E_L$, $v_S < v_L$ b) $E_S < E_L$, $v_S > v_L$ c) $E_S > E_L$, $v_S > v_L$ d). $E_S > E_L$, $v_S < v_L$

Note that working out K using your answer for v is much more difficult (try it)! The form $p^2/2m$ is ***often*** the most convenient way to compute kinetic energy in momentum conservation problems.

Problem 217. problems-1/momentum-mc-baseball-and-bat-icp.tex



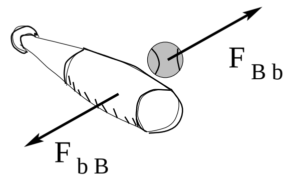
Michelle is playing baseball and hits a home run with a solid wood bat (mass of 3 kg). The baseball (mass of 0.5 kg) is knocked clean out of the park. The ***magnitude*** of the force exerted by the bat on the baseball is:

a) Greater than the magnitude of the force exerted by the baseball on the bat.

b) Less than the magnitude of the force exerted by the baseball on the bat;

c) The same as the magnitude of the force exerted by the baseball on the bat.

Problem 218. problems-1/momentum-mc-baseball-and-bat-icp-soln.tex



Solution: This is pure Newton's Third Law! The following should be passing through your mind as you answer:

"The bat exerts a (normal) force on the baseball during the collision. The baseball therefore exerts an (equal and opposite normal) force on the bat during the collision."

Note well that ***the named force is the same***, the ***two objects are the same*** (but in opposite order), and the force acts ***along the line connecting the two objects*** which are the exact conditions requied for Newton's Third Law.

The answer ***does not depend on the mass, velocity, time, or any other property of ball and bat, or car and truck, or any other pair of objects that push on one another via a named force law***. If it were not true, ***momentum would not be conserved***, objects made out of many particles would not behave like "particles" themselves, and physics would be very sad...

The magnitude of the force exerted by the bat on the baseball is thus:

a) Greater than the magnitude of the force exerted by the baseball on the bat.

b) Less than the magnitude of the force exerted by the baseball on the bat;

c) The same as the magnitude of the force exerted by the baseball on the bat.

Problem 219. problems-1/momentum-mc-cement-truck-and-bug.tex

A cement truck with a mass M_t is travelling at speed v_t collides with a bug of mass m_b that is hovering above the road (so $v_b \approx 0$). One can safely assume that $M_t \gg m_b$. Which of the following statements are unambiguously true (circle all definitely true statements)?

a) If the bug recoils off of the windshield ***elastically***, its final speed is roughly $2v_t$ (in the same direction as the truck).

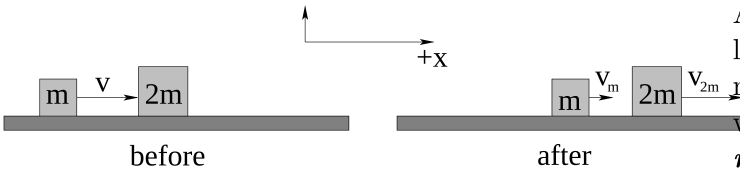
b) The magnitude of the momentum change of the truck is much smaller than the magnitude of the momentum change of the bug.

c) If the bug splatters and sticks to the windshield of the truck, the total kinetic energy of the bug and truck will be conserved.

d) At all times during the collision, the bug exerts exactly the same magnitude of force on the truck that the truck exerts on the bug.

e) The final speed of the bug as it recoils off of the windshield is roughly $\frac{M_t}{m_b}v_t$ (in the same direction as the truck).

Problem 220. problems-1/momentum-mc-elastic-collision-blocks-2.tex

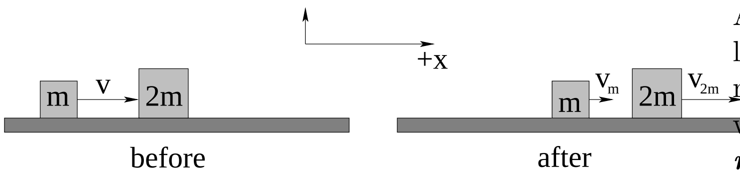


A block of mass m slides on a frictionless table at velocity $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{x}}$ to the right (positive x-direction) to collide with a block of mass $2m$ initially ***at rest*** as shown.

Assuming that the collision is ***one dimensional*** and ***elastic***, the velocities of the two blocks after the collision are:

- ☐ $\vec{\boldsymbol{v}}_m = -\frac{v}{2}\hat{\boldsymbol{x}} \quad \vec{\boldsymbol{v}}_{2m} = \frac{3v}{4}\hat{\boldsymbol{x}}$
- ☐ $\vec{\boldsymbol{v}}_m = 0\hat{\boldsymbol{x}} \quad \vec{\boldsymbol{v}}_{2m} = \frac{v}{6}\hat{\boldsymbol{x}}$
- ☐ $\vec{\boldsymbol{v}}_m = -\frac{v}{3}\hat{\boldsymbol{x}} \quad \vec{\boldsymbol{v}}_{2m} = \frac{4v}{3}\hat{\boldsymbol{x}}$
- ☐ $\vec{\boldsymbol{v}}_m = \frac{v}{3}\hat{\boldsymbol{x}} \quad \vec{\boldsymbol{v}}_{2m} = \frac{2v}{3}\hat{\boldsymbol{x}}$
- ☐ $\vec{\boldsymbol{v}}_m = -\frac{v}{3}\hat{\boldsymbol{x}} \quad \vec{\boldsymbol{v}}_{2m} = \frac{2v}{3}\hat{\boldsymbol{x}}$

Problem 221. problems-1/momentum-mc-elastic-collision-blocks-2-soln.tex



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Assuming that the collision is ***one dimensional*** and ***elastic***, the velocities of the two blocks after the collision are:

- ☐ $\vec{\boldsymbol{v}}_m = -\frac{v}{2}\hat{\boldsymbol{x}}$ $\quad \vec{\boldsymbol{v}}_{2m} = \frac{3v}{4}\hat{\boldsymbol{x}}$
- ☐ $\vec{\boldsymbol{v}}_m = 0\hat{\boldsymbol{x}}$ $\quad \vec{\boldsymbol{v}}_{2m} = \frac{v}{6}\hat{\boldsymbol{x}}$
- ☐ $\vec{\boldsymbol{v}}_m = -\frac{v}{3}\hat{\boldsymbol{x}}$ $\quad \vec{\boldsymbol{v}}_{2m} = \frac{4v}{3}\hat{\boldsymbol{x}}$
- ☐ $\vec{\boldsymbol{v}}_m = \frac{v}{3}\hat{\boldsymbol{x}}$ $\quad \vec{\boldsymbol{v}}_{2m} = \frac{2v}{3}\hat{\boldsymbol{x}}$
- ☒ $\vec{\boldsymbol{v}}_m = -\frac{v}{3}\hat{\boldsymbol{x}}$ $\quad \vec{\boldsymbol{v}}_{2m} = \frac{2v}{3}\hat{\boldsymbol{x}}$

Solution: Use the solution to the 1D elastic collision problem we derived in class using the center of mass reference frame:

$$v_f = -v_i + 2v_{\mathrm{cm}}$$

for each mass! First we have to find $M_{\mathrm{tot}} = m + 2m = 3m$ and:

$$v_{\mathrm{cm}} = \frac{1}{M_{\mathrm{tot}}}\left(mv + 2m * 0\right) = \frac{1}{3}v$$

The x-velocity of the smaller block m after the collision is then:

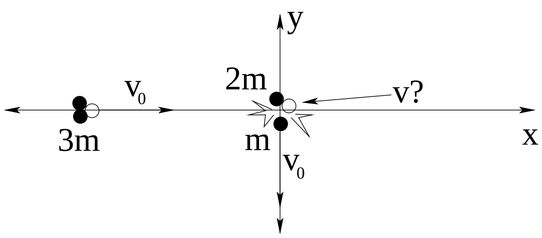
$$v_m = -v + 2 * \frac{1}{3}v = -\frac{1}{3}v$$

That of the large block $2m$ is:

$$v_{2m} = -0 + 2 * \frac{1}{3}v = +\frac{2}{3}v$$

I checked the correct answer above, which (since the problem asks for velocity needs to include direction) contains the appropriate unit vectors in the ***provided*** coordinate frame and the correct signs and not just the magnitudes. Note that the smaller mass *recoils to the left!* This is typical of elastic collisions where the incoming mass has *less* mass than the stationary target, something well worth remembering!

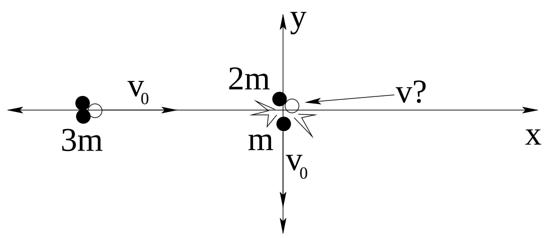
Problem 222. problems-1/momentum-mc-fission-1.tex



An atomic nucleus of mass $3m$ is travelling to the right at velocity $\vec{\boldsymbol{v}}_{\rm initial} = v_0\hat{\boldsymbol{\imath}}$ as shown. It spontaneously fissions into two fragments of mass m and $2m$. The smaller fragment m travels straight down at velocity $\vec{\boldsymbol{v}}_m = -v_0\hat{\boldsymbol{\jmath}}$ after the fission. What is the velocity of the larger fragment?

- ☐ $\vec{\boldsymbol{v}}_{2m} = \frac{3}{2}v_0\hat{\boldsymbol{\imath}}$
- ☐ $\vec{\boldsymbol{v}}_{2m} = 2v_0\hat{\boldsymbol{\jmath}}$
- ☐ $\vec{\boldsymbol{v}}_{2m} = \frac{3}{2}v_0\hat{\boldsymbol{\imath}} + \frac{1}{2}v_0\hat{\boldsymbol{\jmath}}$
- ☐ $\vec{\boldsymbol{v}}_{2m} = -\frac{3}{2}v_0\hat{\boldsymbol{\imath}} - \frac{1}{2}v_0\hat{\boldsymbol{\jmath}}$
- ☐ $\vec{\boldsymbol{v}}_{2m} = 3v_0\hat{\boldsymbol{\imath}} + 2v_0\hat{\boldsymbol{\jmath}}$

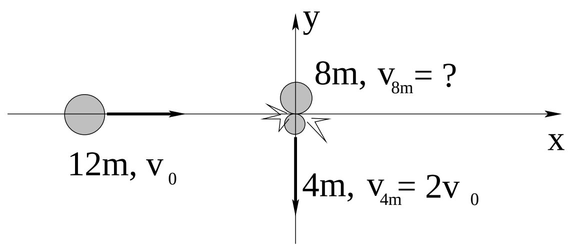
Problem 223. problems-1/momentum-mc-fission-1-soln.tex



An atomic nucleus of mass $3m$ is travelling to the right at velocity $\vec{v}_{\text{initial}} = v_0\hat{\imath}$ as shown. It spontaneously fissions into two fragments of mass m and $2m$. The smaller fragment m travels straight down at velocity $\vec{v}_m = -v_0\hat{\jmath}$ after the fission. What is the velocity of the larger fragment?

- ☐ $\vec{v}_{2m} = \frac{3}{2}v_0\hat{\imath}$
- ☐ $\vec{v}_{2m} = 2v_0\hat{\jmath}$
- ☒ $\vec{v}_{2m} = \frac{3}{2}v_0\hat{\imath} + \frac{1}{2}v_0\hat{\jmath}$
- ☐ $\vec{v}_{2m} = -\frac{3}{2}v_0\hat{\imath} - \frac{1}{2}v_0\hat{\jmath}$
- ☐ $\vec{v}_{2m} = 3v_0\hat{\imath} + 2v_0\hat{\jmath}$

Problem 224. problems-1/momentum-mc-fission-review.tex



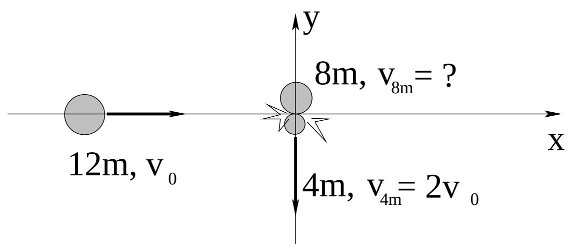
An atomic nucleus of mass $12m$ is travelling to the right at velocity $\vec{\boldsymbol{v}}_{\rm initial} = v_0\hat{\boldsymbol{x}}$ as shown. It spontaneously fissions into two fragments of mass $4m$ and $8m$ (releasing energy). The smaller fragment $4m$ travels straight down at velocity $\vec{\boldsymbol{v}}_{4m} = -2v_0\hat{\boldsymbol{y}}$ after the fission. What is the velocity of the larger fragment?

a) $\vec{\boldsymbol{v}}_{8m} = \frac{1}{2}v_0\hat{\boldsymbol{x}} + v_0\hat{\boldsymbol{y}}$ b) $\vec{\boldsymbol{v}}_{8m} = 2v_0\hat{\boldsymbol{x}} + 2v_0\hat{\boldsymbol{y}}$

c) $\vec{\boldsymbol{v}}_{8m} = -\frac{3}{2}v_0\hat{\boldsymbol{x}} - v_0\hat{\boldsymbol{y}}$ d) $\vec{\boldsymbol{v}}_{8m} = \frac{3}{2}v_0\hat{\boldsymbol{x}} + 2v_0\hat{\boldsymbol{y}}$

e) $\vec{\boldsymbol{v}}_{8m} = \frac{3}{2}v_0\hat{\boldsymbol{x}} + v_0\hat{\boldsymbol{y}}$

Problem 225. problems-1/momentum-mc-fission-review-soln.tex



Use momentum conservation:

$$\vec{\boldsymbol{p}}_i = 12mv_0\hat{\boldsymbol{x}} = 8mv_{8x}\hat{\boldsymbol{x}} + 8mv_{8y}\hat{\boldsymbol{y}} - 4m(2v_0)\hat{\boldsymbol{y}} = \vec{\boldsymbol{p}}_f$$

and equate the x and y components:

$$12mv_0 = 8mv_{8x} \Longrightarrow v_{8x} = \frac{3}{2}v_0 \qquad 8mv_{8y} = 8mv_0 \Longrightarrow v_{8y} = v_0$$

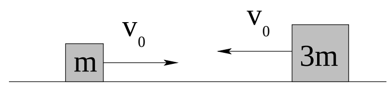
or:

a) $\vec{\boldsymbol{v}}_{8m} = \frac{1}{2}v_0\hat{\boldsymbol{x}} + v_0\hat{\boldsymbol{y}}$ b) $\vec{\boldsymbol{v}}_{8m} = 2v_0\hat{\boldsymbol{x}} + 2v_0\hat{\boldsymbol{y}}$

c) $\vec{\boldsymbol{v}}_{8m} = -\frac{3}{2}v_0\hat{\boldsymbol{x}} - v_0\hat{\boldsymbol{y}}$ d) $\vec{\boldsymbol{v}}_{8m} = \frac{3}{2}v_0\hat{\boldsymbol{x}} + 2v_0\hat{\boldsymbol{y}}$

d) $\vec{\boldsymbol{v}}_{8m} = \frac{3}{2}v_0\hat{\boldsymbol{x}} + v_0\hat{\boldsymbol{y}}$

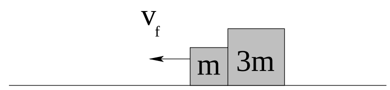
Problem 226. problems-1/momentum-mc-simple-inelastic-collision-icp.tex



A mass m travelling at (one-dimensional) velocity v_0 to the right collides with mass $3m$ travelling at velocity $-v_0$ to the left and sticks to it. The final velocity v_f of the blocks after the collision is:

a) $v_f = -2v_0$ b) $v_f = v_0/2$ c) $v_f = -v_0$ d) $v_f = -2v_0/3$ e) $v_f = -v_0/2$

Problem 227. problems-1/momentum-mc-simple-inelastic-collision-icp-soln.tex



A mass m travelling at (one-dimensional) velocity v_0 to the right collides with mass $3m$ travelling at velocity $-v_0$ to the left and sticks to it.

Solution: Momentum is conserved in *all* collisions (in the impulse approximation), elastic and partially or fully inelastic. This is a fully inelastic collsion as the two blocks stick together, but we only need to use momentum conservation to answer the question.

The initial total momentum (positive to the right) is:

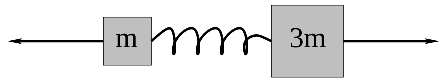
$$p_{\rm tot} = p_i = mv_0 - 3mv_0 = -2mv_0 = (4m)v_f = p_f$$

so the final velocity v_f of the blocks after the collision is:

a) $v_f = -2v_0$ b) $v_f = v_0/2$ c) $v_f = -v_0$ d) $v_f = -2v_0/3$ e) $v_f = -v_0/2$

(to the left, as drawn above).

Problem 228. problems-1/momentum-mc-two-masses-spring-2.tex



Two masses, m and $3m$, are separated by a compressed spring as shown above. At time $t = 0$ they are released from rest and the (massless) spring expands. There is no gravity or friction. As they move apart, which statement about the **magnitude** of each mass's relative kinetic energy K_i and the **magnitude** of each mass's relative momentum p_i is true?

a) $K_m = 3K_{3m}, \quad p_m = p_{3m}/3$

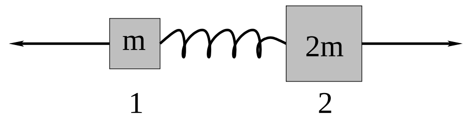
b) $K_m = K_{3m}/3, \quad p_m = 3p_{3m}$

c) $K_m = K_{3m}/3, \quad p_m = p_{3m}$

d) $K_m = 3K_{3m}, \quad p_m = p_{3m}$

e) $K_m = K_{3m}/4, \quad p_m = p_{3m}$

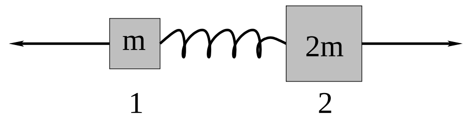
Problem 229. problems-1/momentum-mc-two-masses-spring.tex



Two masses, $m_2 = 2m_1$ are separated by a compressed spring. At time $t = 0$ they are released from rest and the (massless) spring expands. There is no gravity or friction. As they move apart, which statement about the ***magnitude*** of each mass's kinetic energy K_i and momentum p_i is true?

- ☐ $K_1 = 2K_2$, $p_1 = 2p_2$
- ☐ $K_1 = 2K_2$, $p_1 = p_2$
- ☐ $K_1 = K_2$, $p_1 = 2p_2$
- ☐ $K_1 = K_2/2$, $p_1 = p_2/2$
- ☐ $K_1 = K_2/4$, $p_1 = p_2$

Problem 230. problems-1/momentum-mc-two-masses-spring-soln.tex



Two masses, $m_2 = 2m_1$ are separated by a compressed spring. At time $t = 0$ they are released from rest and the (massless) spring expands. There is no gravity or friction. As they move apart, which statement about the ***magnitude*** of each mass's kinetic energy K_i and momentum p_i is true?

- ☐ $K_1 = 2K_2$, $p_1 = 2p_2$
- ☒ $K_1 = 2K_2$, $p_1 = p_2$
- ☐ $K_1 = K_2$, $p_1 = 2p_2$
- ☐ $K_1 = K_2/2$, $p_1 = p_2/2$
- ☐ $K_1 = K_2/4$, $p_1 = p_2$

Solution: Recall momentum conservation (no external forces act) in 1D:

$$p_i = 0 = p_2 - p_1 = p_f$$

or

$$p_1 = p_2 = p$$

Also recall the following expression for kinetic energy of a particle:

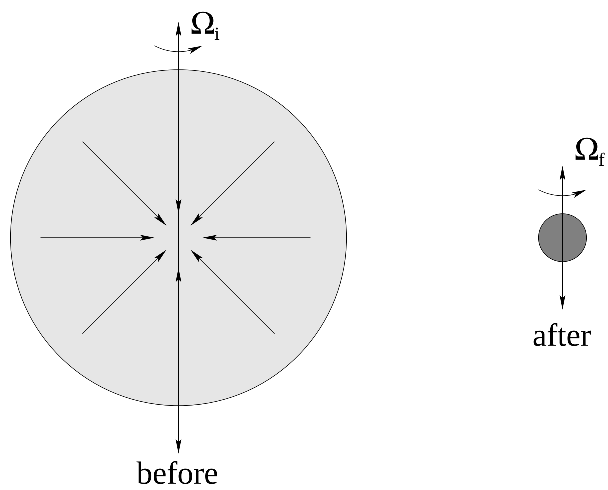
$$K = \frac{p^2}{2m}$$

Hence:

$$K_1 = \frac{p^2}{2m_1}$$

$$K_2 = \frac{p^2}{2m_2} = \frac{p^2}{4m_1} = \frac{1}{2}K_1 \implies K_1 = 2K_2$$

Problem 231. problems-1/angular-momentum-mc-collapsing-star.tex



When a star rotating with an angular speed Ω_i (eventually) exhausts its fuel, escaping light energy can no longer oppose gravity throughout the star's volume and it ***suddenly*** shrinks, with most of its outer mass falling in towards the center all at the same time.

As this happens, does the magnitude of the angular speed of rotation Ω_f:

a) increase

b) decrease

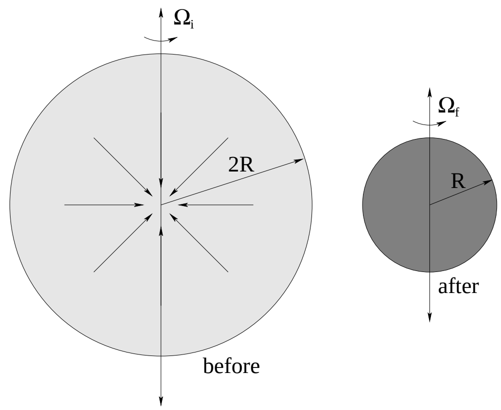
c) remain about the same

Why (state the principle used to answer the question)?

Problem 232. problems-1/angular-momentum-mc-collapsing-star-soln.tex

Angular momentum is (approximately) conserved and the moment of inertial decreases, so Ω increases.

Problem 233. problems-1/angular-momentum-mc-forming-star.tex

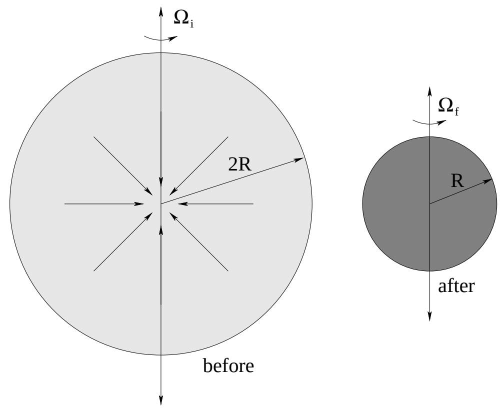


Gravity gradually assembles a star by pulling a cloud of rotating gas together into a rotating ball that then gradually shrinks. The figure above represents a star at two different stages in its formation, the first where a gas of total mass M has formed a ball of radius $2R$ rotating at angular speed Ω_i, the second where the ball has collapsed to a radius R (compressing the nuclear fuel inside closer to the point of fusion and ignition), rotating at a possibly new angular speed Ω_f.

Assuming that the mass is uniformly distributed in both cases, what is the best estimate for Ω_f in terms of Ω_i?

☐ $\Omega_f = \Omega_i/4$ ☐ $\Omega_f = \Omega_i/2$ ☐ $\Omega_f = \Omega_i$ ☐ $\Omega_f = 2\Omega_i$ ☐ $\Omega_f = 4\Omega_i$

Problem 234. problems-1/angular-momentum-mc-forming-star-soln.tex

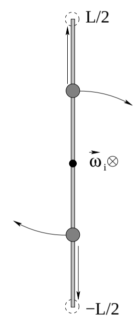


Gravity gradually assembles a star by pulling a cloud of rotating gas together into a rotating ball that then gradually shrinks. The figure above represents a star at two different stages in its formation, the first where a gas of total mass M has formed a ball of radius $2R$ rotating at angular speed Ω_i, the second where the ball has collapsed to a radius R (compressing the nuclear fuel inside closer to the point of fusion and ignition), rotating at a possibly new angular speed Ω_f.

Assuming that the mass is uniformly distributed in both cases, what is the best estimate for Ω_f in terms of Ω_i?

☐ $\Omega_f = \Omega_i/4$ ☐ $\Omega_f = \Omega_i/2$ ☐ $\Omega_f = \Omega_i$ ☐ $\Omega_f = 2\Omega_i$ ☒ $\Omega_f = 4\Omega_i$

Problem 235. problems-1/angular-momentum-mc-rotating-rod-sliding-beads.tex



In the figure above, a massless rod of length L is rotating around a frictionless pivot through its center at angular speed ω_i. Two beads, each with mass m, are stuck a distance $L/4$ from the center. The rotating system initially has a total kinetic energy K_i (which you could actually calculate if you needed to). At a certain time, the beads are released and slide smoothly to the ends of the rod where again, they stick. Which statement about the final angular speed and rotational kinetic energy of the rotating system is true:

a) $\omega_f = \omega_i/2$ and $K_f = K_i$.

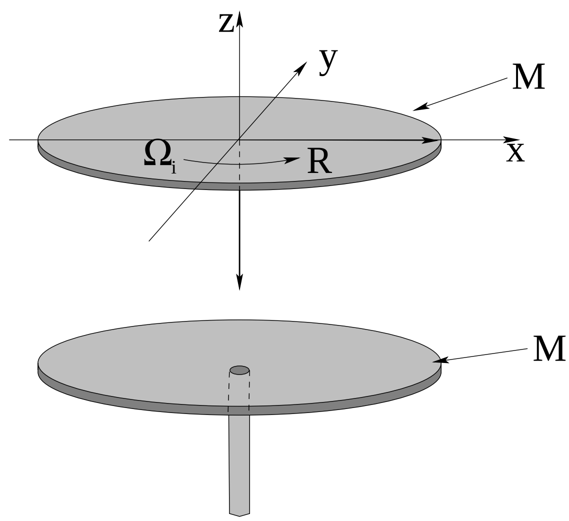
b) $\omega_f = \omega_i/4$ and $K_f = K_i/4$.

c) $\omega_f = \omega_i/4$ and $K_f = K_i/2$.

d) $\omega_f = \omega_i/2$ and $K_f = K_i/4$.

e) $\omega_f = \omega_i/2$ and $K_f = K_i/2$.

Problem 236. problems-1/angular-momentum-mc-two-circular-plates-collide.tex



A **disk** of uniformly distributed mass M and radius R sits at rest on a turntable that permits it to rotate freely. A second uniform disk of mass M with the same radius, centered on the same axis of rotation, is rotating at an (initial) angular speed Ω_i and is dropped gently onto it so that (after sliding for an instant) they stick together and rotate together as one.

How do the final angular velocity and final kinetic energy relate to the initial angular velocity and initial kinetic energy?

a) $\Omega_f = \Omega_i, \quad K_f = K_i$

b) $\Omega_f = 2\Omega_i, \quad K_f = K_i/2$

c) $\Omega_f = \Omega_i/2, \quad K_f = K_i/2$

d) $\Omega_f = \Omega_i/4, \quad K_f = K_i/4$

e) We cannot tell from the information given.

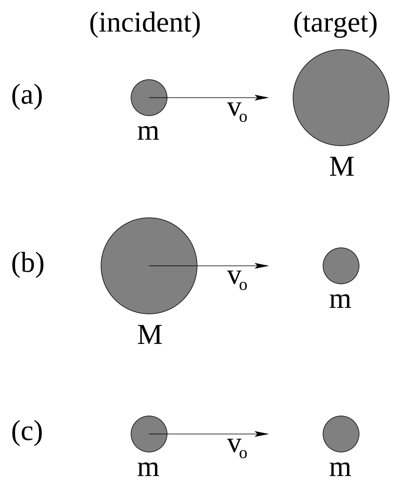
Problem 237. problems-1/angular-momentum-mc-two-circular-plates-collide-soln.tex

Angular momentum is conserved, inelastic collision: In z/axial direction, $L = L_i = \frac{1}{2}MR^2\Omega_0 = 2(\frac{1}{2}MR^2)\Omega_f = L_f$, so $\Omega_f = \Omega_i/2$.

Only one answer has this, but just in case, $K_i = \frac{L^2}{2(\frac{1}{2}MR^2)}$, $K_f = \frac{L^2}{2(MR^2)} = K_i/2$, so answer is c) on both counts.

6.2.2 Short Answer Problems

Problem 238. problems-1/momentum-sa-elastic-recoil.tex



In the three figures above, mass $M > m$. The mass on the left is incident at speed v_0 on the target mass (initially at rest in all three cases) on the right . The two particles undergo an **elastic** collision in one dimension and the target mass recoils to the **right** in all three cases. In the spaces provided below you are asked to provide a *qualitative* estimate of the speed *and* direction of the *incident* particle **after the collision**.

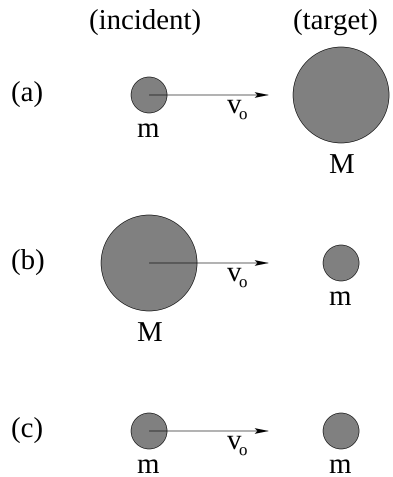
Your answer should be given relative to v_0 and should look like "$v_x > v_0$, to the left" or "$v_x = 0$" or $v_x < v_0$, to the right" where $x = a, b, c$. In other words, specify the speed qualitatively compared to v_0 and then the direction, per figure.

a)

b)

c)

Problem 239. problems-1/momentum-sa-elastic-recoil-soln.tex



You should remember/know (or even be able to derive):

$$v_f = -v_i + 2v_{\rm cm}$$

for 1 dimensional elastic collisions. In the first case $v_{\rm cm} \approx 0 \ll v_0$. In the second, $v_{\rm cm} \approx v_0$. In the third, $v_{\rm cm} = v_0/2$. So:

a)

$$v_x \approx -v_0 \qquad \text{to the left}$$

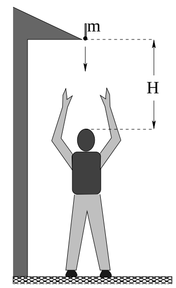
b)

$$v_x \approx v_0 \qquad \text{to the right}$$

c)

$$v_x = 0 \qquad \text{(other particle recoils to the right at } v_0\text{)}$$

Problem 240. problems-1/momentum-sa-hammer-impacts-head.tex

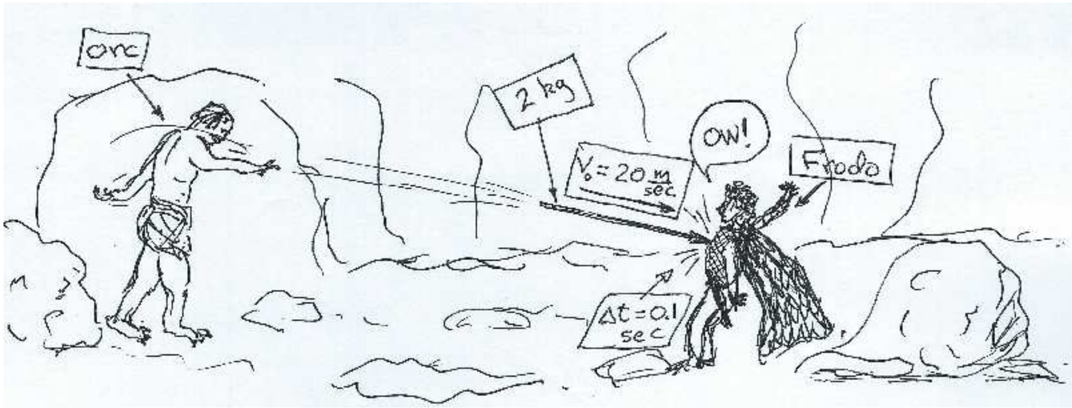


A hammer of mass m falls from rest off of a roof and drops a height H onto your head. Ouch!

a) Assuming that the tool is in actual contact with your head for a time Δt before it stops (thud!) and slides off, what is the algebraic expression for the average force it exerts on your hapless skull while ***stopping***?

b) *Estimate* the magnitude of this force using $m = 1$ kg, $H = 1.25$ meters, $\Delta t = 10^{-2}$ seconds and $g = 10 \frac{\text{m}}{\text{sec}^2}$. Compare this force to the weight of the hammer of 10 Newtons!

Problem 241. problems-1/momentum-sa-impact-orc-spears-frodo.tex



An Orc throws a 2 kg spear at Frodo Baggins at point blank range, but it is stopped by his hidden *mithril* mail shirt. Assuming that the spear was travelling at 20 m/sec when it hit and that it stopped in 0.1 seconds, what was the average force exerted on the spear by the mail coat (and the hobbit underneath)? Ouch!

Problem 242. problems-1/momentum-sa-shark-eats-fish.tex

A great white shark of mass m_1, coasting through the water in a nearly frictionless way at speed v_1, engulfs a tuna of mass $m_2 < m_1$ travelling in the same direction at speed $v_2 < v_1$, swallowing it in one bite.

a) What is the speed of the shark after its tasty meal, sadly eaten without wasabi (mmm, sashimi!)?

b) Did the shark gain (kinetic) energy, lose energy, or have its energy remain the same in the process.

Problem 243. problems-1/angular-momentum-sa-bug-on-rotating-disk.tex

A disk of mass M and radius R is rotating about its axis with initial angular velocity Ω_0. A rhinoceros beetle with mass m is standing on its outer rim as it does so. The beetle decides to walk in to the very center of the disk and stand on the axis as it feels less pseudoforce there and it is easier to hold on. What is the angular velocity of the disk when it gets there?

(Ignore friction and drag forces).

Problem 244. problems-1/angular-momentum-sa-conserved-quantities.tex

For each of the collisions described below, say whether the total mechanical energy, total momentum, and total angular momentum of the system consisting of the two colliding objects are conserved or not. Indicate your answer by writing "**C**" (for "is ***definitely*** conserved") or "**N**" (for "not ***necessarily*** conserved") in each box. You may write a brief word of explanation if you think there is any ambiguity in the answer.

	Total Energy	Linear Momentum	Angular Momentum
A hard ball (point particle) bounces off of a rigid wall that cannot move, returning at the same speed it had before the collision.	☐	☐	☐
A piece of space junk strikes the orbiting space shuttle and sticks to it.	☐	☐	☐

Problem 245. problems-1/angular-momentum-sa-conserved-quantities-soln.tex

For each of the collisions described below, say whether the total mechanical energy, total momentum, and total angular momentum of the system consisting of the two colliding objects are conserved or not. Indicate your answer by writing "**C**" (for "is ***definitely*** conserved") or "**N**" (for "not ***necessarily*** conserved") in each box. You may write a brief word of explanation if you think there is any ambiguity in the answer.

	Total Energy	Linear Momentum	Angular Momentum
A hard ball (point particle) bounces off of a rigid wall that cannot move, returning at the same speed it had before the collision.	C		N

Explanation: It bounces off at the same speed (given) so the collision *is* elastic *as given.* Linear momentum is not conserved, period (so blank). Angular momentum is conserved *only if the pivot is on the line of motion of the particle so it is zero before and after the collision.* This is rather unlikely (and not helpful in solving any sort of problem) but is enough for this to earn an N as it *could* happen if a coordinate system of this sort was given.

	Total Energy	Linear Momentum	Angular Momentum
A piece of space junk strikes the orbiting space shuttle and sticks to it.		C	C

Explanation: The collision is "fully inelastic" as they stick together and lose all of the kinetic energy initially present in the center of mass reference frame, so total mechanical energy is definitely not conserved. *In the (usual) impulse approximation* gravity (holding the shuttle "in orbit" exerts a negligible force during the *short* time of the collision and no other forces are present (it's in a vacuum so no drag etc). Hence *both* momentum *and* angular momentum are conserved, as no external torques act either.

Problem 246. problems-1/angular-momentum-sa-conserved-quantities-soln.tex

For each of the collisions described below, say whether the total mechanical energy, total momentum, and total angular momentum of the system consisting of the two colliding objects are conserved or not. Indicate your answer by writing "**C**" (for "is ***definitely*** conserved") or "**N**" (for "not ***necessarily*** conserved") in each box. You may write a brief word of explanation if you think there is any ambiguity in the answer.

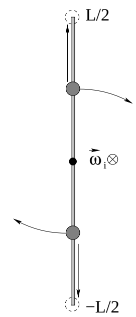
	Total Energy	Linear Momentum	Angular Momentum
A hard ball (point particle) bounces off of a rigid wall that cannot move, returning at the same speed it had before the collision.	$\boxed{\text{C}}$	$\boxed{}$	$\boxed{\text{N}}$

Explanation: It bounces off at the same speed (given) so the collision *is* elastic *as given.* Linear momentum is not conserved, period (so blank). Angular momentum is conserved *only if the pivot is on the line of motion of the particle so it is zero before and after the collision.* This is rather unlikely (and not helpful in solving any sort of problem) but is enough for this to earn an N as it *could* happen if a coordinate system of this sort was given.

	Total Energy	Linear Momentum	Angular Momentum
A piece of space junk strikes the orbiting space shuttle and sticks to it.	$\boxed{}$	$\boxed{\text{C}}$	$\boxed{\text{C}}$

Explanation: The collision is "fully inelastic" as they stick together and lose all of the kinetic energy initially present in the center of mass reference frame, so total mechanical energy is definitely not conserved. *In the (usual) impulse approximation* gravity (holding the shuttle "in orbit" exerts a negligible force during the *short* time of the collision and no other forces are present (it's in a vacuum so no drag etc). Hence *both* momentum *and* angular momentum are conserved, as no external torques act either.

Problem 247. problems-1/angular-momentum-sa-rotating-rod-sliding-beads.tex



In the figure above, a massless rod of length L is rotating around a frictionless pivot through its center at angular speed ω_i. Two beads, each with mass m, are stuck a distance $L/4$ from the center. The rotating system initially has a total kinetic energy K_i (which you could actually calculate if you needed to). At a certain time, the beads are released and slide smoothly to the ends of the rod where again, they stick.

A) What quantities of the system (rod plus two beads) are conserved by this process? (Place a Y or N in the provided answer boxes.)

☐ Total Kinetic Energy

☐ Total Linear Momentum

☐ Total Angular Momentum

B) Determine the ratio of the following quantities:

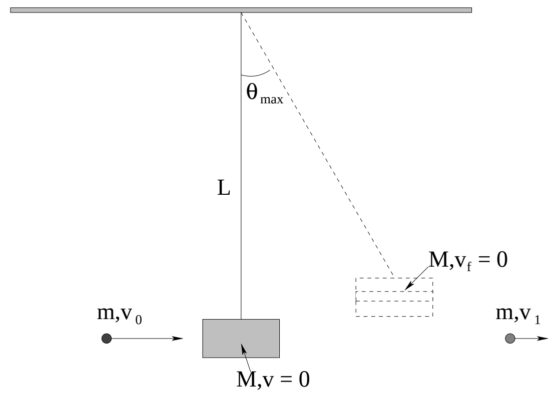
$$\frac{I_f}{I_i} =$$

$$\frac{\omega_f}{\omega_i} =$$

$$\frac{K_f}{K_i} =$$

6.2.3 Regular Problems

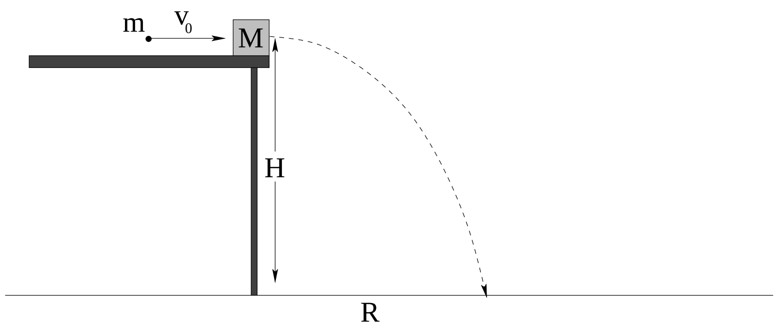
Problem 248. problems-1/momentum-pr-ballistic-pendulum-partially-inelastic.tex



In the figure above, a bullet of mass m and initial velocity v_0 passes through a block of mass M suspended by an unstretchable, massless string of length L from an overhead support as shown. It emerges from the **collision** on the far side travelling at $v_1 < v_0$. This happens extremely quickly (before the block has time to swing up) and the mass of the block is **unchanged** by the passage of the bullet (the mass removed making the hole is negligible, in other words). After the collision, the block swings up to a maximum angle $\theta_{\max}$ and then stops.

Find $\theta_{\max}$.

Problem 249. problems-1/momentum-pr-bullet-block-free-fall-1.tex



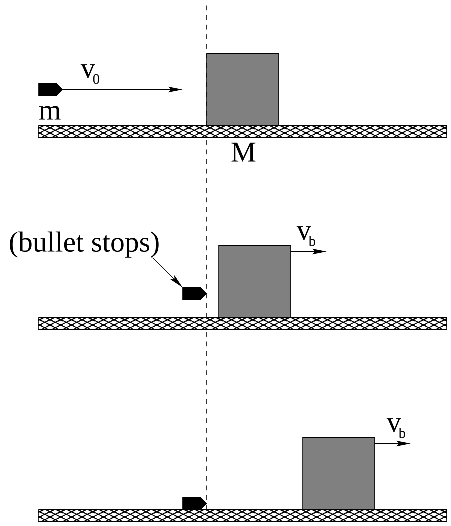
A bullet of mass m travelling at speed v_0 in the direction shown above strikes a block of mass M and embeds itself in it. The block is sitting on the edge of a frictionless table of height H and is knocked off of the table by the collision.

a) What is the speed v_b of the block immediately after the bullet sticks?

b) What distance R from the base of the table does the block land?

Note: If you cannot solve a), just use the symbol v_b where needed to get possibly full credit for b). Do not just use a memorized formula for b): Clearly state the physical principle(s) you are using and work out the answers.

Problem 250. problems-1/momentum-pr-bullet-stops-at-block.tex



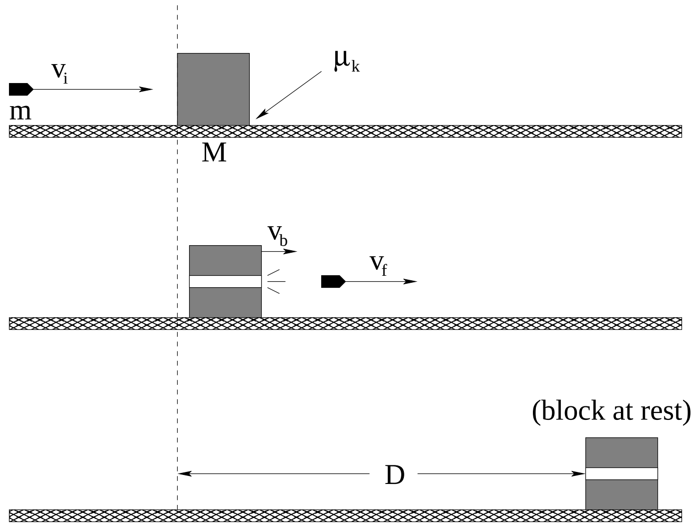
In the figure above a bullet of mass m is travelling at initial speed v_0 to the right when it strikes a larger block of mass M that is resting on a smooth (frictionless) horizontal table.

Instead of "sticking" in the block, the bullet is stopped cold by the block and falls to the ground, while the block recoils from the collision to the right. Note that this collision is *partially inelastic*, so *some* mechanical energy will be lost.

a) What is the velocity of the block v_b immediately after the collision.

b) How much energy is lost in the collision?

Problem 251. problems-1/momentum-pr-bullet-through-block-rough-surface.tex



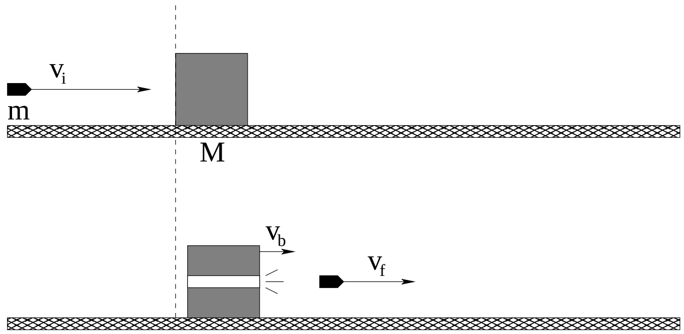
In the figure above a bullet of mass m is travelling at initial speed v_i to the right when it strikes a larger block of mass M that is resting on a rough horizontal table (with coefficient of friction between block and table of μ_k). Instead of "sticking" in the block, the bullet blasts its way *through* the block (without changing the mass of the block significantly in the process). It emerges with the smaller speed v_f, still to the right.

a) Find the speed of the block v_b *immediately after* the collision (but before the block has had time to slide any significant distance on the rough surface).

b) Find the (kinetic) energy lost during this collision. Where did this energy go?

c) How far down the rough surface D does the block slide before coming to rest?

Problem 252. problems-1/momentum-pr-bullet-through-block.tex

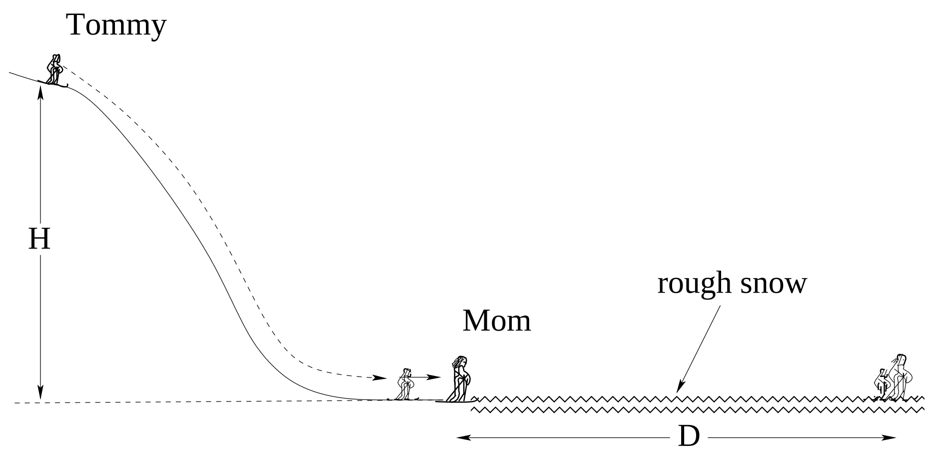


In the figure above a bullet of mass m is travelling at initial speed v_i to the right when it strikes a larger block of mass M that is resting on a horizontal table. Instead of "sticking" in the block, the bullet blasts its way *through* the block (without changing the mass of the block significantly in the process). It emerges with the smaller speed v_f, still to the right.

a) What is the velocity of the block v_b immediately after the collision.

b) How much energy is lost in the collision?

Problem 253. problems-1/momentum-pr-collision-icy-hill-to-rough-snow.tex



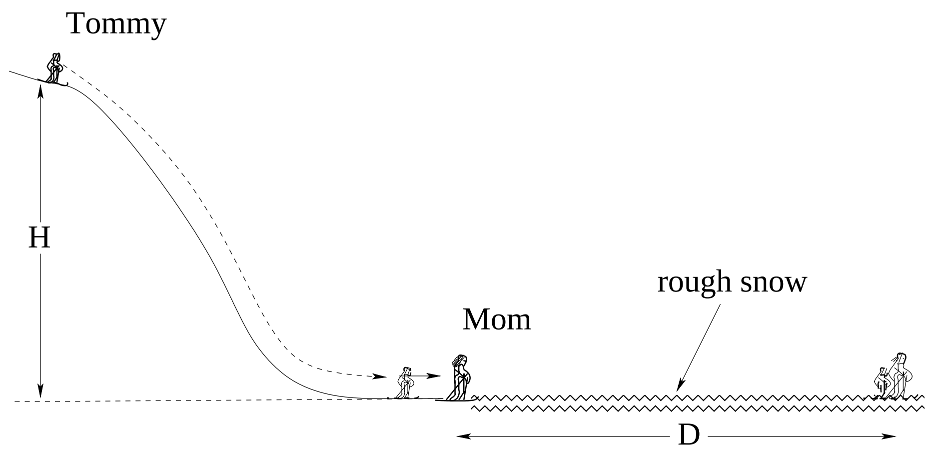
Tommy is learning to ski, but he isn't very good at it. Starting from rest, he skiis down a ***frictionless*** slope of height H above a flat plane, where he runs into Mom. The two of them stick together and slide forward a distance D across a patch of rough snow with coefficient of kinetic friction μ_k until they come to rest. The mass of Tommy is m_t, the mass of his mother is M_m. Answer the following algebraic questions in terms of H, μ_k, m_t, M_m, and g:

a) How fast is Tommy going immediately before he collides with his mother?

b) Find D.

c) How much energy is gained or lost ***during the collision*** between Tommy and his mother? Indicate clearly whether the energy is gained or lost.

Problem 254. problems-1/momentum-pr-collision-icy-hill-to-rough-snow-soln.tex



Tommy is learning to ski, but he isn't very good at it. Starting from rest, he skiis down a ***frictionless*** slope of height H above a flat plane, where he runs into Mom. The two of them stick together and slide forward a distance D across a patch of rough snow with coefficient of kinetic friction μ_k until they come to rest. The mass of Tommy is m_t, the mass of his mother is M_m. Answer the following algebraic questions in terms of H, μ_k, m_t, M_m, and g:

a) How fast is Tommy going immediately before he collides with his mother?

b) Find D.

c) How much energy is gained or lost ***during the collision*** between Tommy and his mother? Indicate clearly whether the energy is gained or lost.

Solution: For part a), use conservation of mechanical energy and solve it for Tommy's speed at the bottom of the hill:

$$m_t g H = \frac{1}{2} m_t v_b^2 \quad \Rightarrow \quad v_t = \boxed{\sqrt{2gH}}$$

For b) we have to solve the *inelastic* collision using conservation of momentum:

$$p_i = m_t v_t = m_t \sqrt{2gH} = (m_t + M_m) v_f = p_f$$

Note that I'm not bothering to solve for v_f as we don't really need it to answer the questions. *After* the collision (and using things like $N - (m_t + M_m)g = 0$, $f_k = -\mu_k N$), we use non-conservative work, mechanical energy:

$$W_{nc} = -\mu_k (m_t + M_m) g D = E_f - E_i = 0 - \frac{p_f^2}{2(m_t + M_m)} = -\frac{m_t^2 2gH}{2(m_t + M_m)}$$

We rearrange to solve for:

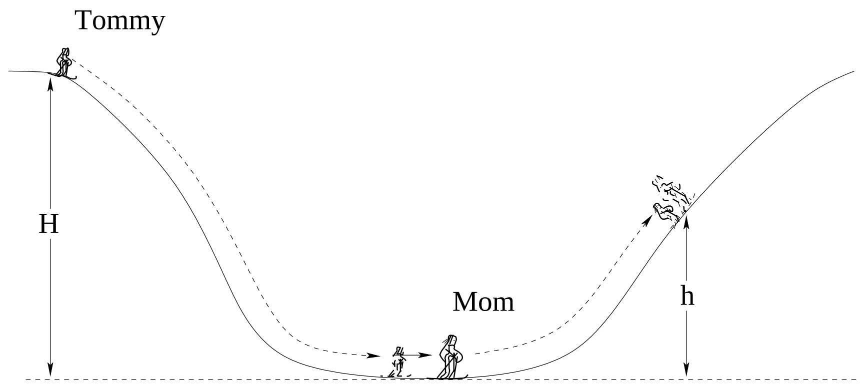
$$D = \boxed{\frac{m_t^2}{\mu_k (m_t + M_m)^2} H}$$

Finally, c) is easy:

$$\Delta K = K_f - K_i = \frac{p_f^2}{2(m_t + M_m)} - \frac{p_f^2}{2m_t} = -\frac{M_m}{m_t + M_m}\left(\frac{1}{2}m_t v_t^2\right) = \boxed{-\frac{M_m}{m_t + M_m} m_t g H}$$

Note that this is (as it must be!) expressed in terms of the *givens*, and indicates the fraction of Tommy's initial *potential* energy that was lost in the collision.

Problem 255. problems-1/momentum-pr-collision-on-icy-hills.tex



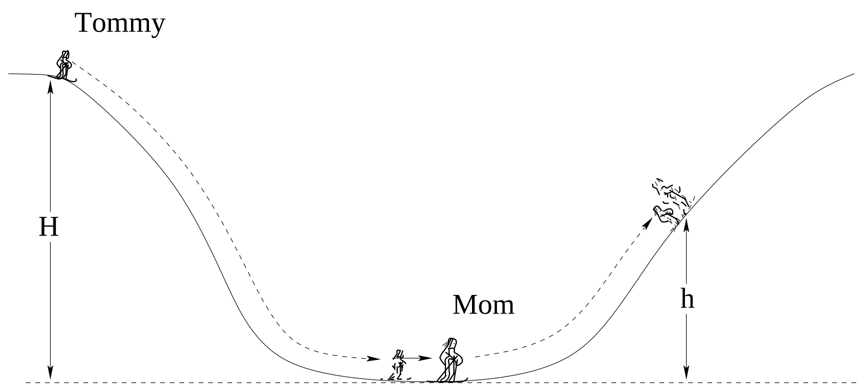
Tommy is learning to ski, but he isn't very good at it. Starting from rest, he skiis down a ***frictionless*** slope of height H above a small valley, where he runs into Mom. The two of them stick together and slide up the slope on the far side to a new height h. The mass of Tommy is m_t, the mass of his mother is M_m. ***Ignore all drag and friction***, and answer the following algebraic questions in terms of H, m_t, M_m, and g:

a) How fast is Tommy going immediately before he collides with his mother?

b) Find h.

c) How much energy is gained or lost ***during the collision*** between Tommy and his mother? Indicate clearly whether the energy is gained or lost.

Problem 256. problems-1/momentum-pr-collision-on-icy-hills-soln.tex



Tommy is learning to ski, but he isn't very good at it. Starting from rest, he skiis down a ***frictionless*** slope of height H above a small valley, where he runs into Mom. The two of them stick together and slide up the slope on the far side to a new height h. The mass of Tommy is m_t, the mass of his mother is M_m. ***Ignore all drag and friction***, and answer the following algebraic questions in terms of H, m_t, M_m, and g:

a) How fast is Tommy going immediately before he collides with his mother?

b) Find h.

c) How much energy is gained or lost ***during the collision*** between Tommy and his mother? Indicate clearly whether the energy is gained or lost.

Solution: For a), use Conservation of Mechanical Energy, as we are neglecting all non-conservative forces like drag and friction:

$$E_i = m_t g H = \frac{1}{2} m_t v_t^2 = E_i$$

so

$$\boxed{v_t = \sqrt{2gH}}$$

For b), momentum ***only*** is conserved during the collision, so:

$$p_i = m_t v_t = (m_t + M_m) v_f = p_f$$

Energy is lost as it is a ***fully inelastic collision***, and c) below has you explicitly compute this. Using the final combined momentum as the initial state, though, energy is once again conserved on the way up the hill, so:

$$E_i = \frac{p_f^2}{2(m_t + M_m)} = \frac{m_t^2 2gH}{2(m_t + M_m)} = \frac{m_t^2 gH}{(m_t + M_m)} = (m_t + M_m) gh = E_f$$

We solve algebraically, substituting as needed to make all answers appear in terms of the givens, for:

$$\boxed{h = \left(\frac{m_t^2}{(m_t + M_m)^2}\right) H}$$

Finally for c), we simply find the change in (kinetic) energy during the collision directly:

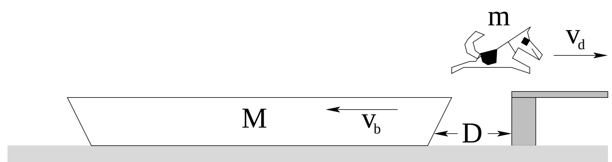
$$\Delta E = E_f - E_i = \frac{m_t^2 gH}{(m_t + M_m)} - m_t gH = \left(\frac{m_t}{(m_t + M_m)} - 1\right) m_t gH$$

Or

$$\boxed{\Delta E = -\left(\frac{M_m}{(m_t + M_m)}\right) m_t gH}$$

where the minus sign means energy is ***lost***.

Problem 257. problems-1/momentum-pr-dog-jumps-from-boat.tex

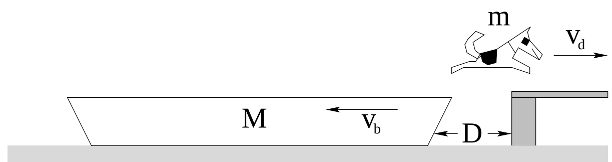


A dog of mass m gets hungry while ***sitting*** at the end of a boat of mass M and length L that is ***at rest*** on the water of a lake. He jumps out onto the dock to go get some tasty dog chunks that are waiting for him at home when the boat is a distance D away from the dock as shown. The dog travels at a horizontal speed v_d ***relative to the ground/lake*** as he flies through the air.

a) What is the recoil speed of the boat, v_b, while the dog is in the air? Assume that dog and boat are both at rest before the jump.

b) How much work did the dog's legs do during the jump?

Problem 258. problems-1/momentum-pr-dog-jumps-from-boat-soln.tex



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a) What is the recoil speed of the boat, v_b, while the dog is in the air? Assume that dog and boat are both at rest before the jump.

b) How much work did the dog's legs do during the jump?

The initial momentum of the boat is zero. Hence remains zero while the dog is in the air:

$$p_i = 0 = -Mv_b + mv_d = p_f \Longrightarrow v_b = \boxed{\frac{m}{M}v_d}$$

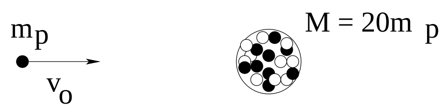
(where I put in the minus sign so the answer would be positive, the boat's speed is the magnitude of the boat's velocity).

The work done by the dog's legs is the total kinetic energy of the boat ***and*** the dog after the dog jumps; nothing else does work in the system. Hence:

$$W = \frac{1}{2}Mv_b^2 + \frac{1}{2}mv_d^2 = \boxed{\left(1 + \frac{m}{M}\right)\frac{1}{2}mv_d^2}$$

in terms of the givens (several other ways to write this answer, all of them OK).

Problem 259. problems-1/momentum-pr-elastic-collision-proton-neon.tex



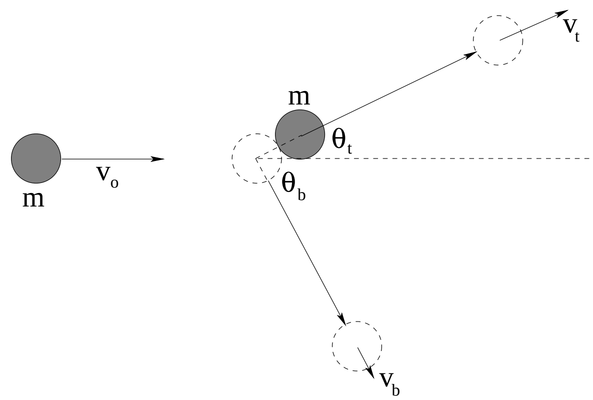
A Proton of mass m_p is directly incident on a Neon nucleus with mass $20m_p$. It is initially (far away from the nucleus) travelling with speed v_0. The two particles repel each other (like charges repel) as they approach, and the force of repulsion is strong enough to prevent the particles from touching. The "collision" that takes place gradually between the two particles is elastic.

a) At some distant time in the future (after the collision) is the proton moving to the left or to the right?

b) What is the speed of the proton when it and the Neon nucleus are at the distance of closest approach?

c) What is the speed of the Neon nucleus at a distant time in the future (after the collision) when they are once again far apart.

Problem 260. problems-1/momentum-pr-elastic-two-balls.tex



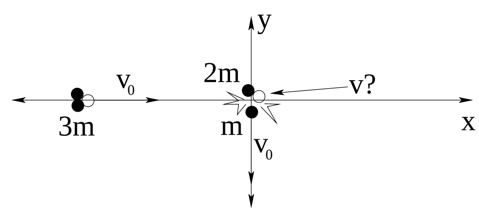
In the figure above, a ball with mass $m = 1kg$ and speed $v_0 = 5$ m/sec ***elastically*** collides with a stationary, identical ball (all resting on a frictionless surface so gravity is irrelevant). A student measures the top ball emerging from the collision at a speed $v_t = 4$ m/sec at an angle $\theta_t \approx 37°$ as shown.

a) Find the speed v_b of the other ball.

b) Find the angle θ_b of the other ball. (Hint: Draw a triangle with sides of length v_0, v_t, v_b.)

c) What does $\theta_t + \theta_b$ add up to? (This is a characteristic of all elastic collisions between identical masses in 2 dimensions.)

Problem 261. problems-1/momentum-pr-fission.tex

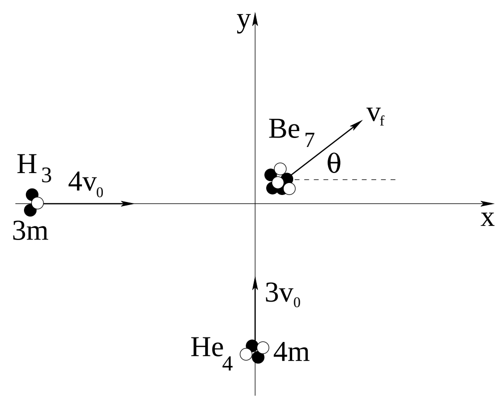


An atomic nucleus of mass $3m$ is travelling to the right at velocity $\vec{v}_{\text{initial}} = v_0\hat{x}$ as shown. It spontaneously fissions into two fragments of mass m and $2m$. The smaller fragment m travels straight down at velocity $\vec{v}_m = -v_0\hat{y}$ after the fission.

a) What is the ***velocity*** of the larger fragment?

b) What is the ***net*** energy ***gain or loss*** (indicate which!) from the fission process, in terms of the initial kinetic energy?

Problem 262. problems-1/momentum-pr-h3-he4-fusion.tex



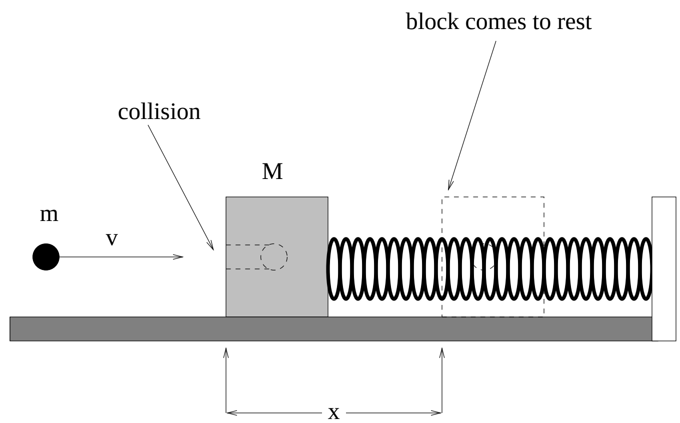
In the figure above, a Tritium (H_3) nucleus and a Helium (He_4) nucleus collide and fuse inelastically into Beryllium (Be_7), an important nucleosynthesis process in the early Universe. The velocity of the H_3 is $4v_0\hat{\boldsymbol{i}}$, the velocity of the He_4 is $3v_0\hat{\boldsymbol{j}}$ as drawn. **Show your work and reasoning** to answer the following questions in terms of the given quantities m and v_0:

a) Find the final velocity vector of the combined object, expressed using vector notation (e.g. $\vec{A} = A_x\hat{\boldsymbol{i}} + A_y\hat{\boldsymbol{j}}$);

b) Find the magnitude of the final velocity v_f and its angle θ with respect to the x-direction;

c) Find the change of momentum vectors $\Delta\vec{\boldsymbol{p}}_H$ for the H_3 nucleus ***and*** $\Delta\vec{\boldsymbol{p}}_{He}$ for the He_4 nucleus. (Recall that $\Delta\vec{\boldsymbol{p}} == \vec{\boldsymbol{p}}_f - \vec{\boldsymbol{p}}_i$.) Briefly discuss how they are related and what this means.

Problem 263. problems-1/momentum-pr-inelastic-collision-ball-bearing-spring.tex

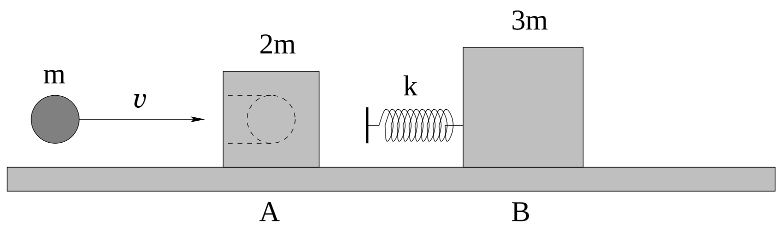


A ball bearing of mass $m = 50$ grams travelling at 200 m/sec smacks into a block of mass $M = 950$ gms and sticks in a hole drilled therein. The block is initially at rest on an Acme frictionless table and is also connected to an Acme spring with spring constant $k = 400$ N/m at its equilibrium position (see figure).

a) What is the maximum distance x the spring is compressed by the recoiling ball bearing-block system?

b) How much mechanical energy is lost in the collision (noting that an answer of 'none' is one possibility)?

Problem 264. problems-1/momentum-pr-inelastic-elastic-collision.tex



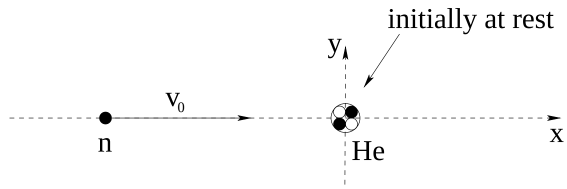
Two masses A and B rest on a frictionless surface, with a massless spring with spring constant k connected to B. A bullet coming from the left with speed v hits A and becomes embedded in it. The masses of the bullet, A and B are m, $2m$ and $3m$ respectively.

a) What is the speed v_{cm} of the center of mass of the system consisting of A, B and the bullet?

b) Immediately after A gets hit by the bullet, what is the speed v_A of A (with the bullet embedded in it) before it hits the spring?

c) In the subsequent motion of the system, what is the maximum compression $\Delta x_{\max}$ of the spring?

Problem 265. problems-1/momentum-pr-neutron-collides-with-helium.tex



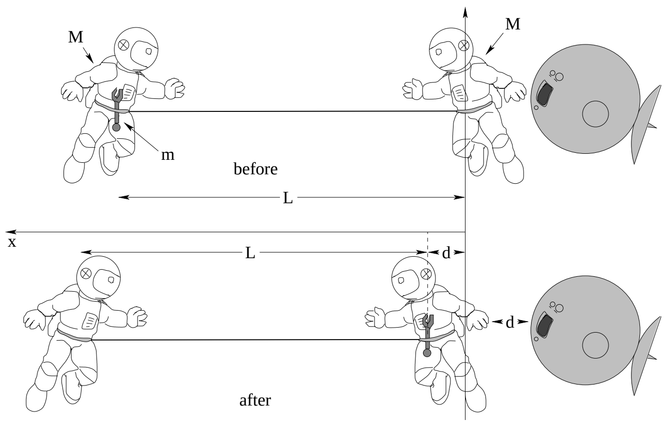
In the figure above, a neutron of mass m collides elastically with a helium nucleus of mass $4m$, striking it head on so that the collision is one dimensional. The initial speed of the the neutron is v_0; the helium nucleus is initially at rest. In answering the following questions you may *either* find *or* just remember the solution for one dimensional elastic collisions – you do not have to derive it, although you may if you wish or cannot remember it.

a) What is the final ***velocity*** of the neutron (magnitude and direction) after the collision.

b) What is the final ***velocity*** of the helium nucleus after the collision.

c) Is the helium nucleus moving faster or slower than the neutron is moving after the collision? (Does your answer make sense?)

Problem 266. problems-1/momentum-pr-two-astronauts.tex



Two astronauts with identical mass M (including their spacesuits) in free-fall are working on a satellite. They are connected by a taut tether rope of length L. The first astronaut on the needs a tool of mass m that the second astronaut is carrying (also initially a distance L away), so the second one tosses the tool to the first at speed v_0.

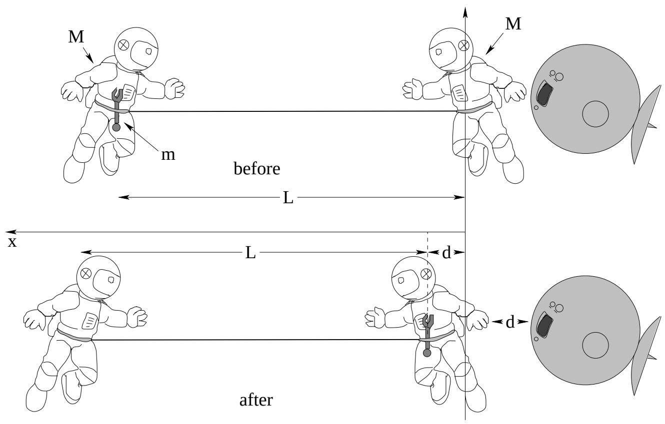
a) What is the speed of the two astronauts while the tool is in space flying freely between them?

b) What is the speed of the two astronauts after the first one catches the tool?

c) The first astronaut cannot reach the satellite if he has drifted a distance d further away while the tool was in flight. Find the maximum length L that the tether can have such that the first astronaut can still reach the satellite. Does the answer depend on v_0?

d) The tool apparently makes a *fully inelastic* collision with the second astronaut. We have learned that inelastic collisions *lose* total mechanical energy. Yet the initial and final energy of the system is the same! Explain how that can be.

Problem 267. problems-1/momentum-pr-two-astronauts-soln.tex



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a) What is the speed of the two astronauts while the tool is in space flying freely between them?

b) What is the speed of the two astronauts after the first one catches the tool?

c) The first astronaut cannot reach the satellite if he has drifted a distance $d_{\max}$ further away while the tool was in flight. Find the maximum length L that the tether can have such that the first astronaut can still reach the satellite. Does the answer depend on v_0?

d) The tool apparently makes a *fully inelastic* collision with the second astronaut. We have learned that inelastic collisions *lose* total mechanical energy. Yet the initial and final energy of the system is the same! Explain how that can be.

Solution: a) is simply a matter of (1D, x-direction) momentum conservation. Initially the momentum is zero. While the tool is in the air, then:

$$p_i = 0 = mv_0 - 2Mv_a = 0 = p_f \quad \Rightarrow \quad \boxed{v_a = \frac{m}{2M}v_0}$$

b) is the same idea:

$$p_i = 0 = (2M + m)v_f = 0 = p_f \quad \Rightarrow \quad \boxed{v_f = 0}$$

(you could have done this one "by inspection" with no algebra at all).

To answer c), we need to find d, the "drift distance", after the second astronaut has caught the tool. Note that:

$$p = 0 = M_{\rm tot}v_{\rm cm} = (2M + m)v_{\rm cm} \quad \Rightarrow \quad v_{\rm cm} = 0 = \frac{dx_{\rm cm}}{dt} \quad \Rightarrow \quad x_{\rm cm} = \textbf{Constant!}$$

at all times, even when the tool is in the, umm, 'air'! We then pop a coordinate frame onto the picture (provided for you this time) and write:

$$x_{\mathrm{cm},i} = \cancel{\frac{1}{2M + m}}\left((m + M)L + M \times 0\right) = \cancel{\frac{1}{2M + m}}\left((m + M)d + M(L + d)\right) = x_{\mathrm{cm},f}$$

$$\cancel{ML} + mL = (m + M)d + \cancel{ML} + Md$$

or (solving for d):

$$\boxed{d = \frac{mL}{2M + m}}$$

The astronaut can't reach the satellite if $d > d_{\rm max}$:

$$\frac{mL}{2M + m} > d_{\rm max}$$

so the *largest* L that allows him to still reach occurs when these two terms are equal:

$$\frac{mL_{\rm max}}{2M + m} = d_{\rm max} \quad \Rightarrow \quad \boxed{L_{\rm max} = \frac{(2M + m)}{m} d_{\rm max}}$$

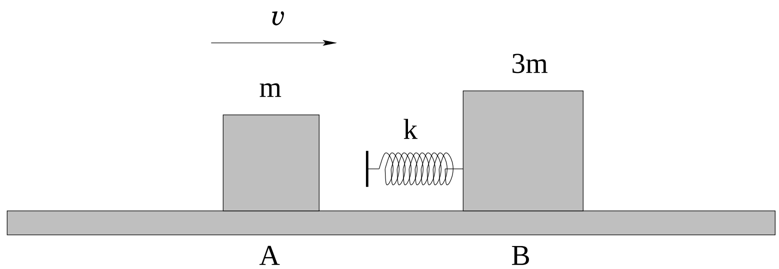
which obviously doesn't depend on v_0. Hopefully it makes *sense* that it would not depend on v_0, since momentum is conserved independent of its (intermediate) value!

Finally, to answer d) we need only note that the first astronaut *did non-conservative work on the system* when he threw the wrench. You could even compute how much work he did:

$$W_{\rm in} = K_a + K_t = \frac{1}{2}(2M)v_a^2 + \frac{1}{2}mv_0^2 = \frac{1}{2}(2M)\frac{m^2}{4M^2}v_0^2 + \frac{1}{2}mv_0^2 = \left(1 + \frac{m}{2M}\right)\frac{1}{2}mv_0^2 = -W_{\rm out}$$

Obviously, this is *also* exactly equal to the work *lost* in the collision when the second astronaut catches the tool, so total mechanical energy is still conserved because the non-conservative work being done a *both* ends cancels!

Problem 268. problems-1/momentum-pr-two-block-elastic-collision.tex



Two blocks A and B collide elastically on a frictionless surface. A massless spring with spring constant k is connected to block B. Initially, block A moves to the right with a speed v to collide with block B which is initially stationary.

Show your work.

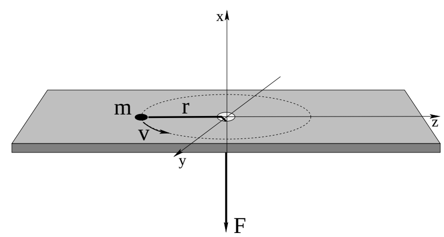
a) Before the collision, what is the speed v_{cm} of the center of mass of the system consisting of blocks A and B?

$v_{\text{cm}} =$ $\boxed{}$

b) During the collision, what is the maximum compression $\Delta x_{\max}$ of the spring?

$\Delta x_{\max} =$ $\boxed{}$

Problem 269. problems-1/angular-momentum-pr-circular-orbit-on-table.tex



A particle of mass M is tied to a string that passes through a hole in a frictionless table and held. The mass is given a push so that it initially moves in a circle of radius r_i at speed v_i. We will now conceptual review and algebraically analyze the physics of its motion in two stages. Please answer the following questions. While the string is fixed (so that r_i is constant):

a) What is the torque exerted on the particle by the string?

b) What is the ***vector angular momentum*** L_i of the particle? Use the provided coordinate system to give the direction.

c) Show that the magnitude of the force (the tension in the string) that must be exerted to keep the particle moving in this circle is:

$$F = \frac{L_i^2}{mr_i^3}$$

Note that ***this is a general result*** for a particle moving in a circle and in no way depends on the fact that the force is being exerted by a string in particular.

d) Show that the kinetic energy of the particle in terms of its angular momentum is:

$$K_i = \frac{L_i^2}{2mr_i^2}$$

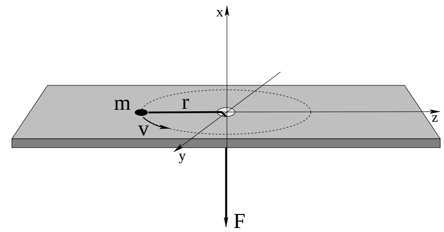
From under the table, the string is *slowly* pulled down (so that the puck is always moving in an approximately circular trajectory and the tension in the string remains radial) to where the particle is moving in a circle of radius r_f.

e) If the tension in the string remains radial, what quantity ought to be conserved?

f) Find its velocity v_f using conservation of that quantity.

g) Compute the work done by the force from part c) above and identify the answer *as* the work-kinetic energy theorem. Use this principle *instead* to to find the velocity v_f. You should get the same answer!

Problem 270. problems-1/angular-momentum-pr-circular-orbit-on-table-soln.tex



A particle of mass M is tied to a string that passes through a hole in a frictionless table and held. The mass is given a push so that it initially moves in a circle of radius r_i at speed v_i. We will now conceptual review and algebraically analyze the physics of its motion in two stages. Please answer the following questions. While the string is fixed (so that r_i is constant):

a) What is the torque exerted on the particle by the string?

b) What is the ***vector angular momentum*** L_i of the particle? Use the provided coordinate system to give the direction.

c) Show that the magnitude of the force (the tension in the string) that must be exerted to keep the particle moving in this circle is:

$$F = \frac{L_i^2}{mr_i^3}$$

Note that ***this is a general result*** for a particle moving in a circle and in no way depends on the fact that the force is being exerted by a string in particular.

d) Show that the kinetic energy of the particle in terms of its angular momentum is:

$$K_i = \frac{L_i^2}{2mr_i^2}$$

From under the table, the string is *slowly* pulled down (so that the puck is always moving in an approximately circular trajectory and the tension in the string remains radial) to where the particle is moving in a circle of radius r_f.

e) If the tension in the string remains radial, what quantity ought to be conserved?

f) Find its velocity v_f using conservation of that quantity.

g) Compute the work done by the force from part c) above and identify the answer *as* the work-kinetic energy theorem. Use this principle *instead* to to find the velocity v_f. You should get the same answer!

Solution:

a) $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times (-T)\hat{\boldsymbol{r}} = 0 = \frac{d\vec{\boldsymbol{L}}}{dt}$

b) $L_i = |\vec{\boldsymbol{r}}_i \times m\vec{\boldsymbol{v}}_i| = mv_ir_i$. The direction, given by the right hand rule, is $\hat{\boldsymbol{x}}$ (or "up", although using coordinates is better).

c) Using Newton's Second Law and centripetal acceleration:

$$F(=T) = \frac{mv_i^2}{r_i} = \frac{mv_i^2}{r_i} \times \frac{mr_i^2}{mr_i^2} = \frac{(mv_ir_i)^2}{mr_i^3} = \frac{L_i^2}{mr_i^3}$$

Note that we multipled by "1" in a convenient form in the middle.

d) We'll start with the standard "easy" version of the kinetic energy:

$$K_i = \frac{1}{2}mv_i^2 = \frac{1}{2}mv_i^2 \times \frac{mr_i^2}{mr_i^2} = \frac{(mv_ir_i)^2}{2mr_i^2} = \frac{L_i^2}{2mr_i^2}$$

Note that we multipled ***again*** by "1" in that ***same*** convenient form in the middle.

e) Because the force exerted by the tension is almost always perpendicular to the velocity of the particle, the torque exerted by the tension remains (almost exactly) zero. We therefore *expect* the angular momentum to be (almost exactly) conserved.

f) This permits us to use angular momentum conservation to find v_f given (in the problem) r_f:

$$L_i = mv_ir_i = mv_fr_f = L_f \quad \Rightarrow \quad v_f = \frac{r_i}{r_f}v_i$$

g) We have to be very careful here. The tension T is *also* nearly perpendicular to $\vec{\boldsymbol{v}}$ throughout the motion, so one might conclude that no work is done and energy is also conserved. However, *one's hand, pulling down on the string, absolutely* does work and that work is not dissipated by any non-conservative forces, so it must appear as a change in kinetic energy! Expressing F in terms of (constant) $L = mv_ir_i$:

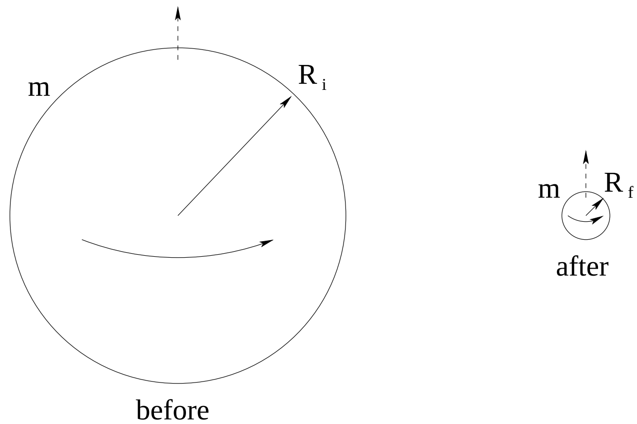
$$W = \int_{r_i}^{r_f} F dr = -\int_{r_i}^{r_f} \frac{L^2}{m} r^{-3} dr = \left.\frac{L^2}{2mr^2}\right|_{r_i}^{r_f} = fracL^2 2mr_f^2 - fracL^2 2mr_i^2 = \Delta K$$

which is the WKE theorem. The final step is easy – note that:

$$K_f = \frac{\cancel{1}}{\cancel{2}}\cancel{m}v_f^2 = \frac{(\cancel{m}v_ir_i)^2}{\cancel{2}\cancel{m}r_f^2} \quad \Rightarrow \quad v_f = \frac{r_i}{r_f}v_i$$

as before. Pay careful attention to this problem, as it is a conceptual key to steps involved in solving *several* problems in this course as well as deriving things like the "angular momentum barrier" in the chapter on gravitation.

Problem 271. problems-1/angular-momentum-pr-collapse-of-sun.tex



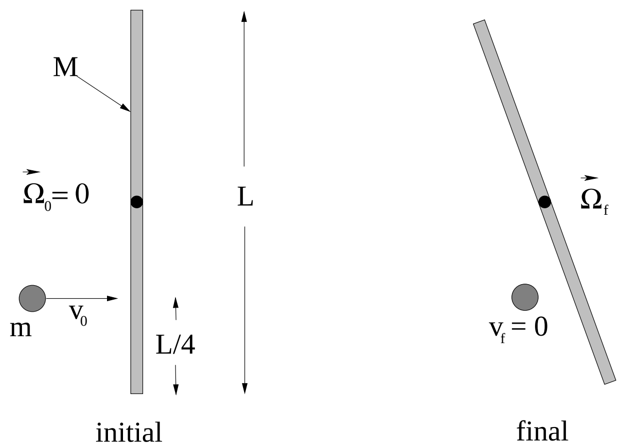
The sun reaches the end of its life and gravitationally collapses quite suddenly, forming a white dwarf. Before it collapses, it has a mass m, a radius R_i, and a period of rotation T_i. After it collapses, its radius is $R_f \ll R_i$ and we will assume that its mass is unchanged. We will also assume that before and after the moment of inertia of the sun is given by $I = \beta m R^2$ where R is the appropriate radius.

a) What is its final period of rotation T_f after the collapse?

b) Evaluate the escape velocity from the surface of the sun before and after its collapse.

For 2 points of extra credit, evaluate the numbers associated with these expressions given $\beta = 0.25$, $m = 2 \times 10^{30}$ kg, $R_i = 5 \times 10^5$ km, $R_f = 100$ km, and $T_i = 108,000$ seconds. These numbers are actually quite interesting in cosmology, as the escape velocity from the surface of the white dwarf approaches the speed of light...

Problem 272. problems-1/angular-momentum-pr-disk-collides-with-pivoted-rod.tex



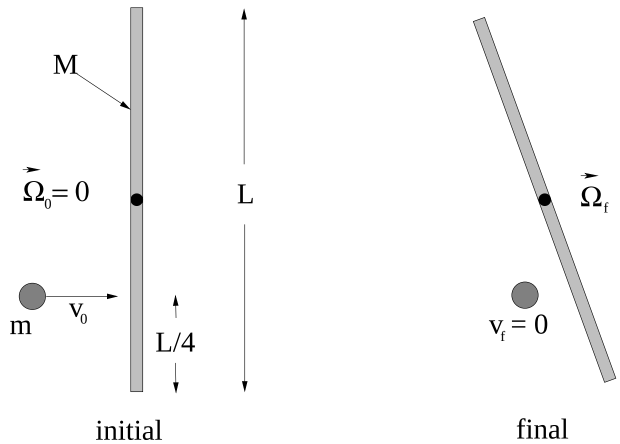
A steel rod of mass M and length L with a frictionless pivot in the center and moment of inertia $\frac{1}{12}ML^2$ sits on a frictionless table at rest. The pivot is attached to the table. A steel disk of mass m approaches with velocity v_0 from the left and strikes the rod a distance $L/4$ from the lower end as shown. This ***elastic*** collision instantly brings the disk to rest and causes the rod to rotate with angular velocity $\vec{\Omega_f}$ (rotating counterclockwise as drawn).

a) What quantities are conserved in this collision?

b) Find the angular velocity $\vec{\Omega_f}$ of the rod about the pivot after the collision (don't forget direction).

c) Find the ratio m/M such that the collision occurs ***elastically***, as described.

Problem 273. problems-1/angular-momentum-pr-disk-collides-with-pivoted-rod-soln.tex



A steel rod of mass M and length L with a frictionless pivot in the center and moment of inertia $\frac{1}{12}ML^2$ sits on a frictionless table at rest. The pivot is attached to the table. A steel disk of mass m approaches with velocity v_0 from the left and strikes the rod a distance $L/4$ from the lower end as shown. This ***elastic*** collision instantly brings the disk to rest and causes the rod to rotate with angular velocity $\vec{\Omega_f}$ (rotating counterclockwise as drawn).

a) What quantities are conserved in this collision?

b) Find the angular velocity $\vec{\Omega_f}$ of the rod about the pivot after the collision (don't forget direction).

c) Find the ratio m/M such that the collision occurs ***elastically***, as described.

Solution: a) **Energy** (because ***the problem states that the collision is elastic!***); **Angular momentum** (because the frictionless pivot exerts no torque); but **Not** linear momentum. The pivot **can** (and in this case obviously does, see below) exert an external impulse force on the rod+disk system during the collision!

b) *Using* angular momentum conservation (with $\vec{L}$ out of the page before and after from the RHR):

$$L = L_i = mv_0\frac{\ell}{4} = \frac{1}{12}M\ell^2\Omega_f = L_f \quad \Rightarrow \quad \boxed{\Omega_f = 3\frac{m}{M}\frac{v_0}{\ell}}$$

c) We need $\Delta K = K_f - K_i = 0$ for the collision to be elastic as ***given***. Using $K_f = L^2/2I_f$:

$$\Delta K = \frac{m^2v_0^2\ell^2}{32\frac{1}{12}M\ell^2} - \frac{1}{2}mv_0^2 = \left(\frac{3}{4}\frac{m}{M} - 1\right)\frac{1}{2}mv_0^2$$

This is clearly zero when:

$$\boxed{\frac{m}{M} = \frac{4}{3}}$$

Note: Consider the initial momentum, $\vec{\boldsymbol{p}}_i = mv_0\hat{\boldsymbol{x}} \neq 0$ (with $\hat{\boldsymbol{x}}$ to the right as usual). The *final* momentum is clearly ***zero!*** The bullet is at rest and the center of mass of the rod is not moving as the rod rotates! The change in the momentum of the system is thus:

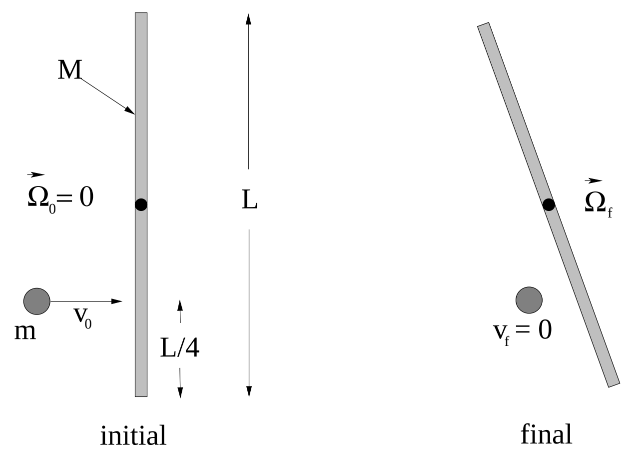
$$\Delta\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}_f - \vec{\boldsymbol{p}}_i = -mv_0\hat{\boldsymbol{x}}$$

If we were given, or could estimate, the time of actual contact in the collision as (say) $\Delta t = t_c$, we could evaluate the ***average force exerted by the pivot during the collision*** as:

$$\vec{\boldsymbol{F}}_{\text{avg}} = \frac{\Delta\vec{\boldsymbol{p}}}{\Delta t} = \frac{-mv_0}{t_c}\hat{\boldsymbol{x}} \neq 0$$

Momentum is clearly not conserved, and the average force exerted by the pivot during the collision is *not negligible*.

Problem 274. problems-1/angular-momentum-pr-disk-collides-with-pivoted-rod-soln.tex



A steel rod of mass M and length L with a frictionless pivot in the center and moment of inertia $\frac{1}{12}ML^2$ sits on a frictionless table at rest. The pivot is attached to the table. A steel disk of mass m approaches with velocity v_0 from the left and strikes the rod a distance $L/4$ from the lower end as shown. This ***elastic*** collision instantly brings the disk to rest and causes the rod to rotate with angular velocity $\vec{\Omega_f}$ (rotating counterclockwise as drawn).

a) What quantities are conserved in this collision?

b) Find the angular velocity $\vec{\Omega_f}$ of the rod about the pivot after the collision (don't forget direction).

c) Find the ratio m/M such that the collision occurs ***elastically***, as described.

Solution: a) **Energy** (because ***the problem states that the collision is elastic!***); **Angular momentum** (because the frictionless pivot exerts no torque); but **Not** linear momentum. The pivot **can** (and in this case obviously does, see below) exert an external impulse force on the rod+disk system during the collision!

b) *Using* angular momentum conservation (with $\vec{\boldsymbol{L}}$ out of the page before and after from the RHR):

$$L = L_i = mv_0\frac{\ell}{4} = \frac{1}{12}M\ell^2\Omega_f = L_f \quad \Rightarrow \quad \boxed{\Omega_f = 3\frac{m}{M}\frac{v_0}{\ell}}$$

c) We need $\Delta K = K_f - K_i = 0$ for the collision to be elastic as ***given***. Using $K_f = L^2/2I_f$:

$$\Delta K = \frac{m^2v_0^2\ell^2}{32\frac{1}{12}M\ell^2} - \frac{1}{2}mv_0^2 = \left(\frac{3}{4}\frac{m}{M} - 1\right)\frac{1}{2}mv_0^2$$

This is clearly zero when:

$$\boxed{\frac{m}{M} = \frac{4}{3}}$$

Note: Consider the initial momentum, $\vec{\boldsymbol{p}}_i = mv_0\hat{\boldsymbol{x}} \neq 0$ (with $\hat{\boldsymbol{x}}$ to the right as usual). The *final* momentum is clearly ***zero!*** The bullet is at rest and the center of mass of the rod is not moving as the rod rotates! The change in the momentum of the system is thus:

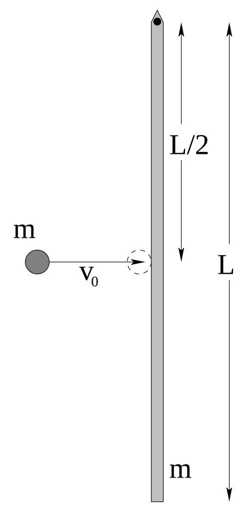
$$\Delta\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}_f - \vec{\boldsymbol{p}}_i = -mv_0\hat{\boldsymbol{x}}$$

If we were given, or could estimate, the time of actual contact in the collision as (say) $\Delta t = t_c$, we could evaluate the ***average force exerted by the pivot during the collision*** as:

$$\vec{\boldsymbol{F}}_{\rm avg} = \frac{\Delta\vec{\boldsymbol{p}}}{\Delta t} = \frac{-mv_0}{t_c}\hat{\boldsymbol{x}} \neq 0$$

Momentum is clearly not conserved, and the average force exerted by the pivot during the collision is *not negligible*.

Problem 275. problems-1/angular-momentum-pr-marble-and-rod.tex



In the figure above, a marble with mass m travelling to the right at speed v_0 collides with a rigid rod of length L pivoted about one end, also of mass m, . The marble strikes the rod $L/2$ down from the pivot and comes **precisely to rest** in the collision. Ignore gravity, drag forces, and any friction in the pivot.

a) What is the ***rotational velocity*** $\boldsymbol{\Omega_f}$ of the rod after the collision?

b) What is the change in ***linear momentum in the x direction*** $\boldsymbol{\Delta p_x}$ (to the right) during this collision?

c) What is the ***change in kinetic energy*** $\boldsymbol{\Delta K}$ in this collision? The sign of your answer should indicate whether energy was gained or lost.

Problem 276. problems-1/angular-momentum-pr-marble-and-rod-soln.tex

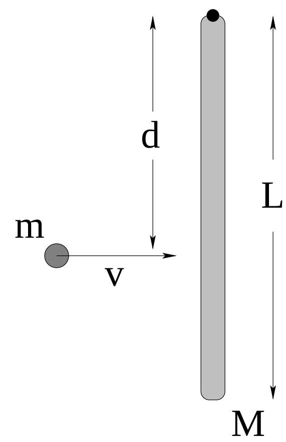
a) Angular momentum is conserved, so $L = L_i = mv_0 L/2 = I_{\rm rod}\Omega = \frac{1}{3}mL^2\Omega = L_f$. Hence

$$\Omega = \frac{3mv_0}{2mL} = \boxed{\frac{3v_0}{2L}}$$

b) $p_{xi} = mv_0$ initially. $p_{xf} = mv_{\rm cm} = m\Omega L/2$ finally. So use answer to a) and form $\Delta p_x = p_{xf} - p_{xi}$.

c) Easiest to use $K_i = \frac{1}{2}mv_0^2$, $K_f = \frac{L^2}{2I_{\rm rod}}$, and subtract.

Problem 277. problems-1/angular-momentum-pr-putty-sticks-to-pivoted-rod-gravity.tex



A rod of mass M and length L is hanging vertically from a frictionless pivot (where gravity is "down"). A blob of putty of mass m approaches with velocity v from the left and strikes the rod a distance d from its center of mass as shown, sticking to the rod.

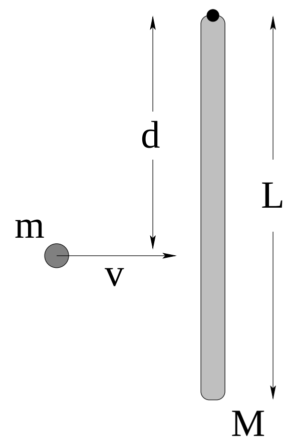
a) Find the angular velocity ω_f of the system about the pivot (at the top of the rod) after the collision.

b) Find the distance $x_{\rm cm}$ from the pivot of the center of mass of the rod-putty system immediately after the collision.

c) After the collision, the rod swings up to a maximum angle $\theta_{\max}$ and then comes momentarily to rest. Find $\theta_{\max}$.

All answers should be in terms of M, m, L, v, g and d as needed. The moment of inertia of a rod pivoted about one end is $I = \frac{1}{3}ML^2$, in case you need it.

Problem 278. problems-1/angular-momentum-pr-putty-sticks-to-pivoted-rod.tex



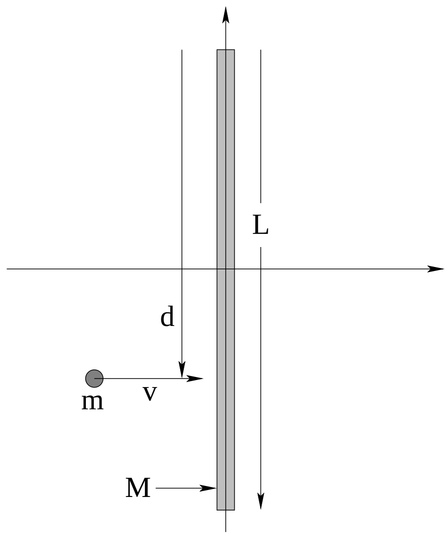
A rod of mass M and length L rests on a frictionless table and is pivoted on a frictionless nail at one end as shown. A blob of putty of mass m approaches with velocity v from the left and strikes the rod a distance d from the end as shown, sticking to the rod.

a) Find the angular velocity ω of the system about the nail after the collision.

b) Is the linear momentum of the rod/blob system conserved in this collision for a general value of d? If not, why not?

c) Is there a value of d for which it *is* conserved? If there were such a value, it would be called the *center of percussion* for the rod for this sort of collision.

Problem 279. problems-1/angular-momentum-pr-putty-sticks-to-unpivoted-rod.tex



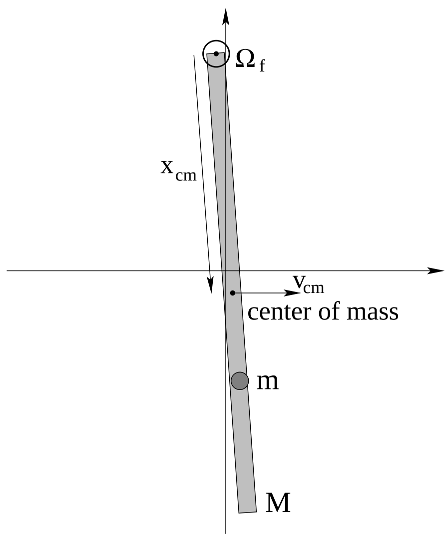
A rod of mass M and length L rests on a ***frictionless table***. A blob of putty of mass m approaches with velocity v from the left and strikes the rod a distance d from the end as shown, ***sticking to the rod***.

a) Find the angular velocity Ω_f of the system after the collision. Note that the rod and putty will be rotating about the center of mass of the ***system***, not the center of mass of the *rod by itself*!

b) Is the linear momentum of the rod/blob system conserved in this collision for a general value of d? If not, why not?

All answers should be in terms of M, m, L, v and d as needed.

Problem 280. problems-1/angular-momentum-pr-putty-sticks-to-unpivoted-rod-soln.tex



A rod of mass M and length L rests on a ***frictionless table***. A blob of putty of mass m approaches with velocity v from the left and strikes the rod a distance d from the ***center of the rod*** as shown, ***sticking to the rod***.

a) Find the angular velocity Ω_f of the system after the collision. Note that the rod and putty will be rotating about the center of mass of the ***system***, not the center of mass of the *rod by itself*!

b) Is the linear momentum of the rod/blob system conserved in this collision for a general value of d? If not, why not?

All answers should be in terms of M, m, L, v and d as needed.

Solution: In this problem ***there are no meaningful external forces acting on the system!*** Gravity is canceled by the (frictionless) normal force of the table. Consequently we expect ***linear momentum to be conserved in the collision***. However, there are also no external *torques* acting, so we expect ***angular momentum to be conserved as well!*** Which one should we use to answer the questions? What ***coordinate system*** should we use to answer the questions?

If we just consider momentum conservation:

$$p_i = mv = (m+M)v_cm = p_f$$

(to the right, say $\hat{x}$). This would make it very easy to find:

$$v_{\rm cm} = \frac{mv}{m+M}$$

as usual, but doesn't help us find Ω_f. It seems that angular momentum conservation is our best bet here. The problem remaining is *choosing a good pivot!*. After the collision, the center of mass will move in a straight line to the right in a predictable way, but every other point in the system will be undergoing somewhat complicated motion *around* the center of mass as it simultaneously moves. It therefore makes sense for us to ***use the center of mass as our pivot*** for conservation of angular momentum. This in turn is made simple by using the center of the rod as the origin of coordinates:

The steps:

$$x_{\rm cm} = \frac{md + M(0)}{m+M} = \frac{md}{m+M} \quad \text{and} \quad d - x_{\rm cm} = \frac{Md}{m+M}$$

(radii of circles of motion of the putty and rod centers of mass around the center of mass of the system) so that:

$$L_i = m(d - x_{\rm cm})v = \left\{ m(d - x_{\rm cm})^2 + \left(\frac{1}{12} M L^2 + M x_{\rm cm}^2 \right) \right\} \Omega_f$$

$$\frac{mM}{m+M} vd = \left\{ \frac{M^2}{(m+M)^2} md^2 + \frac{1}{12} ML^2 \right\} \Omega_f = \frac{mM}{m+M} d^2 \left\{ \frac{M}{m+M} + \frac{1}{12} \frac{m+M}{m} \frac{L^2}{d^2} \right\} \Omega_f$$

Note that we used the parallel axis theorem to find the moment of inertia of the rod rotating around the new center of mass. Now we just solve for:

$$\boxed{\Omega_f = \frac{1}{\left\{ \frac{M}{m+M} + \frac{1}{12} \frac{m+M}{m} \frac{L^2}{d^2} \right\}} \left(\frac{v}{d} \right)}$$

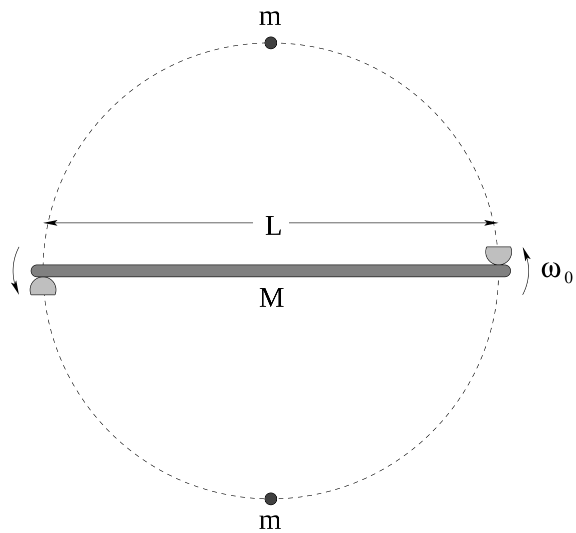
where the final direction is (obviously, RHR) out of the page.

This can be simplified to:

$$\boxed{\Omega_f = \frac{12m(m+M)d^2}{12mMd^2 + (m+M)^2 L^2} \left(\frac{v}{d} \right)}$$

which obviously has the correct dimensions as the entire fraction on the left is dimensionless. If $M \gg m$, $\Omega_f \to 0$ as we might expect as well. It could be wrong – a lot of algebra in there, and I make algebra errors as easily as the next person – but *it isn't crazy!*

Problem 281. problems-1/angular-momentum-pr-spinning-cups-catch-balls.tex

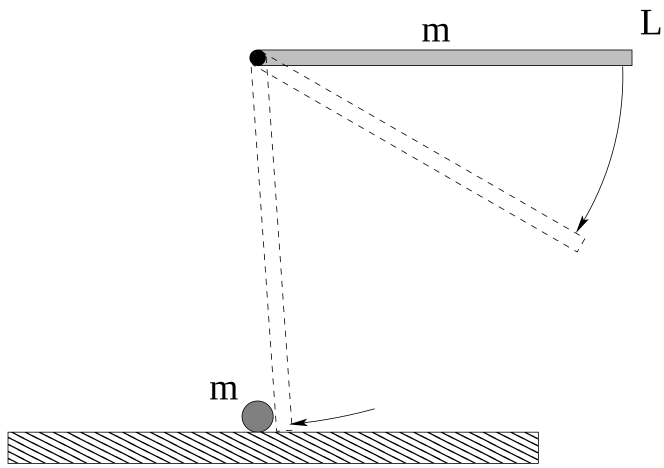


In the figure above, a bar of length L with two cups at the ends is freely rotating (in space – ignore gravity and friction or drag forces) about its center of mass with angular velocity ω_0. The bar and cups together have a mass M and a moment of intertia of $I = \beta M L^2$. When the bar reaches the vertical position, the cups catch two small balls of mass m that are at rest, which stick in the cups. The balls have a negligible moment of inertia about their own center of mass – you may think of them as particles.

a) What is the velocity of the center of mass of the system after the collision?

b) What is the angular velocity of the bar after it has caught the two balls in its cups? Is kinetic energy gained or lost in this process?

Problem 282. problems-1/angular-momentum-pr-swinging-rod-strikes-putty.tex



A uniform rod of mass m and length L swings about a frictionless peg through its ***end***. The rod is held horizontally and released from rest as shown in the figure. At the bottom of its swing the rod strikes a ball of putty of mass m that sits at rest on a frictionless table. In answering the questions take the magnitude of acceleration due to gravity to be g and assume that gravity acts downward (in the usual way). The questions below should be answered in terms of the given quantities.

a) What is the angular speed Ω_i of the rod just before it hits the putty?

b) If the putty ***sticks to the rod***, what is the angular speed Ω_f of the rod-putty system immediately after the collision?

c) What is ΔE, the mechanical energy change of the system in this collision (be sure to specify its sign).

Problem 283. problems-1/angular-momentum-pr-swinging-rod-strikes-putty-soln.tex

a) Energy conservation, with $I = \frac{1}{3}mL^2$:

$$mgL/2 = \frac{1}{2}I\Omega_i^2 \to \Omega_i = \sqrt{\frac{3g}{L}}$$

b) Angular momentum conservation:

$$L = L_i = I\Omega_i = \frac{1}{3}mL^2\sqrt{3g}L = \frac{4}{3}mL^2\Omega_f = L_f$$

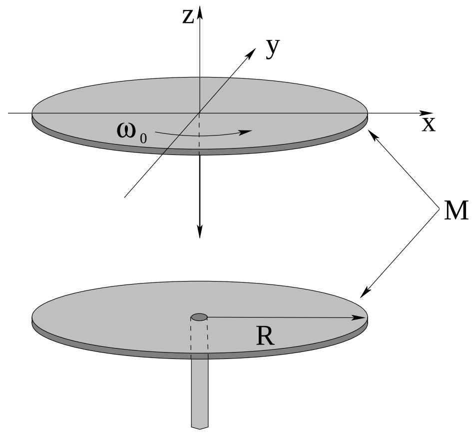
so

$$\Omega_f = \frac{1}{4}\Omega_i = \frac{1}{4}\sqrt{\frac{3g}{L}}$$

c) Subtract initial energy from energy ***after*** collision:

$$\Delta E = \frac{L^2}{2I_f} - \frac{mgL}{2} = -\frac{3mgL}{8}$$

Problem 284. problems-1/angular-momentum-pr-two-circular-plates-collide.tex



A **disk** of mass M and radius R sits at rest on a turntable that permits it to rotate freely. A second identical disk, this one rotating around their mutual axis at an angular speed ω_0, is dropped gently onto it so that (after sliding for an instant) they rotate together. In terms of the givens M, R, ω_0 and known constants:

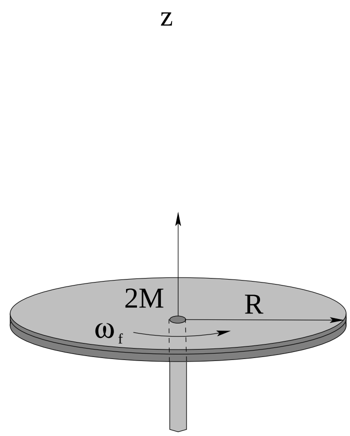
a) Find the ***final*** angular speed ω_f of the two disks moving together ***after*** the collision:

$$\omega_f = \boxed{}$$

b) What ***fraction*** of the original kinetic energy of the system K_0 is gained (+) or lost (-) in this rotational collision?

$$\Delta K = \boxed{} \times K_0$$

Problem 285. problems-1/angular-momentum-pr-two-circular-plates-collide-soln.tex

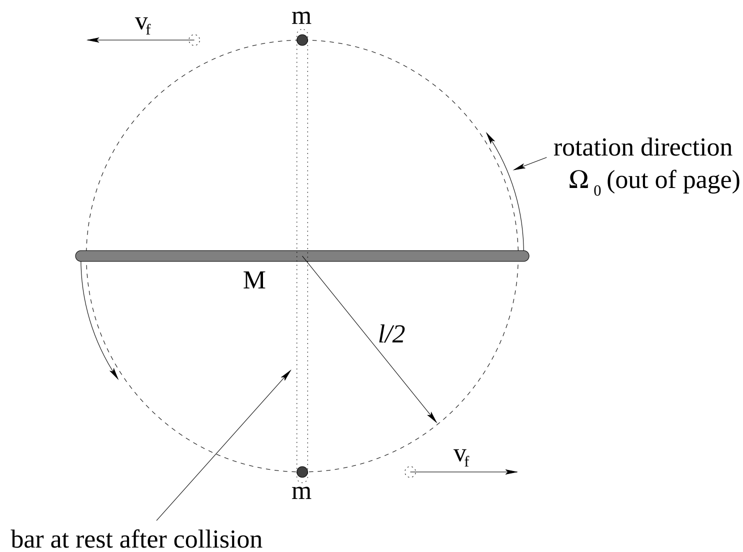


This is a fully inelastic rotational collision. There are no external torques about the axis of rotation, so angular momentum in this direction (relative to a pivot at the orgin of the bottom disk say) is conserved. In this (z) direction:

a)
$$L = L_i = I\omega_0 = 2I\omega_f = L_f$$
where $I = \frac{1}{2}MR^2$ for both disks. Hence:
$$\boxed{\omega_f = \omega_0/2}$$

b) There are many ways to get this, but the easiest (since angular momentum is conserved) is to write:
$$K_0 = \frac{L^2}{2I} \qquad K_f = \frac{L^2}{2(2I)} = \frac{K_0}{2}$$
or
$$\boxed{\Delta K = K_f - K_0 = -\tfrac{1}{2}} \times K_0$$
and energy is $\boxed{\textbf{\textit{lost}}}$ in the collision, as expected.

Problem 286. problems-1/angular-momentum-pr-rotating-bar-elastic-collision-balls.tex

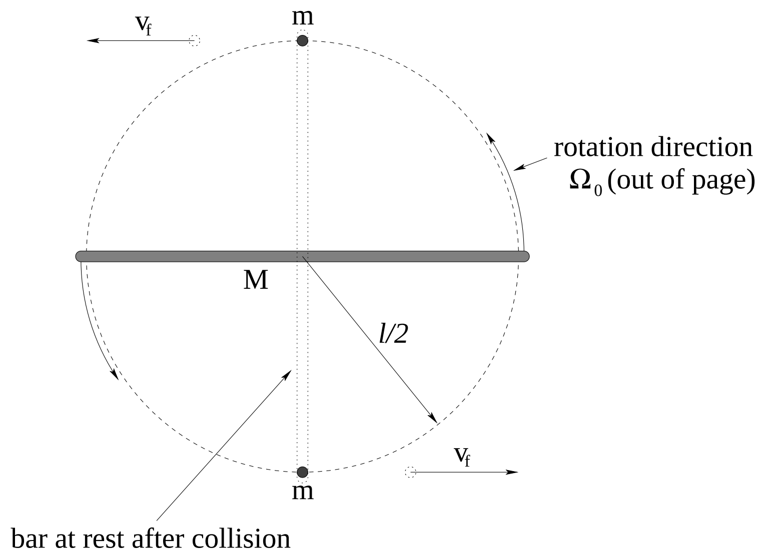


In the figure above, a unpivoted solid rod of length ℓ and mass M is rotating around its center of mass with an angular velocity Ω_0 out of the page. It simultaneously strikes two hard balls of mass m sitting at rest a distance $\ell/2$ from the center of rotation as shown, causing them to recoil to the left and right respectively. After the collision ***the rod is at rest***.

a) Is momentum conserved in this collision? b) Find the final ***speed*** of either ball, v_f.

c) Find the ratio of masses m/M such that the collision as described is ***elastic***.

Problem 287. problems-1/angular-momentum-pr-rotating-bar-elastic-collision-balls-soln.tex



In the figure above, a unpivoted solid rod of length ℓ and mass M is rotating around its center of mass with an angular velocity Ω_0 out of the page. It simultaneously strikes two hard balls of mass m sitting at rest a distance $\ell/2$ from the center of rotation as shown, causing them to recoil to the left and right respectively. After the collision ***the rod is at rest***.

a) Is momentum conserved in this collision? b) Find the final ***speed*** of either ball, v_f.

c) Find the ratio of masses m/M such that the collision as described is ***elastic***.

Solution: a) As it happens, the answer is $\boxed{\textbf{yes}}$, momentum is conserved. Before the center of mass is at rest, and afterwards (from symmetry) it is *still* at rest. But this doesn't really help us solve the problem.

b) To find the speed of the balls, we need to use conservation of ***angular*** momentum.

$$L = L_{i,rod} + L_{i,balls} = \frac{1}{12}M\ell^2\Omega_0\ (+\ 0) = (0\ +)\ \cancel{2}\left(m\frac{\ell}{\cancel{2}}v_f\right) = L_{f,rod} + L_{f,balls}$$

(out of the page, RHR) using $L = mv_f r_\perp = mv_f\ell/2$ for the magnitude of the angular momentum of the two balls, each, as well as $I_{rod} = \frac{1}{12}M\ell^2$ for the moment of inertia of a rod pivoted in the middle. Thus:

$$\boxed{v_f = \frac{M}{12m}\Omega_0\ell}$$

c) We can use $K = L^2/2I$ for both initial and final kinetic energies (with $I_f = 2 \times m(\ell/2)^2 = m\ell^2/2$ for the two balls after the collision), and take their ratio:

$$\frac{K_f}{K_i} = \frac{L^2/2I_f}{L^2/2I_i} = \frac{I_i}{I_f} = \frac{\frac{1}{2}m\ell^2}{\frac{1}{12}M\ell^2} = \frac{6m}{M} = 1$$

or

$$\boxed{\frac{m}{M} = \frac{1}{6}}$$

for the collision to be elastic.

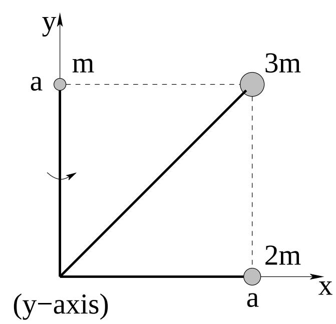
Chapter 7

1D Rotation

7.1 Torque and Moment of Inertia

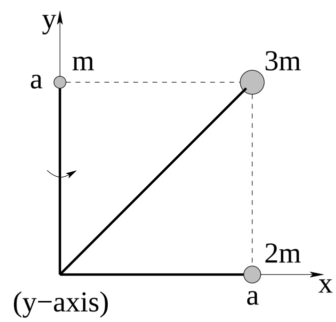
7.1.1 Short Answer Problems

Problem 288. problems-1/moment-of-inertia-sa-3-masses-1.tex



In the figure above, massless rigid rods connect three masses at the origin so that they can freely rotate around the y-axis (rotating initially into the page as shown). The masses are fixed so that they are at three corners of a square of side a. Find the ***moment of inertia*** about the y-axis of this arrangement in terms of m and a.

Problem 289. problems-1/moment-of-inertia-sa-3-masses-1-soln.tex



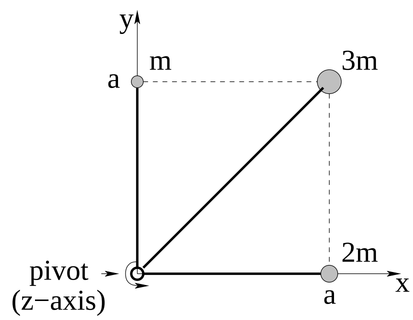
In the figure above, massless rigid rods connect three masses at the origin so that they can freely rotate around the y-axis (rotating initially into the page as shown). The masses are fixed so that they are at three corners of a square of side a. Find the ***moment of inertia*** about the y-axis of this arrangement in terms of m and a.

Solution: We simply sum:

$$I_{\text{tot}} = \sum_i m_i r_i^2 = m \times 0^2 + 2m \times a^2 + 3m \times a^2 = \boxed{5ma^2}$$

Note well that we are treating all three masses as "point masses" with no moment of inertia of their own, and that "r" in this case is the ***radius of the circle each mass moves in around the y-axis***.

Problem 290. problems-1/moment-of-inertia-sa-3-masses.tex

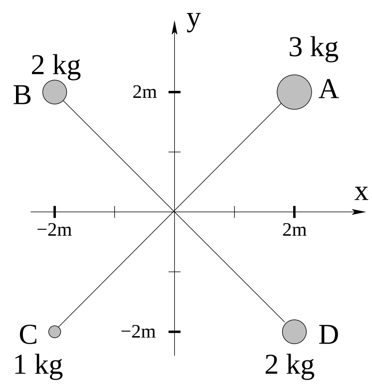


In the figure above, massless rigid rods connect three masses to a pivot at the origin so that they can freely rotate around the z-axis (perpendicular to the page). The masses are fixed so that they are at three corners of a square of side a, with the pivot at the fourth corner as shown. Find the ***moment of inertia*** about the z-axis of this arrangement in terms of m and a.

Problem 291. problems-1/moment-of-inertia-sa-3-masses-soln.tex

$$I = \sum_i m_i r_i^2 = ma^2 + 3m(\sqrt{2}a)^2 + 2ma^2 = 9ma^a$$

Problem 292. problems-1/moment-of-inertia-sa-4-masses-1.tex



In the figure, massless rigid rods connect four point-like masses centered at points A, B, C, D to form a rigid body. The rigid body can rotate about any axis perpendicular to the plane of the figure.

[*Showing your work is recommended, but not mandatory.*]

a) Find the center of mass (CM) location of the rigid body (Note: proper units should be included in your answer).

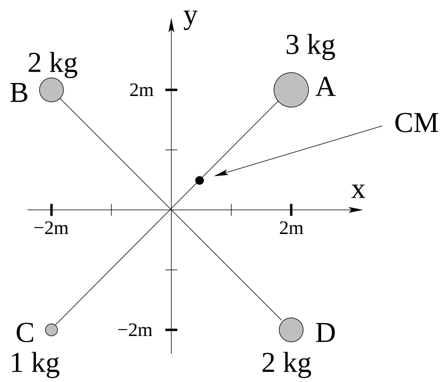
$x_{\rm cm} = \boxed{}$; $y_{\rm cm} = \boxed{}$

b) ***Mark the CM's location*** in the figure.

c) The moment of inertia about an axis perpendicular to the plane of the figure depends on the location of the axis. Answer the following by filling the box using A, B, C, D, or CM.

- The smallest moment of inertia is about an axis going through point $\boxed{}$.
- The next smallest moment of inertia is about an axis going through point $\boxed{}$.

Problem 293. problems-1/moment-of-inertia-sa-4-masses-1-soln.tex



a) Using:

$$x_{\rm cm} = \frac{1}{M_{\rm tot}} \sum_i m_i x_i$$

(and ditto for $y_{\rm cm}$):

$$x_{\rm cm} = \frac{1}{8}\{-2-4+4+6\} = \boxed{\frac{1}{2}\text{ meter}} \qquad y_{\rm cm} = \frac{1}{8}\{-2-4+4+6\} = \boxed{\frac{1}{2}\text{ meter}}$$

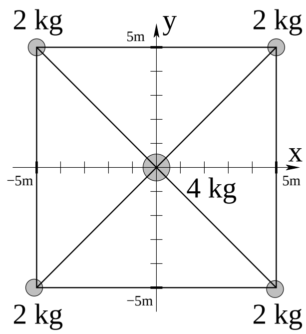
b) ***Mark the CM's location*** in the figure. **(See above.)**

c) The moment of inertia about an axis perpendicular to the plane of the figure depends on the location of the axis. Answer the following by filling the box using A, B, C, D, or CM.

- The smallest moment of inertia is about an axis going through point $\boxed{\text{CM}}$.
- The next smallest moment of inertia is about an axis going through point $\boxed{\text{A}}$.

Scoring: +1 for each *correct* answer, up to 4 (or 5 per final exam) points.

Problem 294. problems-1/moment-of-inertia-sa-4-masses-2.tex



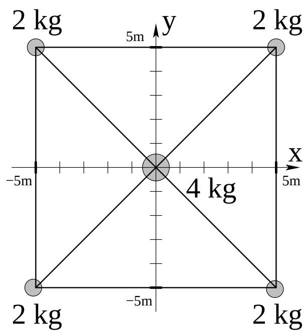
In the figure above, massless rigid rods connect five masses to form a rigid body. The rigid body can be rotated about any axis perpendicular to the plane of the figure. Find a point with coordinates (x_0, y_0) on the provided coordinate frame (units of meters) so that the moment of inertia of the system is ***smallest*** if the axis goes through this point. Then, enter the moment of inertia of the system about this axis.

$x_0 =$ $\boxed{}$ meters

$y_0 =$ $\boxed{}$ meters

$\boldsymbol{I}_{\mathbf{min}} =$ $\boxed{}$ kg-meter^2

Problem 295. problems-1/moment-of-inertia-sa-4-masses-2-soln.tex



In the figure above, massless rigid rods connect five masses to form a rigid body. The rigid body can be rotated about any axis perpendicular to the plane of the figure. Find a point with coordinates (x_0, y_0) on the provided coordinate frame (units of meters) so that the moment of inertia of the system is ***smallest*** if the axis goes through this point. Then, enter the moment of inertia of the system about this axis.

The moment of inertia is minimal when it goes through the center of mass, which is clearly at the origin (from symmetry).

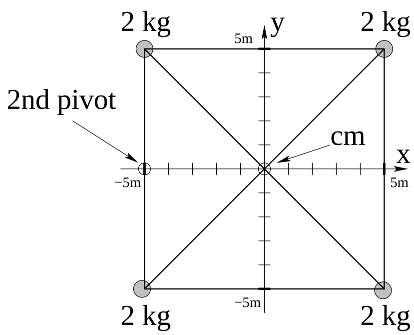
$x_0 = \boxed{0}$ meters

$y_0 = \boxed{0}$ meters

$\boldsymbol{I}_{\mathbf{min}} = \quad \boxed{400}$ kg-meter^2

Problem 296. problems-1/moment-of-inertia-sa-4-masses.tex

In the figure below, four 2 kilogram masses are held at the corners of a rigid square by massless rods as shown. The center of mass of the system is located at the origin of the provided $x-y$ coordinate frame (units in meters). The z-axis points out of the page.



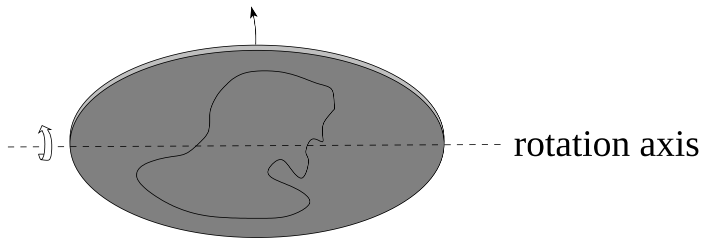
Find the moment of inertia of this system around the z-axis through the center of mass:

$\boldsymbol{I}_{\mathrm{cm}} = \boxed{}$

Now find the moment of inertia of this system around an axis ***parallel*** to the z-axis but passing through a new pivot point at $(-5, 0, 0)$ meters.

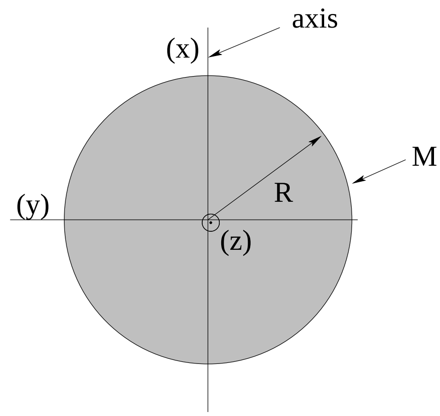
$\boldsymbol{I}_{\mathrm{new}} = \boxed{}$

Problem 297. problems-1/moment-of-inertia-sa-disk-axis-in-plane.tex



You flip a coin with a friend to see who pays for lunch. The flipped coin spins rapidly around an axis *in the plane of the coin* as shown. Assuming the coin to be a uniform disk of mass M and radius R, find the moment of inertia of the coin ***about this axis***.

Problem 298. problems-1/moment-of-inertia-sa-disk-axis-in-plane-soln.tex



We know that the moment of inertia of a disk around its axis of symmetry (in this case z) is $I = \frac{1}{2}MR^2$. We also know that I_x and I_y in the figure above must be identical in magnitude, also from symmetry. The perpendicular axis theorem is thus:

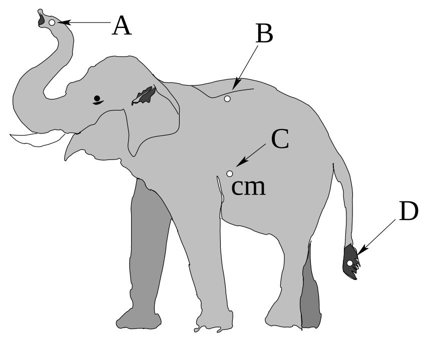
$$2I_x = I_x + I_y = I_z = \frac{1}{2}MR^2$$

or

$$I_x = \frac{1}{4}MR^2$$

7.1.2 Ranking Problems

Problem 299. problems-1/moment-of-inertia-ra-parallel-axes.tex



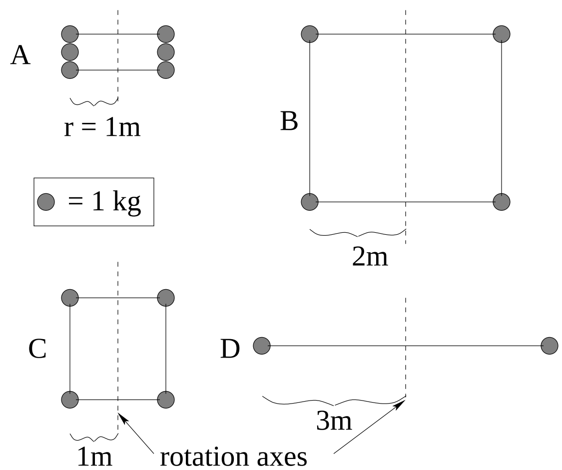
A two-dimensional cardboard cut-out of an elephant is drawn above. Small holes are drilled through it at the points A, B, C and D indicated. Hole C is at the center of mass of the figure. Rank the moment of inertia of the elephant about axes through each of the holes (with equality permitted) so that a possible (but unlikely) answer is $I_A < I_B = I_C < I_D$.

Problem 300. problems-1/moment-of-inertia-ra-parallel-axes-soln.tex

The parallel axis theorem says that the moment of inertia about each axis parallel to the one through point C is$I = I_{\mathrm{cm}} + mh^2$, where h is the distance between the C pivot and the point in question. Hence:

$$I_C < I_B < I_D < I_A$$

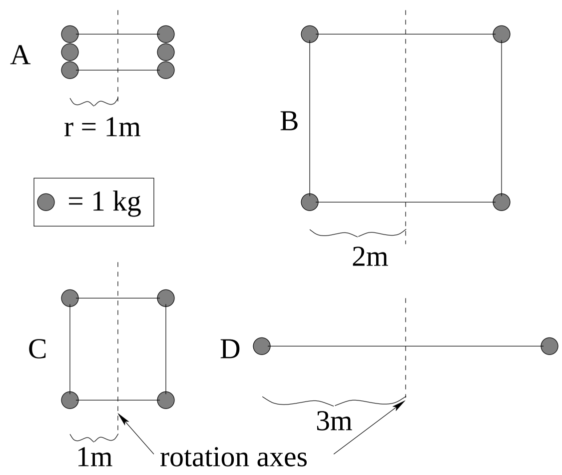
Problem 301. problems-1/moment-of-inertia-ra-point-masses-massless-rods-2.tex



In the figure above, all of the masses m are identical and are connected by ***rigid massless*** rods as drawn. ***Rank the moments of inertia*** of the four objects about the ***rotation axes drawn as dashed lines.*** Equality is permitted, so a possible answer might be $A > C = D > B$.

$\square \quad \square \quad \square \quad \square \quad \square \quad \square \quad \square$

Problem 302. problems-1/moment-of-inertia-ra-point-masses-massless-rods-2-soln.tex

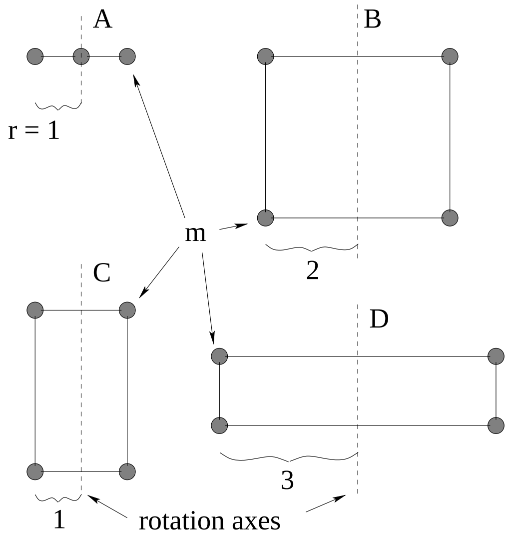


In the figure above, all of the masses m are identical and are connected by ***rigid massless*** rods as drawn. ***Rank the moments of inertia*** of the four objects about the ***rotation axes drawn as dashed lines.*** Equality is permitted, so a possible answer might be $A > C = D > B$.

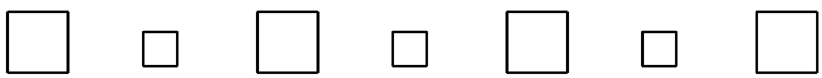
$\boxed{\text{D}}$ $\boxed{>}$ $\boxed{\text{B}}$ $\boxed{>}$ $\boxed{\text{A}}$ $\boxed{>}$ $\boxed{\text{C}}$

Solution: We use $I = mr^2$ for each mass, and just add it up! Hence: $I_A = 6$ kg-m^2. $I_B = 16$ kg-m^2. $I_C = 4$ kg-m^2. $I_D = 18$ kg-m^2.

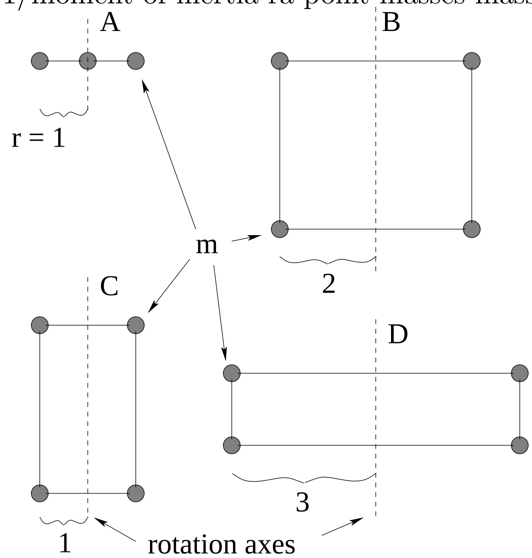
Problem 303. problems-1/moment-of-inertia-ra-point-masses-massless-rods.tex



In the figure above, all of the masses m are identical and are connected by rigid ***massless*** rods as drawn. Rank the moments of inertia of the four objects about the rotation axes drawn as dashed lines. A possible answer could look like $A < C = D < B$ (but probably isn't).



Problem 304. problems-1/moment-of-inertia-ra-point-masses-massless-rods-soln.tex



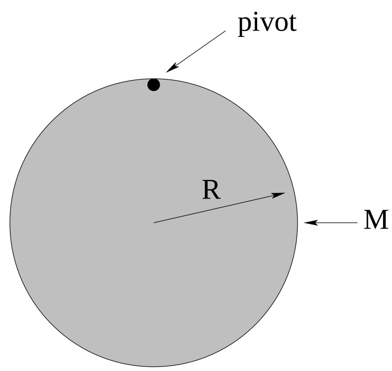
$$I = \sum_i m_i r_i^2$$

so:

$$I_A < I_C < I_B < I_D$$

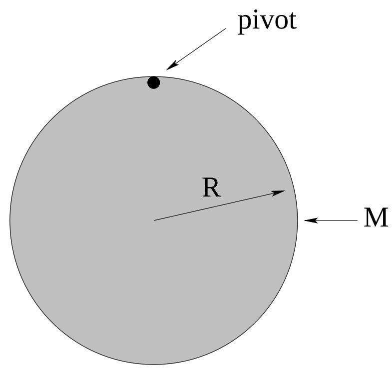
7.1.3 Regular Problems

Problem 305. problems-1/moment-of-inertia-pr-disk-pivoted-at-rim.tex



In the figure above, a disk of mass M and radius R is pivoted about a point on the rim as shown. What is the moment of inertia of the disk about this pivot?

Problem 306. problems-1/moment-of-inertia-pr-disk-pivoted-at-rim-soln.tex



In the figure above, a disk of mass M and radius R is pivoted about a point on the rim as shown. What is the moment of inertia of the disk about this pivot?

Solution Use the ***parallel axis theorem***!

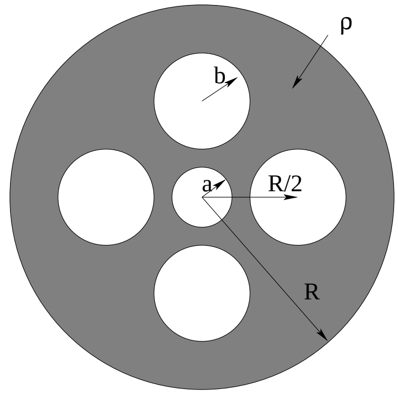
$$I_{\text{new}} = I_{\text{cm}} + Mh^2$$

In this case, direct application, using $I_{\text{cm}} = \frac{1}{2}MR^2$ for a disk, is:

$$\boxed{I_p = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2}$$

Note that this problem is likely to be solvable using direct integration and some fairly nasty calculus involving the law of cosines, but nobody sane would want to do it that way...

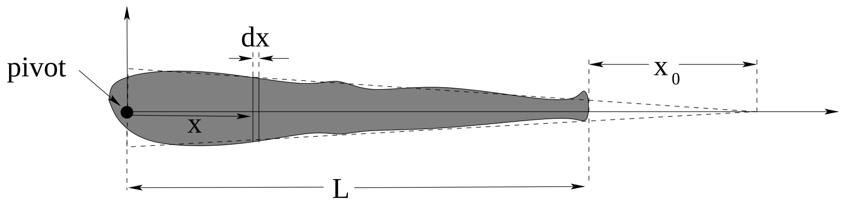
Problem 307. problems-1/moment-of-inertia-pr-disk-with-holes.tex



You are employed by a company that makes cogs, pulleys, and other widgets for lawnmower engines. They have designed a new pulley that is basically an annular disk of thickness t, outer radius R, and inner radius a, with approximately uniform density otherwise, as shown. To save on material costs (and to be able to deliver more torque to the real payload, instead of the pulley itself) they have removed all the material in four large circular holes of radius b through the solid part of the disk, centered on a circle of radius $R/2$ as shown. Your job is to compute the new moment of inertia as a function of ρ, t, R and $a, b < R/2$.

Hints: Note that you SHOULDN'T have to actually do any integrals in this problem if you remember that the moment of inertia of a disk is $\frac{1}{2}MR^2$. You are also welcome to introduce quantities like $M = \rho\pi R^2 t$, $m_a = \rho\pi a^2 t$ and $m_b = \rho\pi b^2 t$ into the problem if it would make the final answer simpler. Explain/show your reasoning regardless.

Problem 308. problems-1/moment-of-inertia-pr-moments-of-a-leg.tex



This problem will help you learn required concepts such as:

- Finding the Center of Mass using Integration
- Finding the Moment of Inertia using Integration

so please review them before you begin.

A *simple* model for the one-dimensional mass distribution of a human leg of length L and mass M is:

$$\lambda(x) = C \cdot (L + x_0 - x)$$

Note that this quantity is maximum at $x = 0$, varies linearly with x, and vanishes smoothly at $x = L + x_0$. That means that it doesn't *reach* $\lambda = 0$ when $x = L$, just as the mass per unit length of your leg doesn't reach zero at your ankles.

a) Find the constant C in terms of M, L, and x_0 by evaluating:

$$M = \int_0^L \lambda(x)\ dx$$

and solving for C.

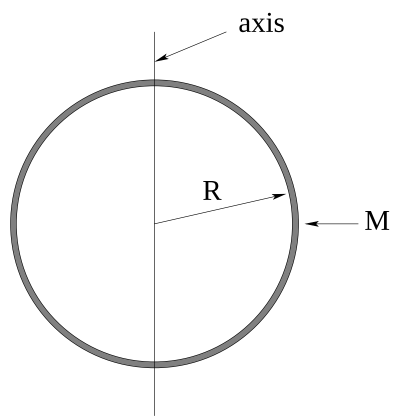
b) Find the center of mass of the leg (as a distance down the leg from the hip/pivot at the origin). You may leave your answer in terms of C (now that you know it) or you can express it in terms of L and x_0 only as you prefer.

c) Find the moment of inertia of the leg about the hip/pivot at the origin. Again, you may leave it in terms of C if you wish or express it in terms of M, L and x_0. Do your answers all have the right units?

d) How might one improve the estimate of the moment of inertia to take into account the foot (as a lump of "extra mass" m_f out there at $x = L$ that doesn't quite fit our linear model)?

This is, as you can see, something that an orthopedic specialist might well need to actually do with a much better model in order to e.g. outfit a patient with an artificial hip. True, they might use a computer to do the actual computations required, but is it plausible that they could possibly do what they need to do without knowing the physics involved in some detail?

Problem 309. problems-1/moment-of-inertia-pr-ring-axis-in-plane.tex

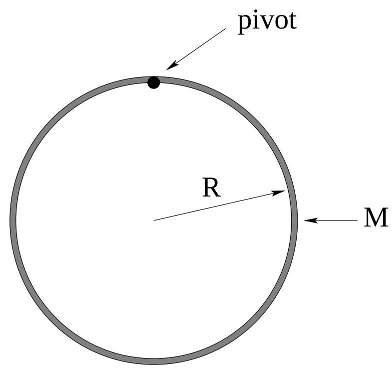


In the figure above, a ring of mass M and radius R is rotated around an axis through the middle in the plane of the ring as shown.

a) Find the moment of inertia of the ring about this axis through direct integration.

b) Find the moment of inertia of the ring about this axis using the perpendicular axis theorem. Which is easier?

Problem 310. problems-1/moment-of-inertia-pr-ring-pivoted-at-rim.tex

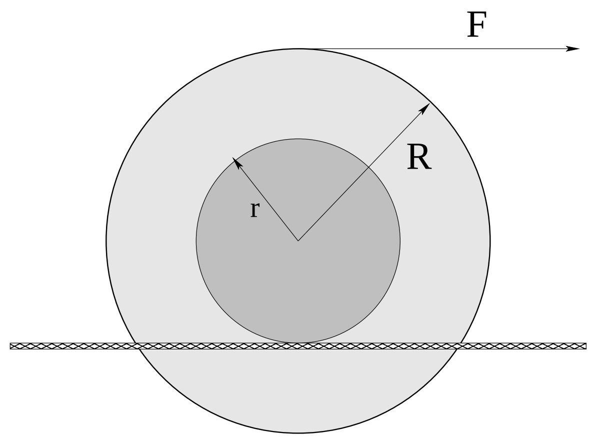


In the figure above, a ring of mass M and radius R is pivoted about a point on the rim as shown. What is the moment of inertia of the ring about this pivot?

7.2 Torque and 1D Rotation

7.2.1 Multiple Choice Problems

Problem 311. problems-1/rotation-mc-cable-spool-rolls-on-line.tex



A cable spool of mass M, radius R and moment of inertia $I = \beta M R^2$ around an axis through its center of mass is wrapped around its ***outer*** disk with fishing line and set on a rough rope as shown. The fishing line is then pulled with a force of magnitude F to the right as shown so that it rolls down the rope on the spool at radius r to the right ***without slipping***.

What is the direction of static friction as it rolls?

a) To the right.

b) To the left.

c) Not enough information to tell (depends on e.g. the size of r relative to R, the numerical value of β, or other unspecified data).

Problem 312. problems-1/rotation-mc-K-scaling.tex

Sphere **A** has mass M and radius R. Sphere **B** has mass M and radius $2R$. In order for the two spheres to have the ***same kinetic energy***, the ratio of their angular velocities must be:

a) $\dfrac{\omega_A}{\omega_B} = 4$

b) $\dfrac{\omega_A}{\omega_B} = 2$

c) $\dfrac{\omega_A}{\omega_B} = 1/2$

d) $\dfrac{\omega_A}{\omega_B} = 1/4$

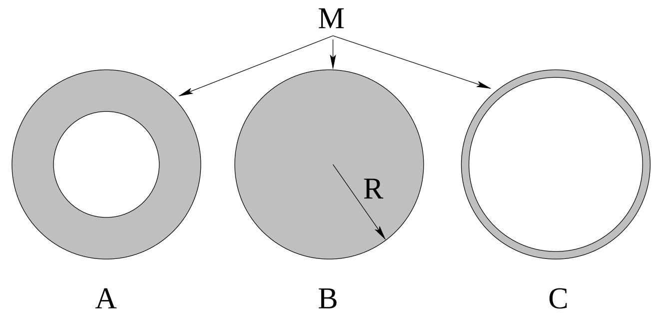
Problem 313. problems-1/rotation-mc-K-scaling-soln.tex

$K = \dfrac{1}{2}I\omega^2$, so we just need to compare I_A to I_B. I goes like the product MR^2. (**Note well:** We don't need to know the ***exact*** formula $I = \frac{2}{5}MR^2$, only that I for rolling objects usually scales with M and R^2 independently.) The spheres have the same mass but B has twice the radius of A, so $I_B = 4I_A$. Hence we need $\omega_B = \frac{1}{2}\omega_A$, or:

$$\frac{\omega_A}{\omega_B} = 2$$

One can also explicitly compute K_A and K_B algebraically using the exact formula for the I's, set them equal, and solve for ω_A in terms of ω_B but that is somewhat more work and has more opportunities to make a mistake.

Problem 314. problems-1/rotation-mc-K-scaling-with-I.tex

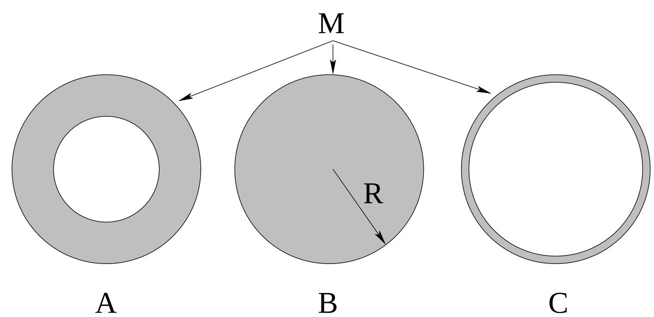


Three wheels have the ***same mass M and outer radius R***. This mass is distributed uniformly (but differently) in each wheel, with the inner radius of the uniform distribution varying as illustrated in the figure above. Each is rotating about its axis of symmetry through its center of mass in the center, and all three have the ***same kinetic energy***. Which wheel is rotating the ***fastest***?

a) ☐ Wheel **A** b) ☐ Wheel **B**

c) ☐ Wheel **C** d) ☐ Two or more of these wheels are tied for fastest

Problem 315. problems-1/rotation-mc-K-scaling-with-I-soln.tex



First we mentally rank the moments of inertia:

$$I_B < I_A < I_C$$

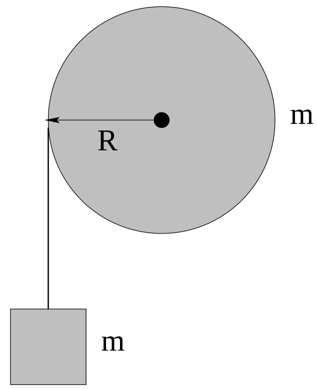
Then we note that $\Omega = \sqrt{2K/I}$, so the angular speed scales *opposite* to I. Hence the smallest I has the largest Ω at constant K:

a) ☐ Wheel **A** b) ☒ Wheel **B**

c) ☐ Wheel **C** d) ☐ Two or more of these wheels are tied for fastest

Scoring: +4 for correct box checked. +2 for any reasonable progress: correct ranking of I, expression such as $K = \frac{1}{2}I\Omega^2$.

Problem 316. problems-1/rotation-mc-K-sharing-with-m.tex



A very light string is wrapped many times around a disk of mass m and radius R that is pivoted in the center and free to rotate. A block with the ***same*** mass m is attached to the end of the string and released from rest so that the disk spins as the string unrolls as it falls.

At the instant that the mass m has fallen to where it has kinetic energy K, the disk has kinetic energy:

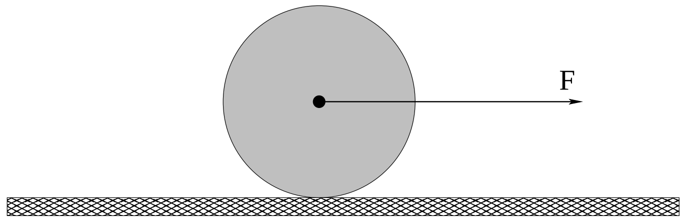
a) K

b) $K/2$

c) $2K$

d) None of these.

Problem 317. problems-1/rotation-mc-rolling-disk-friction-1.tex



In the figure above a force is applied to the center of a disk (initially at rest) sitting on a rough table by means of a rope attached to its frictionless axle ***in the direction shown***. The disk then accelerates and ***rolls without slipping***. The net horizontal force exerted by the table on the disk is:

a) kinetic friction to the right.

b) kinetic friction to the left.

c) static friction to the right.

d) static friction to the left.

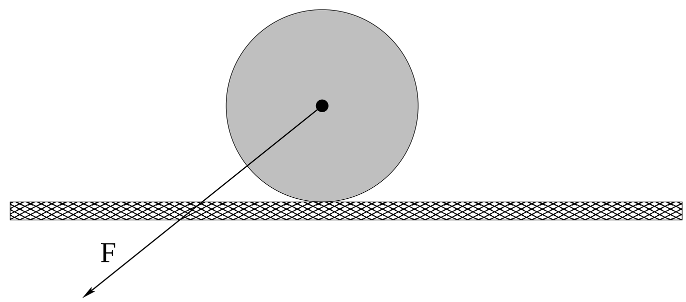
e) Cannot tell from the information given.

Problem 318. problems-1/rotation-mc-rolling-disk-friction-1-soln.tex

The angular acceleration is ***into the page***. $\vec{\boldsymbol{F}}$ is applied at the pivot in the middle and exerts no torque. Gravity and the normal force act along a line through the pivot and exert no torque. Only friction exerts a torque, and this torque ***must*** be in the same direction as α as $\tau = I\alpha$. The disk rolls without slipping which implies static, not kinetic, friction. Hence the right hand rule tells us:

Static friction to the left.

Problem 319. problems-1/rotation-mc-rolling-disk-friction.tex



In the figure above a force is applied to the center of a disk (initially at rest) sitting on a rough table by means of a rope attached to its frictionless axle in the direction shown. The disk then accelerates and ***rolls without slipping***. The net horizontal force exerted by the table on the disk is:

a) kinetic friction to the right.

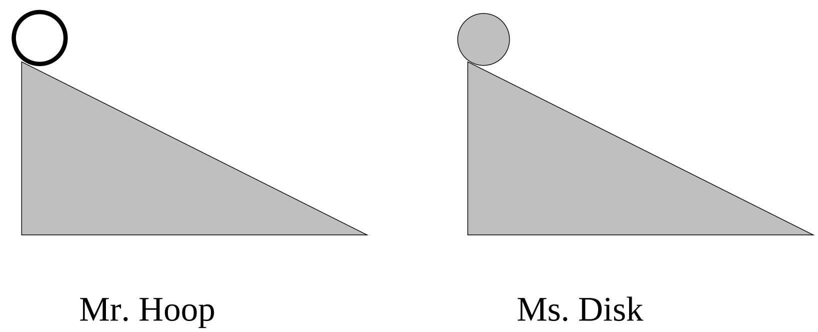
b) kinetic friction to the left.

c) static friction to the right.

d) static friction to the left.

e) Cannot tell from the information given.

Problem 320. problems-1/rotation-mc-rolling-race-1.tex



Mr. Hoop and Ms. Disk had a race rolling down two identical hills ***without slipping***. They both started at the top at the same time. Who won?

a) Mr. Hoop

b) Ms. Disk

Problem 321. problems-1/rotation-mc-rolling-race-1-soln.tex

Lot of ways to see that it is ***Ms. Disk*** that wins. If $I = \beta MR^2$ for some dimensionless scalar β, then its rotational kinetic energy per unit mass scales like β:

$$\frac{K_{\rm rot}}{M} = \frac{1}{2}(\beta R^2)\omega^2 = \beta\frac{1}{2}v_{\rm cm}^2$$

This means that its total kinetic energy per unit mass is:

$$\frac{K_{\rm tot}}{M} = (1+\beta)\frac{1}{2}v_{\rm cm}^2$$

The disk and hoop both descend the same distance (say) H. $\Delta U = MgH = \Delta K = K_{\rm tot}$, so (dividing by the mass M):

$$\frac{\Delta U}{M} = gH = (1+\beta)\frac{1}{2}v_{\rm cm}^2 = \frac{K_{\rm tot}}{M}$$

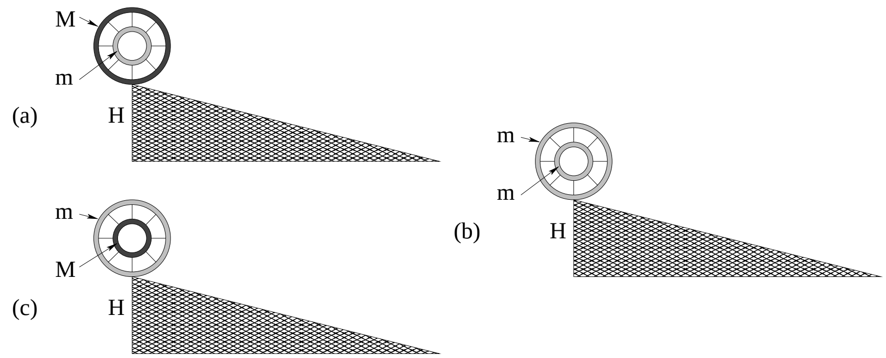
were gH is the same for both, so that the one with the highest β has to have the lowest $v_{\rm cm}$ and loses the race.

An alternative is to look at $a_{\rm cm}$, which will always have the $1+\beta$ in the denominator, so again larger β means smaller $a_{\rm cm}$ and hence smaller $v_{\rm cm}$ throughout (again, losing the race).

Note well that the result is ***independent of the mass and radius*** of the disk and/or hoop – it ***depends only on the relative size of $\boldsymbol{\beta}$***. So a small, massive ring loses to a large, light disk, and a large, light ring still loses to a small, massive disk!

7.2.2 Short Answer Problems

Problem 322. problems-1/rotation-sa-rolling-race-1.tex



Its a race! Three wheels made out of two concentric rings of mass connected by light spokes are identically placed at the top of an inclined plane of height H as shown. At time $t = 0$ they are all three released from rest to roll without slipping down the incline. You are given the following information about each double ring:

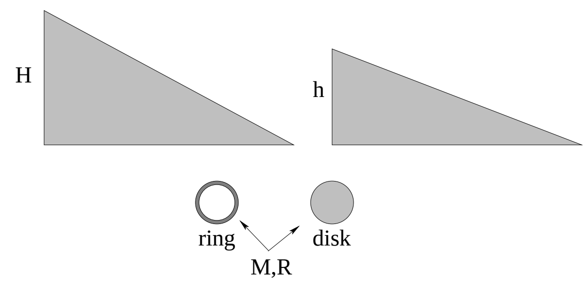
a) Inner ring mass m is less than outer ring mass M.

b) Inner ring mass m is the same as outer ring mass m.

c) Inner ring mass M is greater than outer ring mass m.

In what a, b, c order do the rings arrive at the bottom of the incline? (4 points)

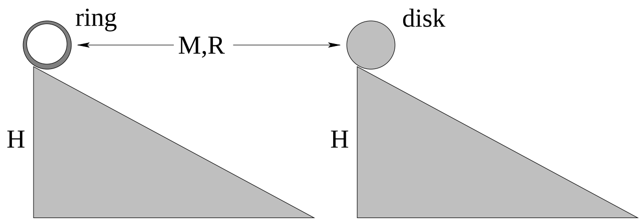
Problem 323. problems-1/rotation-sa-rolling-race-2.tex



You are given two inclined planes with different maximum heights $H > h$ as shown. If a ring and disk of identical radius R and mass M are each placed at the top of one of the two planes and released, they will roll without slipping to arrive at the bottom travelling ***at the same speed***. If placed at the top of the planes in the other order, they will not.

Draw and label the ring and disk at the tops of the correct planes such that they will roll to the bottom and arrive travelling ***at the same speed***.

Problem 324. problems-1/rotation-sa-rolling-race-3.tex



You are given two inclined planes with the same height H as shown. A ring and disk of identical radius R and mass M are each placed at the top of one of the two planes and released at the same instant to roll without slipping to the bottom of their respective inclines..

a) Which one gets to the bottom first? (Circle) **ring** **disk**

b) Which one has the greatest speed at the bottom? **ring** **disk**

c) Which one has the greatest rotational kinetic energy at the bottom? **ring** **disk**

7.2.3 Ranking Problems

Problem 325. problems-1/rotation-ra-hoop-and-disk.tex

A hoop and a disk of identical mass and radius are rolled up two identical inclined planes ***without slipping*** and reach a maximum height of H_{hoop} and H_{disk} respectively before coming momentarily to rest and rolling back down.

Use one of the three signs $<$, $>$ or $=$ in the boxes below to correctly complete each statement.

a) If both hoop and disk start with the same total ***kinetic energy*** then:

$H_{\text{hoop}} \quad \boxed{} \quad H_{\text{disk}}$

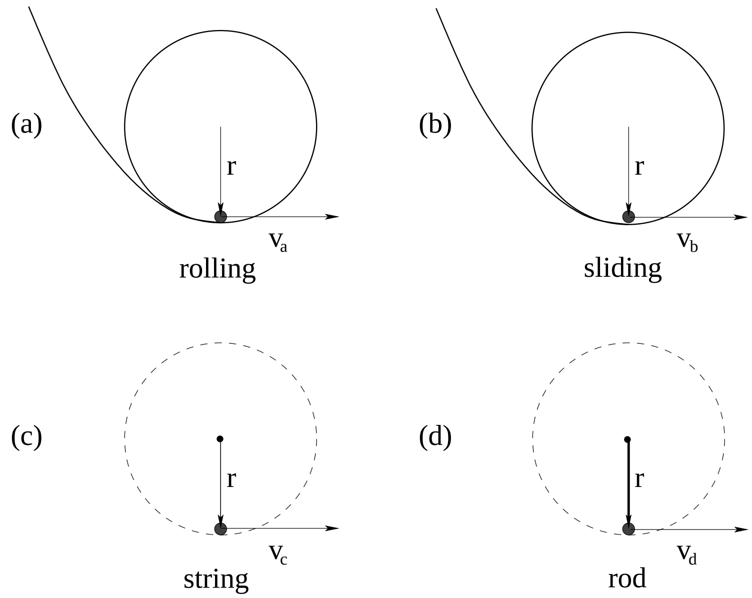
b) If both hoop and disk start with the same total ***center of mass speed*** then:

$H_{\text{hoop}} \quad \boxed{} \quad H_{\text{disk}}$

c) If both hoop and disk start with the same total ***center of mass speed*** then comparing the ***magnitude*** of the ***work done by gravity*** when they have reached their maximum height:

$|W_{\text{gravity,hoop}}| \quad \boxed{} \quad |W_{\text{gravity,disk}}|$

Problem 326. problems-1/rotation-ra-loop-the-loops-balls.tex



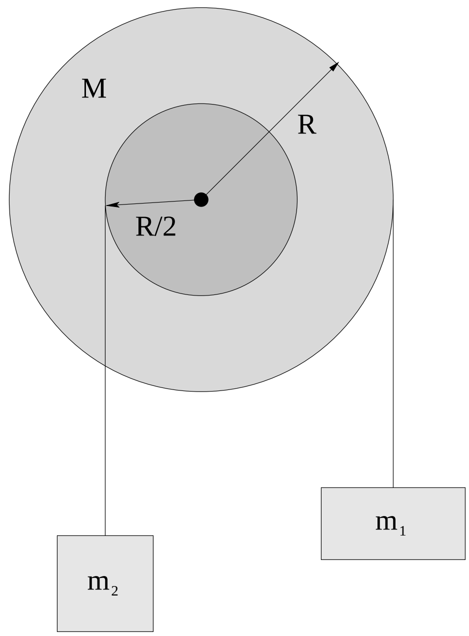
In the figures above, (a) shows a ball rolling without slipping on a track; (b) shows the ball sliding on a frictionless track; (c) shows the ball on a string; (d) shows the ball attached to a rigid massless rod attached to a frictionless pivot. In all four figures the ball has the *smallest* velocity at the bottom of its circular trajectory that will suffice for the ball to reach the top while still moving in a circle (note that the velocities are not drawn to scale).

Correctly ordinally rank these minimum velocities, for example $v_a = v_b < v_c < v_d$ is a possible (but probably incorrect) answer.

Note: You must *either* justify your answer with simple physical arguments *or* just solve for the minimum velocity needed at the bottom in terms of m, r, $\beta = 2/5$ (for a ball), g and then order the results. You can't just put down a "guess" for an order with no valid physical reasoning backing it and have it count, but it is possible to reason your way all or most of the way to an answer without doing all of the algebra.

7.2.4 Regular Problems

Problem 327. problems-1/rotation-pr-asymmetric-atwoods-2.tex

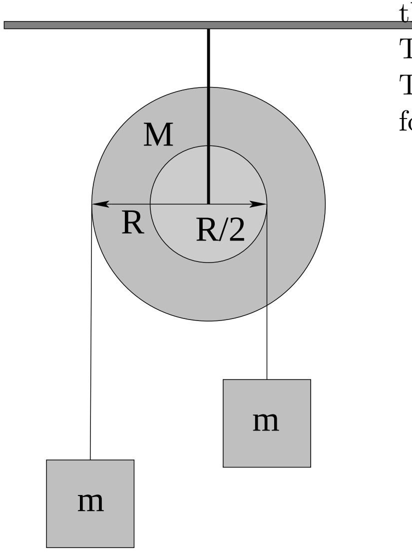


A pulley of mass M, radius R and moment of inertia $I = \beta M R^2$ has a massless, unstretchable string wrapped around it many times and has a mass m_1 suspended from the string. A second massless, unstretchable string is wrapped the opposite way around a massless, frictionless axle with radius $R/2$ as shown and has mass m_2 suspended from the string. The system begins at rest.

a) What must m_2 be in terms of m_1 for the system to remain stationary?

b) Suppose $m_1 = m_2 = M$. Find $\vec{\boldsymbol{\alpha}}$, the angular acceleration of the pulley about its center of mass. This is a vector! ***Indicate the direction*** of the angular acceleration on the figure or in your answer.

Problem 328. problems-1/rotation-pr-asymmetric-atwoods.tex



Two blocks of mass m are attached to massless unstretchable ropes that are wrapped around spools on a common, frictionless axle. The total moment of inertia of the spools is ***given*** as $\boldsymbol{I = \beta M R^2}$. The radii of the wrapped spools are R for the left mass and $R/2$ for the right one as shown in the figure.

a) Draw in and label all *relevant* forces in on the provided figure. You do not need to include the force in the strap that connects the wheel axle to the ceiling – you may assume that it is large enough to keep the axle perfectly fixed.

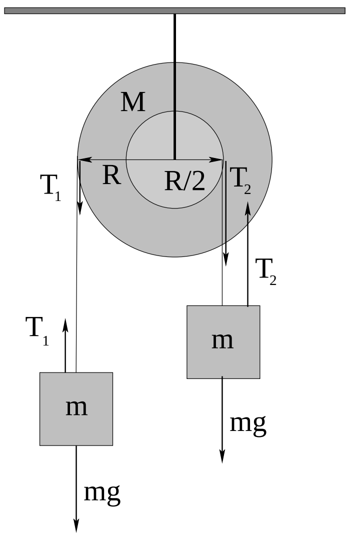
b) When the left mass m e.g. falls a distance x, by what distance x' does the right mass m rise? Use this to relate the accelerations and rolling constraint(s).

c) Find the angular acceleration α of the spools. Is it into or out of the page?

d) Find the speed of the left mass after it falls a distance H.

$\alpha =$ $\boxed{}$ ☐ **In** ☐ **Out** $v_{\text{left}} =$ $\boxed{}$

Problem 329. problems-1/rotation-pr-asymmetric-atwoods-soln.tex



Two blocks of mass m are attached to massless unstretchable ropes that are wrapped around spools on a common, frictionless axle. The total moment of inertia of the spools is ***given*** as $\boldsymbol{I = \beta M R^2}$. The radii of the wrapped spools are R for the left mass and $R/2$ for the right one as shown in the figure.

a) Draw in and label all *relevant* forces in on the provided figure. You do not need to include the force in the strap that connects the wheel axle to the ceiling – you may assume that it is large enough to keep the axle perfectly fixed.

b) When the left mass m e.g. falls a distance x, by what distance x' does the right mass m rise? Use this to relate the accelerations and rolling constraint(s).

c) Find the angular acceleration α of the spools. Is it into or out of the page?

d) Find the speed of the left mass after it falls a distance H.

$\alpha = \boxed{\dfrac{2mg}{(4\beta M + 5m)\,R}}$ ☐**In** ☒**Out** $v_{\text{left}} = \boxed{\sqrt{\dfrac{4mgH}{(4\beta M + 5m)}}}$

Solution: a) (see figure). b) The right mass rises a distance $\boxed{x_r = x/2}$ when the left one descends by x. This is important because it lets us write the following constraints:

$$x_r = x/2 \quad \Rightarrow \quad a_r = a/2 \quad \alpha = \frac{a}{R} = \frac{a_r}{R/2}$$

c) Now to work. We use N2 for translation and rotation, with the constraint(s) (expressing everything for the moment in terms of a of the left mass):

$$\text{left: } mg - T_1 = ma \qquad \text{right: } T_2 - mg = m\frac{a}{2} \qquad \text{spools: } T_1 R - T_2\frac{R}{2} = \beta M R^2 \alpha = \beta M R^2 \frac{a}{R}$$

We divide out and cancel R in the rotation equation, then scale and add all three equations:

$$\begin{aligned} mg - \cancel{T_1} &= ma \\ +\frac{\cancel{T_2}}{\cancel{2}} - \frac{mg}{2} &= m\frac{a}{4} \\ +\cancel{T_1} - \frac{\cancel{T_2}}{\cancel{2}} &= \beta M a \\ =\!=\!=\!=\!=\!=\!=\!=\;&=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!= \\ \frac{1}{2}mg = mg - \frac{1}{2}mg &= \left(\beta M + m + \frac{1}{4}m\right) a \end{aligned}$$

or (using RHR):

$$a = \frac{mg}{2\left(\beta M + m + \frac{1}{4}m\right)} = \frac{2mg}{(4\beta M + 5m)} \quad \Rightarrow \quad \boxed{\alpha = \frac{2mg}{(4\beta M + 5m)\,R} \quad \textbf{Out}}$$

Conservation of energy (**plus the rolling constraints**) yields:

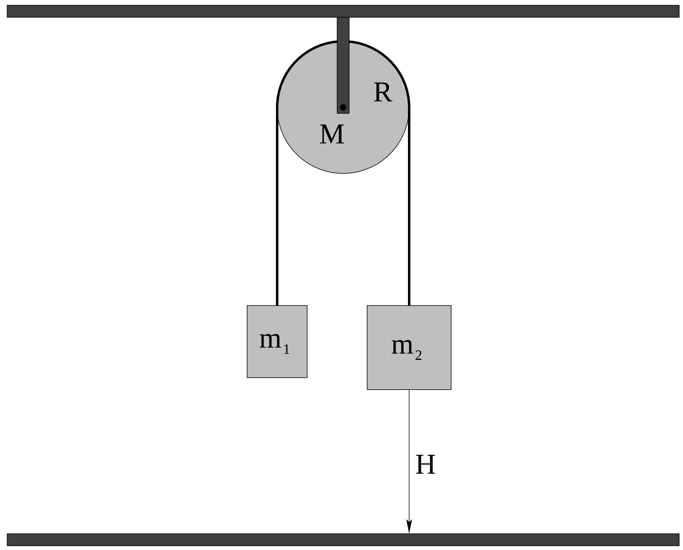
$$E_i = 0 = \frac{1}{2}mgH - mgH + \frac{1}{2}mv^2 + \frac{1}{2}m\frac{v^2}{4} + \frac{1}{2}\beta M R^2 \frac{v^2}{R^2} = E_f \quad \Rightarrow \quad \frac{1}{2}mgH = \left(\frac{5}{8}m + \frac{1}{2}\beta M\right) v^2$$

Or:

$$\boxed{v = \sqrt{\frac{4mgH}{(4\beta M + 5m)}}} = \sqrt{2aH}$$

(the latter, as expected from kinematics as an alternative).

Problem 330. problems-1/rotation-pr-atwoods-machine.tex



In the figure above Atwood's machine is drawn – two masses m_1 and m_2 hanging over a *massive* pulley which you can model as a disk of mass M and radius R, connected by a massless unstretchable string. The string rolls on the pulley without slipping.

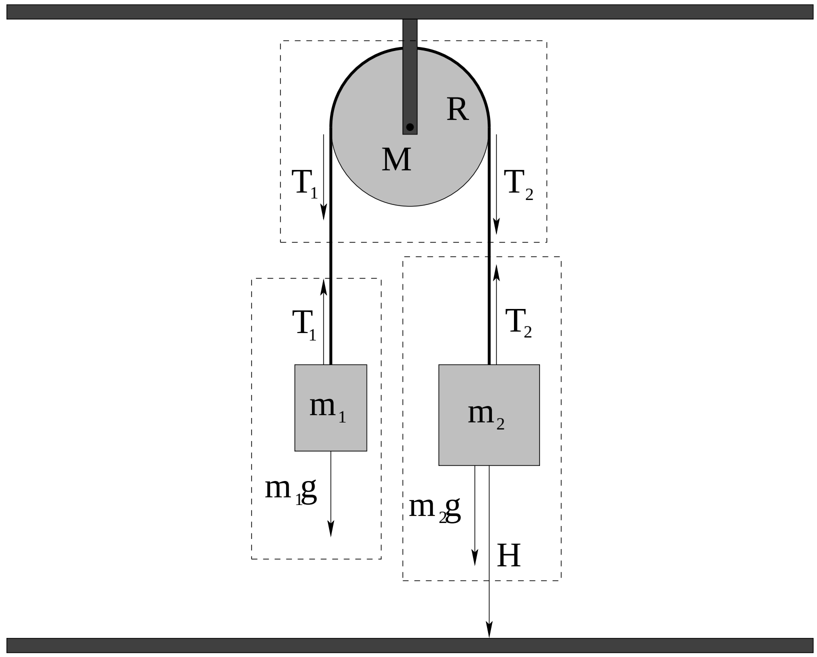
a) Draw three free body diagrams (isolated diagrams for each object showing just the forces acting on that object) for the three masses in the figure above.

b) Convert each free body diagram into a statement of Newton's Second Law (linear or rotational) for that object.

c) Using the *rolling constraint* (that the pulley rolls without slipping as the masses move up or down) find the acceleration of the system and the tensions in the string on *both* sides of the pulley in terms of m_1, m_2, M, g, and R.

d) Suppose mass $m_2 > m_1$ and the system is released from rest with the masses at equal heights. When mass m_2 has descended a distance H, find the velocity of each mass and the angular velocity of the pulley.

Problem 331. problems-1/rotation-pr-atwoods-machine-soln.tex



a) See above.

b)

$$\begin{aligned} F_1 &= T_1 - m_1 g = m_1 a \\ F_2 &= m_2 g - T_2 = m_2 a \\ \tau &= (T_2 - T_1)R = \frac{1}{2}MR^2\frac{a}{R} = I\alpha \end{aligned}$$

where we've put in the rolling constraint $\alpha = a/R$.

c) Divide last equation by R, add all three, get:

$$(m_2 - m_1)g = (m_1 + m_2 + \frac{1}{2}M)a$$

Solve for a, then back substitute into F_1 and F_2 equations to get T_1 and T_2.

d) Using mechanical energy conservation is easiest:

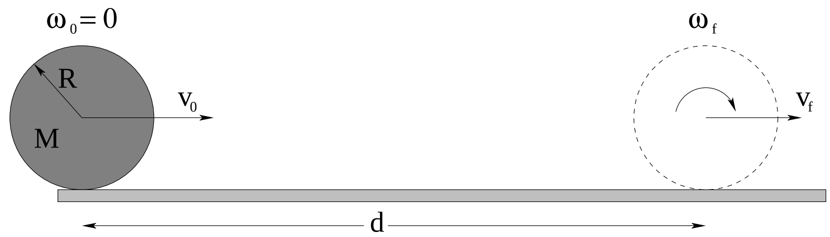
$$E_i = 0 = m_1 gH - m_2 gH + \frac{1}{2}m_1 v^2 + \frac{1}{2}m_2 v^2 + \frac{1}{2}I\omega^2$$

Using the rolling constraint $\omega = v/R$ and the moment of inertia, this turns into:

$$\frac{1}{2}(m_1 + m_2 + \frac{1}{2}M)v^2 = (m_2 - m_1)gH$$

Solve for v, then divide by R to get ω.

Problem 332. problems-1/rotation-pr-bowling-ball-friction.tex



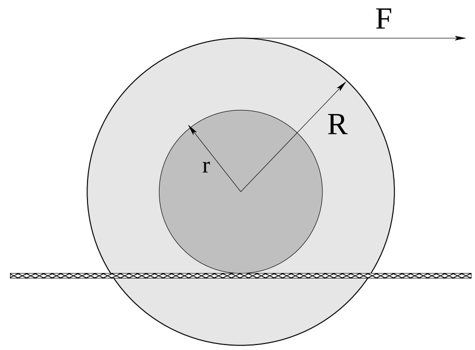
A bowling ball of mass M and radius R is released horizontally moving at a speed v_0 so that it initially slides without rotating on the bowling lane floor. μ_k is the coefficient of kinetic friction between the bowling ball and the lane floor. It slides for a time t and distance d before it rolls without slipping the rest of the way to the pins at speed v_f.

a) Find t.

b) Find d.

c) Find v_f.

Problem 333. problems-1/rotation-pr-cable-spool-rolls-on-line.tex



A cable spool of mass M, radius R and moment of inertia $I = \beta M R^2$ around an axis through its center of mass is wrapped around its OUTER disk with fishing line and set on a rough rope as shown. The fishing line is then pulled with a force F to the right as shown so that it rolls down the rope on the spool at radius r **without slipping**.

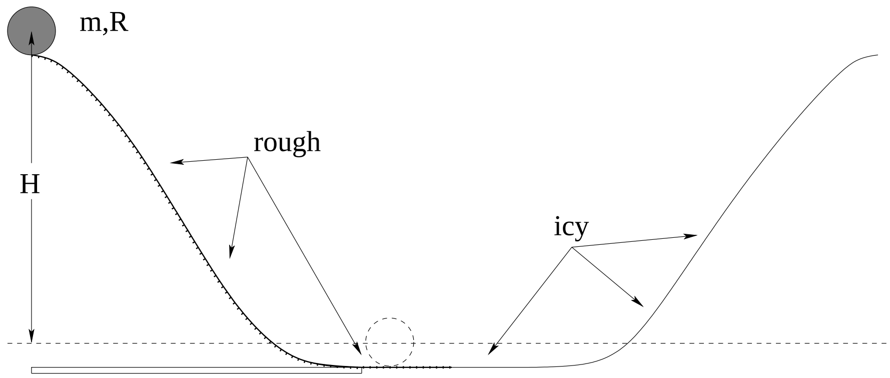
a) Which way does the spool roll (left or right)?

b) Find the magnitude of the acceleration of the spool.

c) Find the force the friction of the rope exerts on the spool.

d) Is there a value of the radius r relative to R for which *friction exerts no force on the spool?* If so, what is it?

Problem 334. problems-1/rotation-pr-disk-on-ice-example.tex



This problem will help you learn required concepts such as:

- Conservation of Mechanical Energy
- Rotational Kinetic Energy
- Rolling Constraint.

so please review them before you begin.

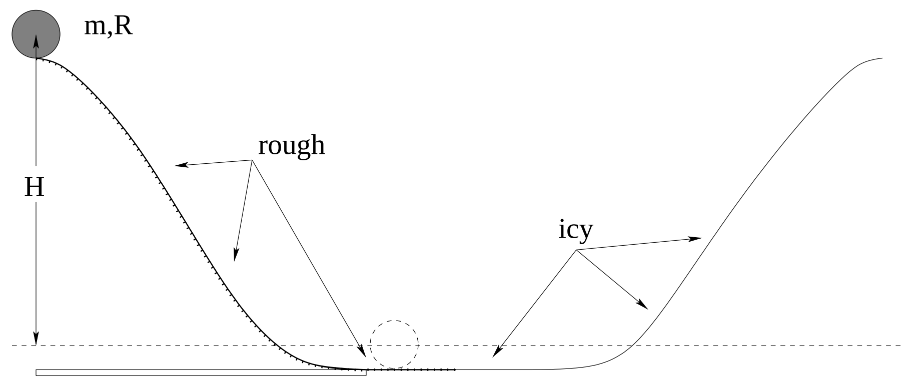
A disk of mass m and radius R rolls without slipping down a rough slope of height H onto an icy (frictionless) track at the bottom that leads up a second icy/frictionless hill as shown.

a) How fast is the disk moving at the bottom of the first incline? How fast is it rotating (what is its angular velocity)?

b) Does the disk's angular velocity change as it leaves the rough track and moves onto the ice (in the middle of the flat stretch in between the hills)?

c) How far up the second hill (vertically, find H') does the disk go before it stops rising?

Problem 335. problems-1/rotation-pr-disk-on-ice-solution.tex



This problem will help you learn required concepts such as:

- Conservation of Mechanical Energy
- Rotational Kinetic Energy
- Rolling Constraint.

so please review them before you begin.

A disk of mass m and radius R rolls without slipping down a rough slope of height H onto an icy (frictionless) track at the bottom that leads up a second icy/frictionless hill as shown.

a) How fast is the disk moving at the bottom of the first incline? How fast is it rotating (what is its angular velocity)?

   As the disk rolls down the incline without slipping, the velocity of its center and the angular velocity with which it rotates are related by the ***rolling without slipping*** constraint:

$$v = \omega R \; . \tag{7.1}$$

   Because it is not slipping, friction does no work, so that the total mechanical energy is conserved during the descent. Initial kinetic energy is zero, so $E_i = U_i = mgH$, where I am setting $U = 0$ at the bottom of the incline. With this choice, final potential energy vanishes and total energy in final state is kinetic. This, in turn, is a sum of a ***translational*** term representing the motion of the center of mass and a ***rotational*** contribution representing the motion as seen by an observer moving with the center of mass. Thus:

$$E_f = K_f = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \; . \tag{7.2}$$

   Inserting the constraint as well as the value of the moment of inertia for a uniform disk $I = \frac{1}{2}mR^2$ we have

$$mgH = \frac{1}{2}mv^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{v}{R}\right)^2 = \frac{1}{2}mv^2\,(1 + 1/2) = \frac{3}{4}mv^2 \; , \tag{7.3}$$

whence

$$v = \sqrt{\frac{4}{3} gH} \ . \tag{7.4}$$

Note that this is less than the $\sqrt{2gH}$ we would find were the disk sliding down a frictionless incline. This makes sense, because friction has been acting to enforce the constraint, but has also slowed the disk. Alternatively, the expression for the kinetic energy above shows that some of the work done by gravity was converted into *rotational* kinetic energy, leaving less of it to be converted to translational kinetic energy.

Using the constraint we then have

$$\omega = \frac{v}{R} = \sqrt{\frac{4gH}{3R^2}}. \tag{7.5}$$

b) Does the disk's angular velocity change as it leaves the rough track and moves onto the ice (in the middle of the flat stretch in between the hills)?

During the disk's accelerating descent down the incline, friction acted to retard the acceleration and increase the angular acceleration, in order to maintain the condition of no slipping, but it ***did no work*** because it acts at the one point on the wheel that is ***always stationary with respect to the ground***. Instead it served to ***redistribute*** the gravitational potential energy between translational and rotational kinetic energy.

Once the horizontal stretch is reached, the disk continues at the *constant* translational and angular velocity given by the values we computed above. Since these satisfy the rolling constraint and no energy is entering the system, friction ***does not act*** on the disk as it rolls along the horizontal rough stretch.

This is an important fact! A perfectly round wheel (with frictionless bearings) rolling without slipping on a level surface experiences no friction and does not slow down. ***This is why we use wheels!***

When the disk moves onto the ice the change in the coefficient of friction thus produces *no change* in its motion, since friction was not applying any force on the rough surface anyway.

c) How far up the second hill (vertically, find H') does the disk go before it stops rising?

As it begins to climb the second incline, the disk's velocity decreases as kinetic energy is converted to potential energy. As its motion acquires a vertical component gravity is doing negative work on the disk and this force slows the disk. On the other hand, with no friction the only forces on the disk, gravity and the normal force, exert no torque about the disk's center so its angular velocity remains constant at the value found above. The disk slows as it climbs but continues spinning at a constant rate. When it comes to a stop at the highest point it can reach, its total mechanical energy is:

$$E_{H'} = mgH' + \frac{1}{2} I\omega^2 \ . \tag{7.6}$$

where its ***rotational*** kinetic energy is ***unchanged***.

This is equal to the total energy found above since during the climb all work was done by gravity, whence we find

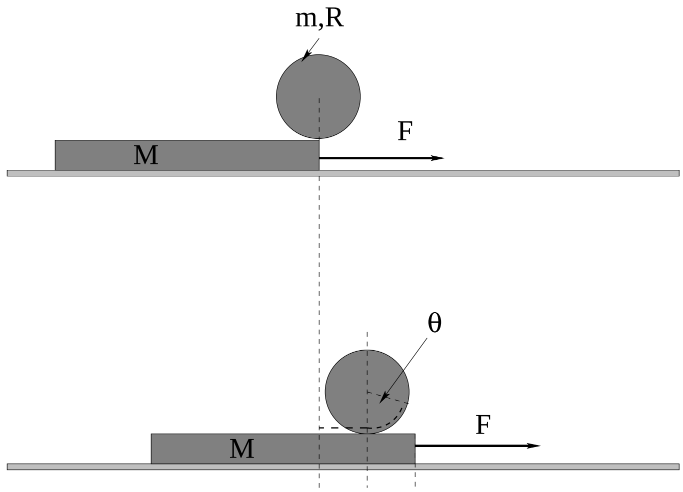
$$mgH' = \frac{1}{2} mv^2 = \frac{1}{2} m \left(\frac{4}{3} gH \right) \ , \tag{7.7}$$

or

$$H' = \frac{2}{3}H \ . \tag{7.8}$$

The disk does not recover its original height, though energy is conserved, because the energy converted to rotational kinetic energy cannot, without friction, be converted to potential energy. If we throw sand on the ice as the disk comes to a halt, the resulting friction will act to propel the disk farther up the hill, slowing its rotation and recovering this stored energy.

Problem 336. problems-1/rotation-pr-disk-rolling-on-slab-difficult.tex



This problem will help you learn required concepts such as:

- Newton's Second Law (linear and rotational)
- Rolling Constraint
- Static and Kinetic Friction

so please review them before you begin.

A disk of mass m is resting on a slab of mass M, which in turn is resting on a frictionless table. The coefficients of static and kinetic friction between the disk and the slab are μ_s and μ_k, respectively. A small force $\vec{F}$ to the right is applied to the slab as shown, then gradually increased.

a) When $\vec{F}$ is small, the slab will accelerate to the right and the disk will roll on the slab without slipping. Find the acceleration of the slab, the acceleration of the disk, and the *angular* acceleration of the disk as this happens, in terms of m, M, R, and the magnitude of the force F.

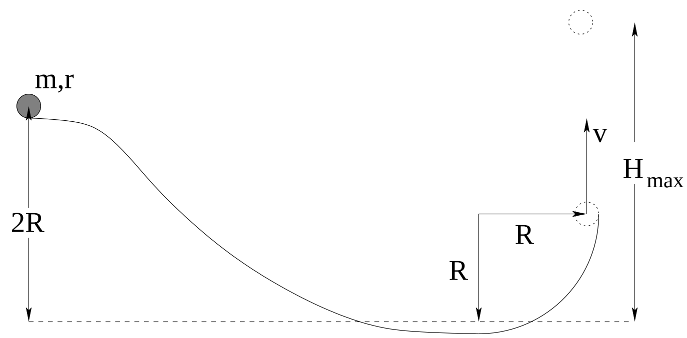
b) Find the maximum force F_{max} such that it rolls without slipping.

c) If F is greater than this, solve once again for the acceleration and angular acceleration of the disk and the acceleration of the slab.

Hint: The hardest single thing about this problem isn't the physics (which is really pretty straightfoward). It is visualizing the coordinates as the center of mass of the disk moves with a different acceleration as the slab. I have drawn *two* figures above to help you with this – the lower figure represents a possible position of the disk after the slab has moved some distance to the right and the disk has rolled back (relative to the slab! It has moved *forward* relative to the ground! Why?) without slipping. Note the dashed radius to help you see the angle through which it has rolled and the various dashed lines to help you relate the distance the slab has moved x_s, the distance the center of the disk has moved x_d, and the angle through which it has rolled θ. Use this relation to connect the acceleration of the slab to the acceleration and angular acceleration of the disk.

If you can do this one, good job!

Problem 337. problems-1/rotation-pr-disk-rolls-to-loop.tex



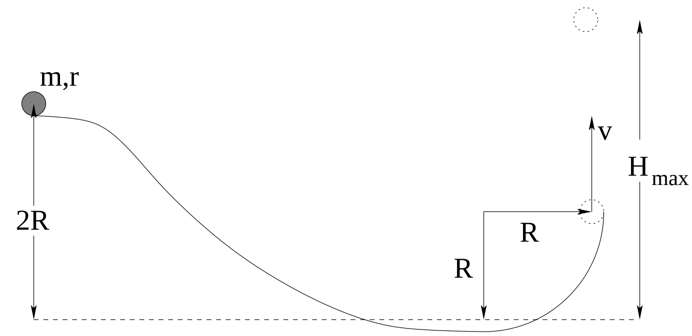
A disk of mass m and radius r (and moment of inertia $I = \frac{1}{2}mr^2$) sits at rest at the top of slope of height $2R$ and ***rolls without slipping*** down the hill to a circular track curving upwards. Ignore drag forces and answer the following questions:

a) How fast is the disk travelling when it reaches the top/end of the curved track (as shown)?

b) Find the normal force acting on the disk due at this point, *just before* it comes off of the circular curve of the track.

c) How high (relative to the lower dashed line) will the disk go above the point where the disk leaves the track before falling back?

Problem 338. problems-1/rotation-pr-disk-rolls-to-loop-soln.tex



A disk of mass m and radius r (and moment of inertia $I = \frac{1}{2}mr^2$) sits at rest at the top of slope of height $2R$ and ***rolls without slipping*** down the hill to a circular track curving upwards. Ignore drag forces and answer the following questions:

a) How fast is the disk travelling when it reaches the top/end of the curved track (as shown)?

b) Find the normal force acting on the disk due at this point, *just before* it comes off of the circular curve of the track.

c) How high (relative to the lower dashed line) will the disk go above the point where the disk leaves the track before falling back?

Solution: Static friction does no work. Hence, our strategy must be to a) use energy conservation plus the ***rolling constraint***, $\Omega = \frac{v}{r}$, to find v as it leaves the track on the right, use this in N2 in the centripetal direction only plus circular motion kinematics to answer b), and then use energy conservation again to answer c, ***noting that it continues to spin as it rises***.

a)

$$E_i = mg2R = mgR + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\Omega^2 + \frac{1}{2}mv^2 = mgR + \frac{3}{4}mv^2 = E_f$$

$$\Rightarrow \quad mv^2 = \frac{4}{3}mgR \quad \Rightarrow \quad v^2 = \frac{4}{3}gR \quad \Rightarrow \quad \boxed{v = \frac{2}{3}\sqrt{3gR}}$$

b)

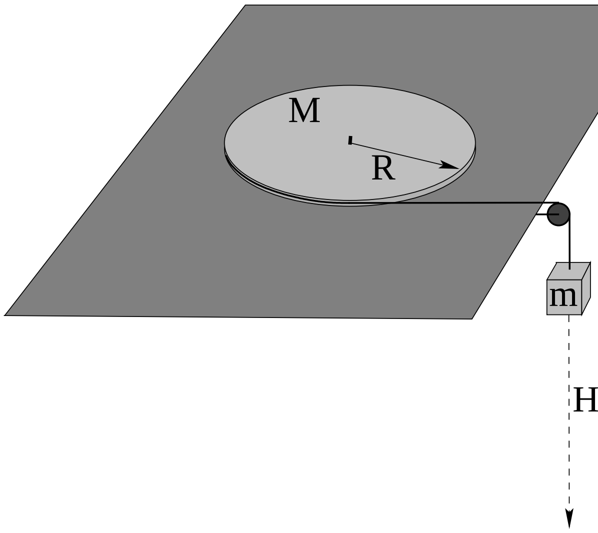
$$\boxed{N = \frac{mv^2}{R} = \frac{4}{3}mg}$$

c)

$$E_i = mgR + \cancel{\frac{1}{2}I\Omega^2} + \frac{1}{2}mv^2 = mgH_{\max} + \cancel{\frac{1}{2}I\Omega^2} = E_f$$

$$\Rightarrow \quad mgR + \frac{2}{3}mgR = \frac{5}{3}\cancel{mg}R = \cancel{mg}H_{\max} \quad \Rightarrow \quad \boxed{H_{\max} = \frac{5}{3}R}$$

Problem 339. problems-1/rotation-pr-falling-mass-spins-disk.tex

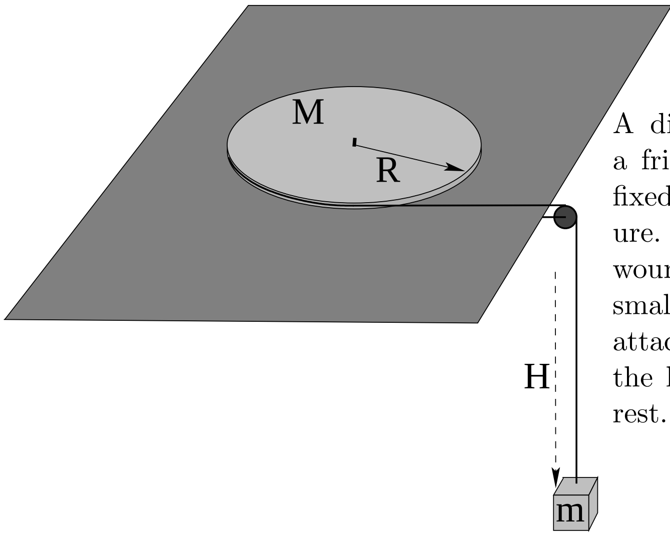


A disk of mass M and radius R placed on a frictionless table can rotate freely about a fixed frictionless spindle as shown in the figure. A massless, unstretchable string is tightly wound around the disk and then passes over a small massless, frictionless pulley, where it is attached to a hanging mass m. At time $t = 0$ the hanging mass and disk are released from rest.

a) Find the tension T in the string ***while the mass is falling and the disk is rotating***.

b) Find the speed v_m of the mass m when it has fallen a height H from its initial position.

Problem 340. problems-1/rotation-pr-falling-mass-spins-disk-soln.tex



A disk of mass M and radius R placed on a frictionless table can rotate freely about a fixed frictionless spindle as shown in the figure. A massless, unstretchable string is tightly wound around the disk and then passes over a small massless, frictionless pulley, where it is attached to a hanging mass m. At time $t = 0$ the hanging mass and disk are released from rest.

a) Find the tension T in the string ***while the mass is falling and the disk is rotating***.

b) Find the speed v_m of the mass m when it has fallen a height H from its initial position.

Solution: Write N2 for the mass and N2 for rotation for the disk, with tension T shared between them and the rolling constraint connecting acceleration to angular acceleration. Eliminate ***acceleration*** (and/or angular acceleration) in favor of the unknown tension T. For b), one can *either* back substitute for the acceleration a and use $v_m = \sqrt{2aH}$, *or* (preferred) use energy conservation plus the rolling constraint.

a) Using down positive:

$$mg - T = ma \qquad RT = I\alpha = \frac{1}{2}MR^2\frac{a}{R} \quad\Rightarrow\quad a = \frac{2T}{M} \quad\Rightarrow\quad mg = T\left(1 + 2\frac{m}{M}\right)$$

or

$$\boxed{T = \left(\frac{M}{M+2m}\right)mg}$$

To find a, it is a bit easier to add the two equations (and use T from the second one to check the answer to a) above):

$$mg = \left(m + \frac{M}{2}\right)a \quad\Rightarrow\quad a = \frac{2mg}{2m+M} \quad\Rightarrow\quad \boxed{T = \frac{1}{2}Ma = \left(\frac{M}{M+2m}\right)mg}$$

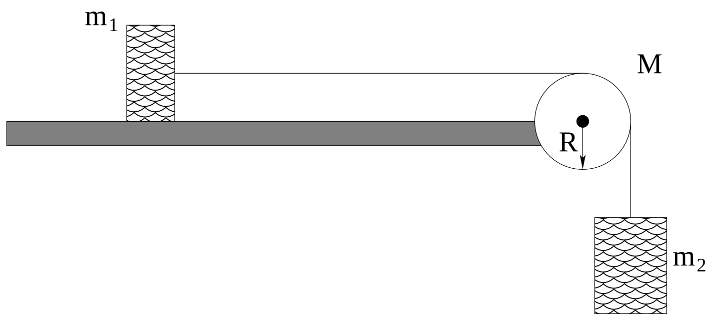
Then b):

$$\boxed{v_m = \sqrt{2aH} = \sqrt{\frac{m}{M+2m}2gH}}$$

or:

$$mgH = \frac{1}{2}mv_m^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\frac{v_m^2}{R^2} = \frac{1}{2}\left(\frac{2m+M}{2}\right)v_m^2 \quad \Rightarrow \quad \boxed{v_m = \sqrt{\frac{m}{M+2m}2gH}}$$

Problem 341. problems-1/rotation-pr-flat-plane-two-blocks-massive-pulley.tex



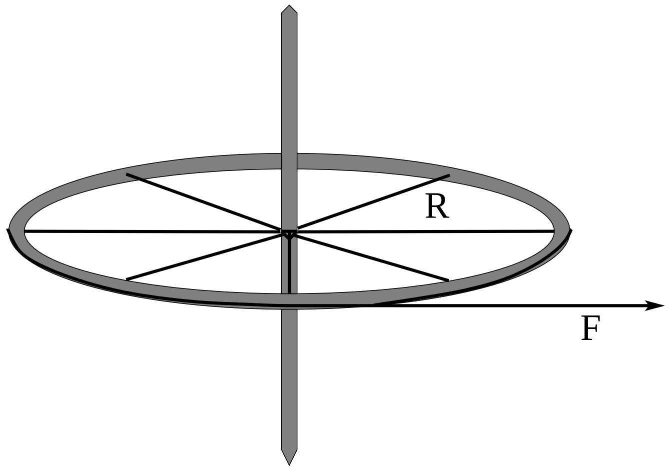
A mass m_1 is attached to a second mass m_2 by an Acme (massless, unstretchable) string. m_1 sits on a frictionless table; m_2 is hanging over the ends of a table, suspended by the taut string from pulley of mass M and radius R. At time $t = 0$ both masses are released.

a) Draw the force/free body diagram for this problem.

b) Find the acceleration of the two masses.

c) Find the angular acceleration of the pulley.

Problem 342. problems-1/rotation-pr-gyroscope-torque.tex

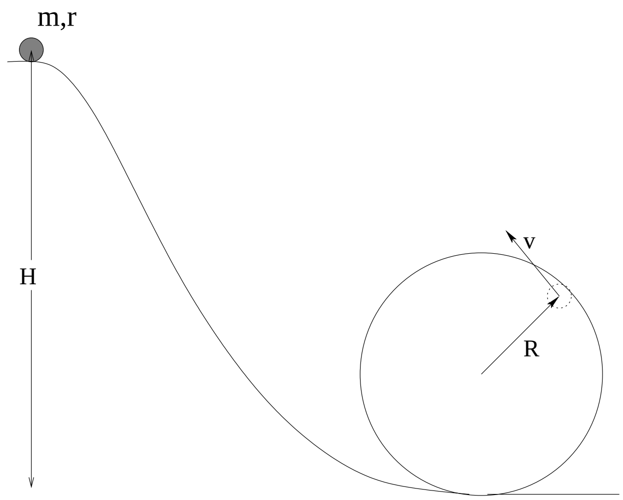


A child spins a gyroscope with moment of inertia I and a frictionless pivot by wrapping a (massless, unstretchable) string of length L around it at a radius R and then pulling the string with a constant force F as shown. Find:

a) The angular acceleration of the gyroscope while the string is being pulled.

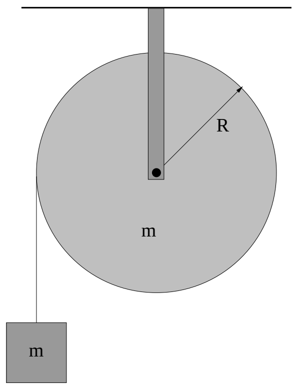
b) The angular speed of the gyroscope as the string comes free (assume that the force F is exerted through the entire distance L).

Problem 343. problems-1/rotation-pr-loop-the-rolling-disk.tex



A disk of mass m and radius r sits at the top of a loop-the-loop of height H. Find the minimum height H such that the disk goes around the loop the loop without coming off of the track, assuming that it rolls without slipping the entire way. Ignore drag forces.

Problem 344. problems-1/rotation-pr-mass-unrolls-spool.tex



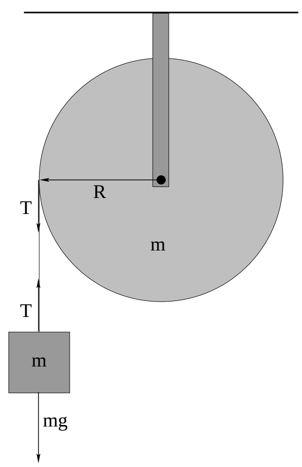
A pulley of mass m, radius R and moment of inertia $I = \beta m R^2$ has a massless, unstretchable string wrapped around it many times with a mass (also) m attached to the end. The system ***begins at rest*** and is released at time $t = 0$ so that the falling mass and the string unrolling ***without slipping*** makes the spool rotate.

a) Find α, the magnitude of the angular acceleration of the pulley about its center of mass as the mass falls.

b) Find the tension T in the string as the mass falls..

c) After the mass m has fallen through a height H, how fast is it moving?

Problem 345. problems-1/rotation-pr-mass-unrolls-spool-soln.tex



A pulley of mass m, radius R and moment of inertia $I = \beta m R^2$ has a massless, unstretchable string wrapped around it many times with a mass (also) m attached to the end. The system ***begins at rest*** and is released at time $t = 0$ so that the falling mass and the string unrolling ***without slipping*** makes the spool rotate.

a) Find α, the magnitude of the angular acceleration of the pulley about its center of mass as the mass falls.

b) Find the tension T in the string as the mass falls..

c) After the mass m has fallen through a height H, how fast is it moving?

Solution: N2 for both masses (only rotation needed for pulley) along with the rolling constraint:

$$F = mg - T = ma \qquad \tau_{\text{out}} = \cancel{R}T = I\alpha = \beta m \cancel{R^2}\frac{a}{\cancel{R}}$$

Add to elimimate T, find a, back substitute to find $\alpha = a/R$ and T:

$$mg = (1+\beta)ma \quad \Rightarrow \quad a = \frac{1}{1+\beta}g \quad \Rightarrow \quad \alpha = \boxed{\frac{g}{(1+\beta)R}} \quad \Rightarrow \quad T = \beta m a = \boxed{\frac{\beta}{1+\beta}mg}$$

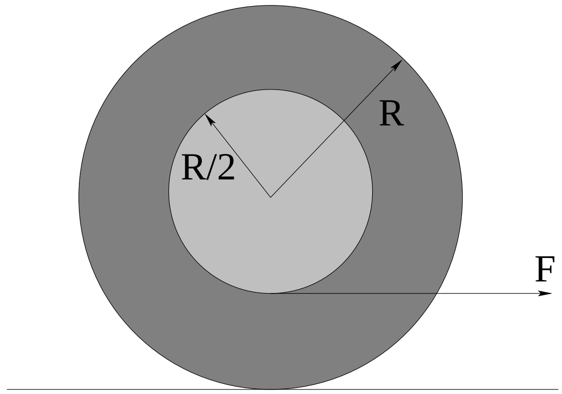
With a in hand, we know that $v = \sqrt{2aH}$ from experience, but the *best* way to actually *show the work* needed to get it is conservation of mechanical energy with the rolling constraint $\Omega = v/R$:

$$E_i = mgH + 0 + 0 = 0 + \frac{1}{2}mv^2 + \frac{1}{2}I\Omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\beta m \cancel{R^2}\left(\frac{v^2}{\cancel{R^2}}\right) = E_f$$

or:

$$\cancel{m}gH = \frac{1}{2}(1+\beta)\cancel{m}v^2 \qquad \Rightarrow \qquad v = \boxed{\sqrt{\frac{2gH}{1+\beta}}} \qquad (= \sqrt{2aH}$$

Problem 346. problems-1/rotation-pr-rolling-spool-pulled-right.tex

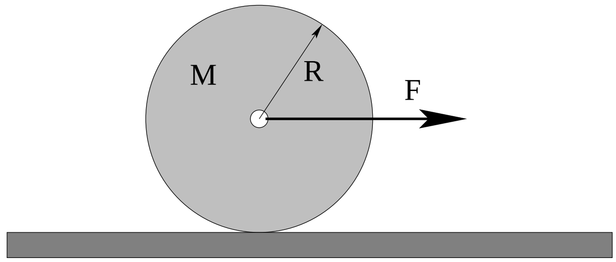


A spool of mass M, radius R and moment of inertia $I = \frac{1}{3}MR^2$ is wrapped around its spindle (radius $R/2$) with fishing line and set on a rough table as shown. The line is then pulled with a force F as shown so that it rolls without slipping.

a) Which way does the spool roll (left or right)? Put another way, does it roll up the string or unroll the string?

b) Find the magnitude of the acceleration of the spool and the force exerted by the table on the spool.

Problem 347. problems-1/rotation-pr-rolling-wheel-static-friction-review.tex



A force of magnitude F (to the right) is applied to the frictionless axle of a wheel made of a ***uniform disk*** with mass M and radius R. It ***rolls without slipping*** on a rough table (with a coefficient of static friction given by μ_s). Find:

a) What is the moment of inertia of this disk about its center of mass? If you cannot remember, use the form $I_cm = \beta M R^2$ to answer the remaining questions.

$$I_{\rm cm} = \boxed{}$$

b) The magnitude of the acceleration of the wheel.

$$a = \boxed{}$$

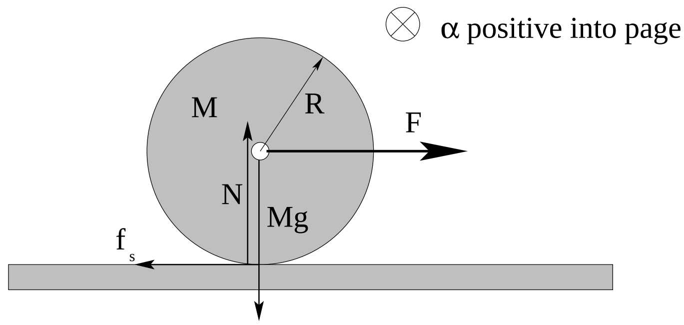
c) The magnitude of the force exerted by static friction. **Indicate its direction on the figure above.**

$$f_s = \boxed{}$$

d) The minimum coefficient of static friction μ_s such that the wheel does not slip for this force.

$$\mu_{s,\rm min} = \boxed{}$$

Problem 348. problems-1/rotation-pr-rolling-wheel-static-friction-review-soln.tex



A force of magnitude F (to the right) is applied to the frictionless axle of a wheel made of a ***uniform disk*** with mass M and radius R. It ***rolls without slipping*** on a rough table (with a coefficient of static friction given by μ_s). Find:

a) What is the moment of inertia of this disk about its center of mass? If you cannot remember, use the form $I_cm = \beta MR^2$ to answer the remaining questions.

$$I_{\rm cm} = \boxed{\frac{1}{2}MR^2}$$

b) The magnitude of the acceleration of the wheel.

This follows from using Newton's Second Law twice, once for translation and once for rotation, plus the rolling constraint $\alpha = a/R$. There is no net vertical force, so $N = Mg$. N, mg, and F exert no torque (about the center of mass – there is an entirely different solution possible using the point of contact with the ground as the pivot, using the parallel axis theorem). So N2 for rotation is:

$$f_sR = I\alpha = \frac{1}{2}MR^2\frac{a}{R}$$

If we divide this by R on both sides and line it up with N2 for translation:

$$F - f_s = Ma$$

$$f_s = \frac{1}{2}Ma$$

and add them to eliminate f_s, we get:

$$a = \boxed{\frac{2}{3}\frac{F}{M}}$$

c) The magnitude of the force exerted by static friction. **Indicate its direction on the figure above.**

Backsubstitute this into the expression above for f_s:

$$f_s = \frac{1}{2}Ma = \boxed{\frac{1}{3}F}$$

Note Well! Any answer such as $f_s = \mu_s Mg$ is ***wrong!***

d) The minimum coefficient of static friction μ_s such that the wheel does not slip for this force.

Here we use:

$$f_s = \frac{1}{3}F < F_s = \mu_s N = \mu_s Mg$$

Note Well the inequality! We rearrange this to obtain:

$$\mu_s > \frac{1}{3}\frac{F}{Mg}$$

or

$$\mu_{s,\mathrm{min}} = \boxed{\frac{1}{3}\frac{F}{Mg}}$$

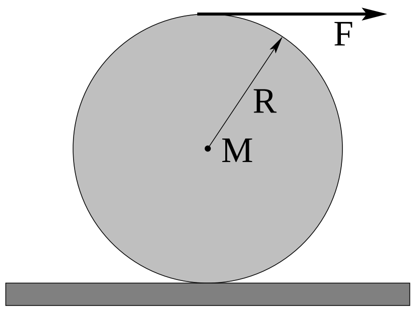
Scoring:

a) +2 points. This is something they were told they should know.

b),c) +12 points together. +4 for each N2, +2 for rolling constraint, +2 for algebra including back substitution. Instant -4 if they assert $f_s = \mu_s Mg$.

d) +6 points. If they wrote the wrong answer for c) above, they may well lose more points here, but we'll cap it at an additional -3 (basically giving them +3 total for knowing that $\mu_s N$ is relevant in some way to the problem. But we'll give successful students +4 for writing down the correct inequality, and save the last +2 for algebra or confusion.

Problem 349. problems-1/rotation-pr-rolling-wheel-static-friction-top.tex



A massless rope that is wrapped around a a uniform disk with mass M and radius R is pulled to the right with a force F as shown. As long as F is less than some maximum value it will ***roll without slipping*** on a rough table (with a coefficient of static friction given by μ_s).

a) Find the **magnitude** of the acceleration of the wheel, assuming that it rolls without slipping.

b) Find the **magnitude** of the force exerted by static friction, assuming that it rolls without slipping.

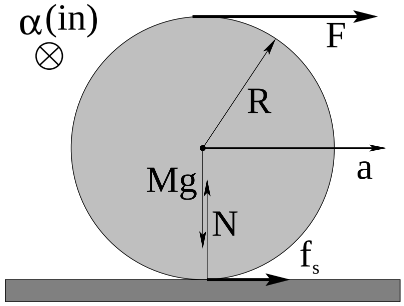
c) **Indicate the actual direction that the static friction points** when it rolls without slipping.

d) What is the largest force F_{max} that can be exerted before the wheel starts to slip?

$a = \boxed{}$ $f_s = \boxed{}$ **Circle:** $\Rightarrow$ $\Leftarrow$

$F_{\mathrm{max}} = \boxed{}$

Problem 350. problems-1/rotation-pr-rolling-wheel-static-friction-top-soln.tex



A massless rope that is wrapped around a a uniform disk with mass M and radius R is pulled to the right with a force F as shown. As long as F is less than some maximum value it will ***roll without slipping*** on a rough table (with a coefficient of static friction given by μ_s).

a) Find the **magnitude** of the acceleration of the wheel, assuming that it rolls without slipping.

b) Find the **magnitude** of the force exerted by static friction, assuming that it rolls without slipping.

c) **Indicate the actual direction that the static friction points** when it rolls without slipping.

d) What is the largest force $F_{\max}$ that can be exerted before the wheel starts to slip?

$a = \boxed{\frac{4}{3}\frac{F}{M}}$ $f_s = \boxed{\frac{1}{3}F}$ **Circle:** $\Rightarrow$ $\Leftarrow$

$F_{\max} = \boxed{3\text{Mg}}$

Solution for a-c: Use N2 for translation and rotation, using the CM as a pivot and the rolling constraint $\alpha = a/R$:

$$F + f_s = Ma \qquad R(F - f_s) = \frac{1}{2}MR^2\frac{a}{R} \quad \Rightarrow \quad 2F = \frac{3}{2}Ma$$

where we divided out the Rs and added the equations. Solving for a and backsubstituting:

$$a = \frac{4}{3}\frac{F}{M} \qquad f_s = Ma - F = +\frac{1}{3}F$$

and the drawn/guessed direction above is apparently correct. The disk accelerates ***faster*** than it would from the force vF alone in empty space!

Solution for d: We know that $f_s = F/3 < \mu_s N$ if it does not slip. We also know that $N - Mg = Ma_y = 0$ in the vertical direction, so $N = Mg$. Hence it will not slip for:

$$F < 3\mu_s N = 3Mg = F_{\max}$$

Alternate Solution for a-c: Same as before, but using the point of contact with the table as the pivot and the parallel axis theorem:

$$F + f_s = Ma \qquad 2RF = (MR^2 + \frac{1}{2}MR^2)\frac{a}{R} = \frac{3}{2}MR^2\frac{a}{R} \quad \Rightarrow \quad a = \frac{4F}{3M}$$

etc. as before. The advantage here is one gets a in one step, but you still have to backsubstitute to get $f_s = F/3$.

Problem 351. problems-1/rotation-pr-sliding-rolling-bowling-ball.tex

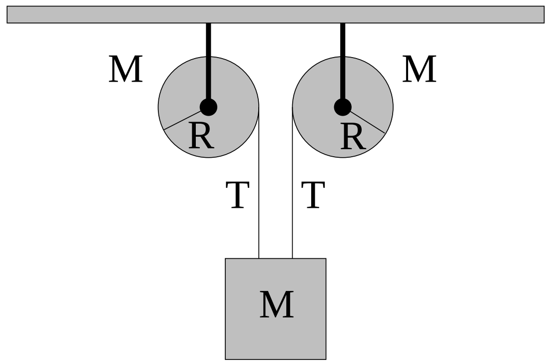
A uniform bowling ball of radius R, mass M, and moment of inertia I about its center of mass is initially launched so that it is sliding with speed v_0 without rolling on an alley with a coefficient of friction μ_k.

a) Analyze the forces acting on the bowling ball to find the acceleration of the center of mass and angular acceleration of the bowling ball about its CM;

b) Find the CM velocity as a function of time (t) and angular velocity of the ball as a function of time (t).

c) Find the CM velocity of the bowling ball when it starts rolling without slipping.

Problem 352. problems-1/rotation-pr-two-spools-one-mass.tex



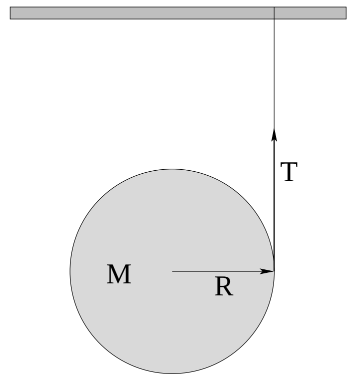
In the figure above, a mass M is connected to two independent massive spools of radius R, also of mass M (each), wrapped with massless unstretchable string. You may consider the spools to be ***disks*** as far as their moment of inertia is concerned. At $t = 0$, the mass M and spools are released from rest and the mass M falls. Find:

a) The magnitude of the acceleration a of the mass M.

b) The tension T in either string (they are the same from symmetry).

c) When the mass M has fallen a distance H, what is its speed?

Problem 353. problems-1/rotation-pr-unrolling-a-falling-spool-algebraic.tex

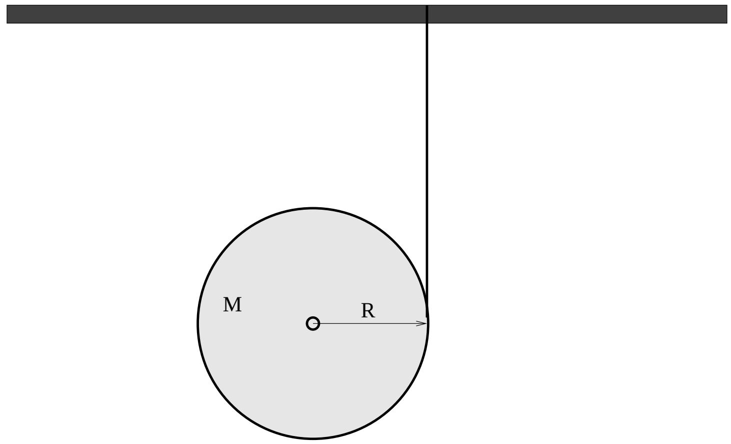


A spool of fishing line is tied to a beam and released from rest in the position shown at time $t = 0$. The spool has a mass M, a radius of R, and a moment of inertial $I = \beta M R^2$. The line itself has negligible mass per unit length. Once released, the disk falls as the taut line unrolls.

a) What is the tension in the line as the disk falls (unrolling the line)?

b) After the disk has fallen a height H, what is its angular velocity ω?

Problem 354. problems-1/rotation-pr-unrolling-a-falling-spool-numeric.tex

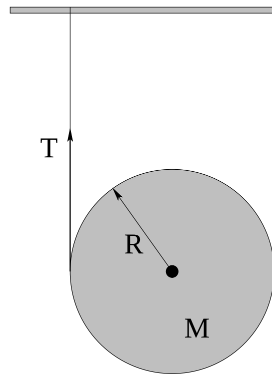


A spool of fishing line is tied to a beam and released from rest in the position shown at time $t = 0$. The spool is a disk and has a mass of 50 grams and a radius of 5 cm. The line itself has negligible mass per unit length. Once released, the disk falls as the taut line unrolls.

a) What is the tension in the line as the disk falls (unrolling the line)?

b) After the disk has fallen 2m, what is its speed?

Problem 355. problems-1/rotation-pr-unrolling-a-falling-spool-reversed.tex



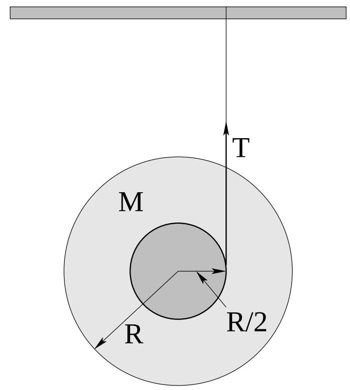
A spool of fishing line is tied to a beam and released from rest in the position shown at time $t = 0$. The spool has a mass M, a radius of R, and a moment of inertial $I = \beta M R^2$. The line itself has negligible mass per unit length. Once released, the spool falls as the taut line unrolls.

a) What is the tension in the line as the spool falls (unrolling the line)?

b) What is the magnitude of the angular acceleration of the spool α about its center of mass as it falls?

c) After the spool has fallen a height H, what is the ***direction*** of its angular velocity, $\vec{\omega}$? Indicate this direction with a labelled arrow symbol on a suitable diagram.

Problem 356. problems-1/rotation-pr-unrolling-a-falling-yo-yo.tex



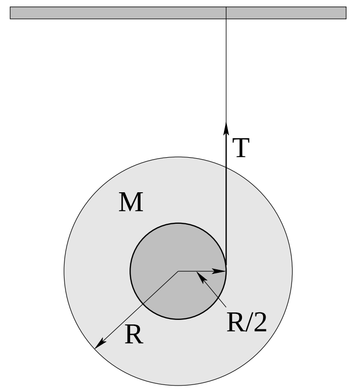
A yo-yo is tied to a beam and released ***from rest*** in the position shown at time $t = 0$. The yo-yo has a mass M, a radius of R, and a moment of inertia $I = \beta M R^2$. The unstretchable line itself has negligible mass per unit length and is wrapped around an inner spindle with radius $R/2$ as shown. Once released, the yo-yo falls as the taut line unrolls.

a) What is the ***angular acceleration*** $\vec{\alpha}$ of the yo-yo as it falls (unrolling the line)? Note that this is a ***vector*** quantity, so please indicate its direction in your answer and/or on the figure.

b) What is the tension T in the line as the yo-yo falls (unrolling the line)?

c) After the yo-yo has fallen a height H, what is its angular velocity ω?

Problem 357. problems-1/rotation-pr-unrolling-a-falling-yo-yo-soln.tex



As always, we start with a free body diagram (easy, only a single body). Then we write ***both, note well, both*** $F = ma$ in the vertical direction ***and*** $\tau = I\alpha$ around a pivot chosen to be the center of mass of the yo-yo:

$$\begin{aligned} F = Mg - T &= Ma \qquad \text{(down +)} \\ \tau = \frac{R}{2}T &= I\alpha \qquad \text{(out of page +)} \end{aligned}$$

Note well that the torque due to tension is exerted at radius $R/2$, not R the radius of the yo-yo.

Next we substitute $I = \beta MR^2$ and the rolling constraint ***at the radius*** $\boldsymbol{R/2}$, that is $\alpha = 2a/R$, into the second equation, multiply the whole equation by 2 (to simplify the algebra), cancel the $R's$ from both sides, and get:

$$T = 4\beta Ma$$

We add this to the first equation:

$$\begin{aligned} F = Mg - T &= Ma \\ T &= 4\beta Ma \end{aligned}$$

to get:

$$Mg = (1 + 4\beta)Ma$$

This gives us $a = g/(1 + 4\beta)$ and we can get the magnitude of α using the expression above. Now we can answer all of the questions easily:

a)
$$\alpha = \frac{2g}{(1 + 4\beta)R} \qquad \text{(out of page)}$$

b) Backsubstitute a to get:
$$T = \frac{4\beta}{(1 + 4\beta)}Mg$$

c) Finally, use mechanical energy conservation plus the rolling constraint, now expressed as $v = R\omega/2$:

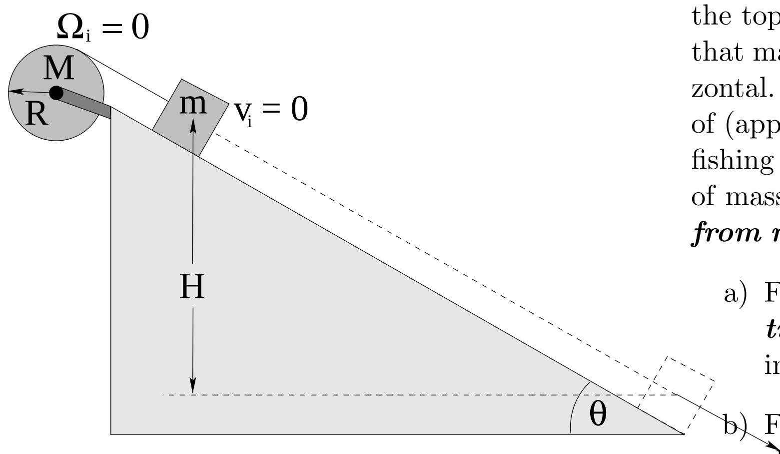
$$MgH = \frac{1}{2}Mv^2 + \frac{1}{2}\beta MR^2\omega^2 = \frac{1}{2}MR^2(\frac{1}{4} + \beta)\omega^2$$

to get:

$$\omega = \sqrt{\frac{2gH}{R^2(\frac{1}{4} + \beta)}} = \frac{2}{R}\sqrt{\frac{2gH}{(1 + 4\beta)}} = \frac{2v}{R}$$

where $v = \sqrt{2aH}$ as usual.

Problem 358. problems-1/rotation-pr-unrolling-spool-down-inclined-plane-1.tex



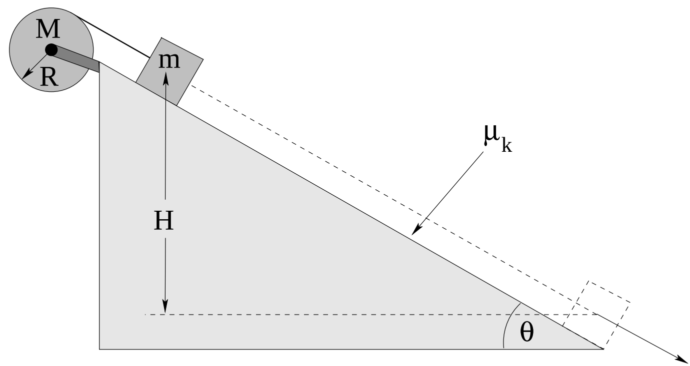
In the figure to the left, a pulley at rest of mass M and radius R with ***frictionless*** bearings and moment of inertia $\boldsymbol{I = \beta M R^2}$ is fixed at the top of a fixed, ***frictionless*** inclined plane that makes an angle θ with respect to the horizontal. The pulley is wrapped with many turns of (approximately massless and unstretchable) fishing line. The line is also attached to a block of mass m. The block and pulley are released ***from rest***: $v_i = 0$ (block) and $\Omega_i = 0$ (pulley).

a) Find the ***magnitude of the acceleration a*** of the block as it slides down the incline.

b) Find the ***tension T*** in the string as it slides.

c) Find the ***speed v*** with which the block reaches the bottom of the incline.

Problem 359. problems-1/rotation-pr-unrolling-spool-down-inclined-plane-2.tex



In the figure above, a pulley of mass M and radius R with frictionless bearings and moment of inertia $I = \beta MR^2$ is fixed at the top of a ***rough*** inclined plane that makes an angle θ with respect to the horizontal that is large enough that the block will definitely overcome static friction and slide. The coefficient of kinetic friction between the block and the plane is μ_k. The pulley is wrapped with many turns of (approximately massless and unstretchable) fishing line. The line is also attached to a block of mass m. At time $t = 0$ the block and pulley are released ***from rest***.

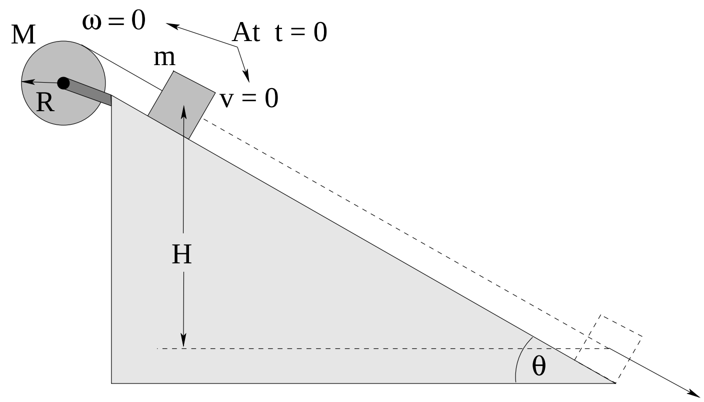
a) ***Draw a force diagram*** for both the block and the pulley separately. You do not have to represent the forces acting at the pivot of the pulley that keep it stationary, only the one(s) relevant to the solution of the problem. Represent *all* the forces on the block.

b) Find both the magnitude of the acceleration a of the block and the tension T in the string as the block slides down the incline in **terms of the givens**.

c) Find the kinetic energy of the block when it reaches the bottom of the incline.

d) Find the kinetic energy of the pulley when the block reaches the bottom of the incline.

Problem 360. problems-1/rotation-pr-unrolling-spool-down-inclined-plane-friction.tex



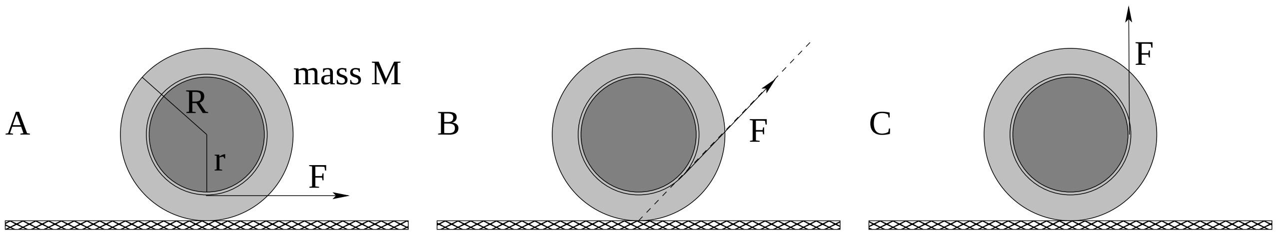
A pulley with frictionless bearings and moment of inertia $I = \beta M R^2$ is at the top of a fixed inclined plane that makes an angle θ with respect to the horizontal. The pulley is wrapped with many turns of (approximately massless and unstretchable) fishing line that is attached to a block of mass m resting on the incline a height H above the bottom. The coefficient of kinetic friction between the block and the incline is μ_k. At time $t = 0$ the block and pulley are released ***from rest*** at an angle θ that is large enough that the block ***will definitely overcome static friction and begin to slide***.

a) On the figure above or in a free body diagram to the side, draw in and label all of the forces acting ***on the block only***.

b) Find the ***magnitude*** of the acceleration a of the block as it slides down the incline.

c) Find the ***speed*** v with which the block reaches the bottom of the incline.

Problem 361. problems-1/rotation-pr-walking-the-spool.tex



In the figure above, a spool of mass M is wrapped with string around the inner spool. The spool is placed on a rough surface and the string is pulled with force F in the three directions shown. The spool, if it rolls at all, ***rolls without slipping***. (Note that if pulled too hard, the spool can both slip and/or roll.)

Find the acceleration and frictional force ***vectors*** (magnitude ***and*** direction) for all three figures. Use $I_{\rm cm} = \beta M R^2$.

Note Well: You can use ***either*** the center of mass ***or*** the point of contact with the ground (with the parallel axis theorem) as a pivot, the latter being ***slightly easier***.

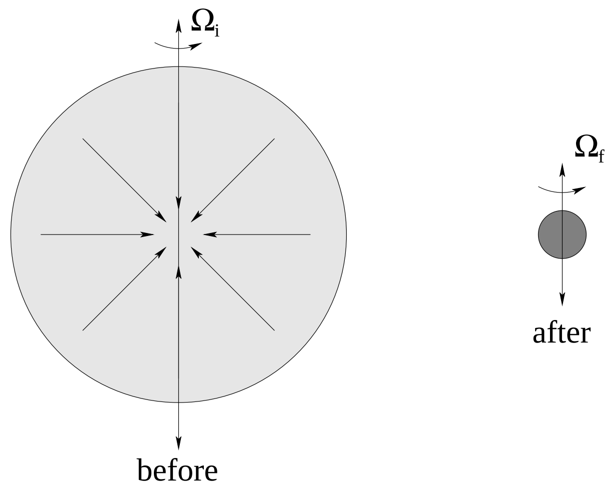
Chapter 8

Vector Torque and Angular Momentum

8.1 Angular Momentum

8.1.1 Multiple Choice Problems

Problem 362. problems-1/angular-momentum-mc-collapsing-star.tex



When a star rotating with an angular speed Ω_i (eventually) exhausts its fuel, escaping light energy can no longer oppose gravity throughout the star's volume and it ***suddenly*** shrinks, with most of its outer mass falling in towards the center all at the same time.

As this happens, does the magnitude of the angular speed of rotation Ω_f:

a) increase

b) decrease

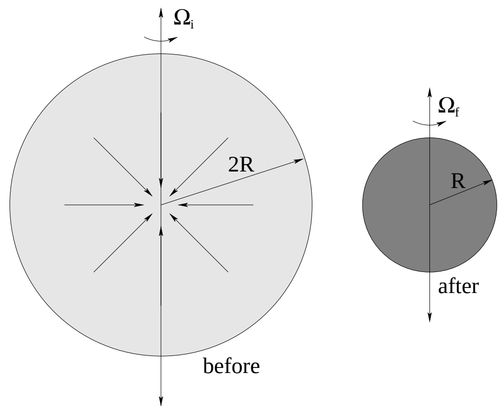
c) remain about the same

Why (state the principle used to answer the question)?

Problem 363. problems-1/angular-momentum-mc-collapsing-star-soln.tex

Angular momentum is (approximately) conserved and the moment of inertial decreases, so Ω increases.

Problem 364. problems-1/angular-momentum-mc-forming-star.tex

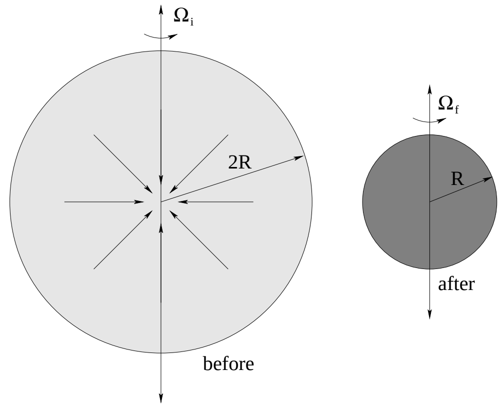


Gravity gradually assembles a star by pulling a cloud of rotating gas together into a rotating ball that then gradually shrinks. The figure above represents a star at two different stages in its formation, the first where a gas of total mass M has formed a ball of radius $2R$ rotating at angular speed Ω_i, the second where the ball has collapsed to a radius R (compressing the nuclear fuel inside closer to the point of fusion and ignition), rotating at a possibly new angular speed Ω_f.

Assuming that the mass is uniformly distributed in both cases, what is the best estimate for Ω_f in terms of Ω_i?

☐ $\Omega_f = \Omega_i/4$ ☐ $\Omega_f = \Omega_i/2$ ☐ $\Omega_f = \Omega_i$ ☐ $\Omega_f = 2\Omega_i$ ☐ $\Omega_f = 4\Omega_i$

Problem 365. problems-1/angular-momentum-mc-forming-star-soln.tex

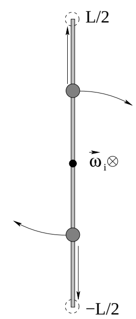


Gravity gradually assembles a star by pulling a cloud of rotating gas together into a rotating ball that then gradually shrinks. The figure above represents a star at two different stages in its formation, the first where a gas of total mass M has formed a ball of radius $2R$ rotating at angular speed Ω_i, the second where the ball has collapsed to a radius R (compressing the nuclear fuel inside closer to the point of fusion and ignition), rotating at a possibly new angular speed Ω_f.

Assuming that the mass is uniformly distributed in both cases, what is the best estimate for Ω_f in terms of Ω_i?

☐ $\Omega_f = \Omega_i/4$ ☐ $\Omega_f = \Omega_i/2$ ☐ $\Omega_f = \Omega_i$ ☐ $\Omega_f = 2\Omega_i$ ☒ $\Omega_f = 4\Omega_i$

Problem 366. problems-1/angular-momentum-mc-rotating-rod-sliding-beads.tex



In the figure above, a massless rod of length L is rotating around a frictionless pivot through its center at angular speed ω_i. Two beads, each with mass m, are stuck a distance $L/4$ from the center. The rotating system initially has a total kinetic energy K_i (which you could actually calculate if you needed to). At a certain time, the beads are released and slide smoothly to the ends of the rod where again, they stick. Which statement about the final angular speed and rotational kinetic energy of the rotating system is true:

a) $\omega_f = \omega_i/2$ and $K_f = K_i$.

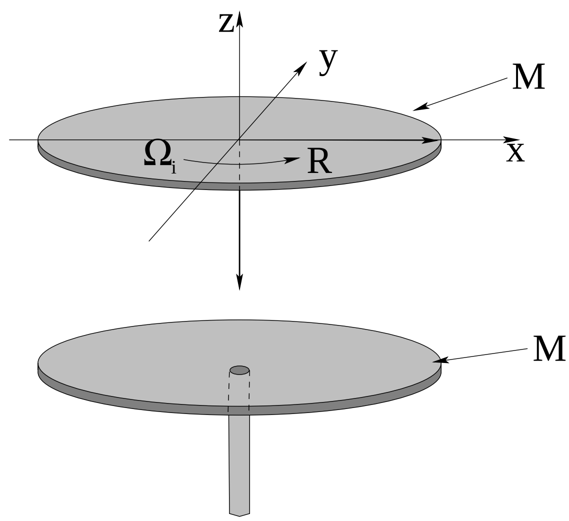
b) $\omega_f = \omega_i/4$ and $K_f = K_i/4$.

c) $\omega_f = \omega_i/4$ and $K_f = K_i/2$.

d) $\omega_f = \omega_i/2$ and $K_f = K_i/4$.

e) $\omega_f = \omega_i/2$ and $K_f = K_i/2$.

Problem 367. problems-1/angular-momentum-mc-two-circular-plates-collide.tex



A **disk** of uniformly distributed mass M and radius R sits at rest on a turntable that permits it to rotate freely. A second uniform disk of mass M with the same radius, centered on the same axis of rotation, is rotating at an (initial) angular speed Ω_i and is dropped gently onto it so that (after sliding for an instant) they stick together and rotate together as one.

How do the final angular velocity and final kinetic energy relate to the initial angular velocity and initial kinetic energy?

a) $\Omega_f = \Omega_i, \quad K_f = K_i$

b) $\Omega_f = 2\Omega_i, \quad K_f = K_i/2$

c) $\Omega_f = \Omega_i/2, \quad K_f = K_i/2$

d) $\Omega_f = \Omega_i/4, \quad K_f = K_i/4$

e) We cannot tell from the information given.

Problem 368. problems-1/angular-momentum-mc-two-circular-plates-collide-soln.tex

Angular momentum is conserved, inelastic collision: In z/axial direction, $L = L_i = \frac{1}{2}MR^2\Omega_0 = 2(\frac{1}{2}MR^2)\Omega_f = L_f$, so $\Omega_f = \Omega_i/2$.

Only one answer has this, but just in case, $K_i = \frac{L^2}{2(\frac{1}{2}MR^2)}$, $K_f = \frac{L^2}{2(MR^2)} = K_i/2$, so answer is c) on both counts.

8.1.2 Short Answer Problems

Problem 369. problems-1/angular-momentum-sa-bug-on-rotating-disk.tex

A disk of mass M and radius R is rotating about its axis with initial angular velocity Ω_0. A rhinoceros beetle with mass m is standing on its outer rim as it does so. The beetle decides to walk in to the very center of the disk and stand on the axis as it feels less pseudoforce there and it is easier to hold on. What is the angular velocity of the disk when it gets there?

(Ignore friction and drag forces).

Problem 370. problems-1/angular-momentum-sa-conserved-quantities.tex

For each of the collisions described below, say whether the total mechanical energy, total momentum, and total angular momentum of the system consisting of the two colliding objects are conserved or not. Indicate your answer by writing "**C**" (for "is ***definitely*** conserved") or "**N**" (for "not ***necessarily*** conserved") in each box. You may write a brief word of explanation if you think there is any ambiguity in the answer.

	Total Energy	Linear Momentum	Angular Momentum
A hard ball (point particle) bounces off of a rigid wall that cannot move, returning at the same speed it had before the collision.	☐	☐	☐
A piece of space junk strikes the orbiting space shuttle and sticks to it.	☐	☐	☐

Problem 371. problems-1/angular-momentum-sa-conserved-quantities-soln.tex

For each of the collisions described below, say whether the total mechanical energy, total momentum, and total angular momentum of the system consisting of the two colliding objects are conserved or not. Indicate your answer by writing "**C**" (for "is ***definitely*** conserved") or "**N**" (for "not ***necessarily*** conserved") in each box. You may write a brief word of explanation if you think there is any ambiguity in the answer.

	Total Energy	Linear Momentum	Angular Momentum
A hard ball (point particle) bounces off of a rigid wall that cannot move, returning at the same speed it had before the collision.	C		N

Explanation: It bounces off at the same speed (given) so the collision *is* elastic *as given.* Linear momentum is not conserved, period (so blank). Angular momentum is conserved *only if the pivot is on the line of motion of the particle so it is zero before and after the collision.* This is rather unlikely (and not helpful in solving any sort of problem) but is enough for this to earn an N as it *could* happen if a coordinate system of this sort was given.

	Total Energy	Linear Momentum	Angular Momentum
A piece of space junk strikes the orbiting space shuttle and sticks to it.		C	C

Explanation: The collision is "fully inelastic" as they stick together and lose all of the kinetic energy initially present in the center of mass reference frame, so total mechanical energy is definitely not conserved. *In the (usual) impulse approximation* gravity (holding the shuttle "in orbit" exerts a negligible force during the *short* time of the collision and no other forces are present (it's in a vacuum so no drag etc). Hence *both* momentum *and* angular momentum are conserved, as no external torques act either.

Problem 372. problems-1/angular-momentum-sa-conserved-quantities-soln.tex

For each of the collisions described below, say whether the total mechanical energy, total momentum, and total angular momentum of the system consisting of the two colliding objects are conserved or not. Indicate your answer by writing "**C**" (for "is ***definitely*** conserved") or "**N**" (for "not ***necessarily*** conserved") in each box. You may write a brief word of explanation if you think there is any ambiguity in the answer.

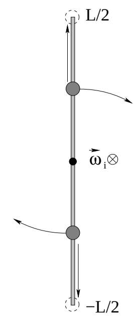
	Total Energy	Linear Momentum	Angular Momentum
A hard ball (point particle) bounces off of a rigid wall that cannot move, returning at the same speed it had before the collision.	$\boxed{\text{C}}$	$\boxed{}$	$\boxed{\text{N}}$

Explanation: It bounces off at the same speed (given) so the collision *is* elastic *as given.* Linear momentum is not conserved, period (so blank). Angular momentum is conserved *only if the pivot is on the line of motion of the particle so it is zero before and after the collision.* This is rather unlikely (and not helpful in solving any sort of problem) but is enough for this to earn an N as it *could* happen if a coordinate system of this sort was given.

	Total Energy	Linear Momentum	Angular Momentum
A piece of space junk strikes the orbiting space shuttle and sticks to it.	$\boxed{}$	$\boxed{\text{C}}$	$\boxed{\text{C}}$

Explanation: The collision is "fully inelastic" as they stick together and lose all of the kinetic energy initially present in the center of mass reference frame, so total mechanical energy is definitely not conserved. *In the (usual) impulse approximation* gravity (holding the shuttle "in orbit" exerts a negligible force during the *short* time of the collision and no other forces are present (it's in a vacuum so no drag etc). Hence *both* momentum *and* angular momentum are conserved, as no external torques act either.

Problem 373. problems-1/angular-momentum-sa-rotating-rod-sliding-beads.tex



In the figure above, a massless rod of length L is rotating around a frictionless pivot through its center at angular speed ω_i. Two beads, each with mass m, are stuck a distance $L/4$ from the center. The rotating system initially has a total kinetic energy K_i (which you could actually calculate if you needed to). At a certain time, the beads are released and slide smoothly to the ends of the rod where again, they stick.

A) What quantities of the system (rod plus two beads) are conserved by this process? (Place a Y or N in the provided answer boxes.)

☐ Total Kinetic Energy

☐ Total Linear Momentum

☐ Total Angular Momentum

B) Determine the ratio of the following quantities:

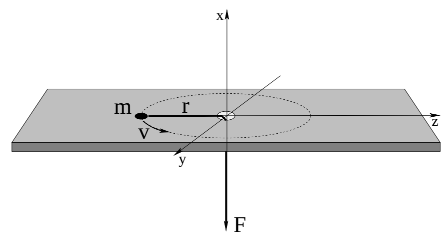
$$\frac{I_f}{I_i} =$$

$$\frac{\omega_f}{\omega_i} =$$

$$\frac{K_f}{K_i} =$$

8.1.3 Regular Problems

Problem 374. problems-1/angular-momentum-pr-circular-orbit-on-table.tex



A particle of mass M is tied to a string that passes through a hole in a frictionless table and held. The mass is given a push so that it initially moves in a circle of radius r_i at speed v_i. We will now conceptual review and algebraically analyze the physics of its motion in two stages. Please answer the following questions. While the string is fixed (so that r_i is constant):

a) What is the torque exerted on the particle by the string?

b) What is the ***vector angular momentum*** L_i of the particle? Use the provided coordinate system to give the direction.

c) Show that the magnitude of the force (the tension in the string) that must be exerted to keep the particle moving in this circle is:

$$F = \frac{L_i^2}{mr_i^3}$$

Note that ***this is a general result*** for a particle moving in a circle and in no way depends on the fact that the force is being exerted by a string in particular.

d) Show that the kinetic energy of the particle in terms of its angular momentum is:

$$K_i = \frac{L_i^2}{2mr_i^2}$$

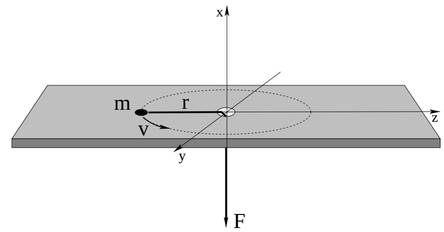
From under the table, the string is *slowly* pulled down (so that the puck is always moving in an approximately circular trajectory and the tension in the string remains radial) to where the particle is moving in a circle of radius r_f.

e) If the tension in the string remains radial, what quantity ought to be conserved?

f) Find its velocity v_f using conservation of that quantity.

g) Compute the work done by the force from part c) above and identify the answer *as* the work-kinetic energy theorem. Use this principle *instead* to to find the velocity v_f. You should get the same answer!

Problem 375. problems-1/angular-momentum-pr-circular-orbit-on-table-soln.tex



A particle of mass M is tied to a string that passes through a hole in a frictionless table and held. The mass is given a push so that it initially moves in a circle of radius r_i at speed v_i. We will now conceptual review and algebraically analyze the physics of its motion in two stages. Please answer the following questions. While the string is fixed (so that r_i is constant):

a) What is the torque exerted on the particle by the string?

b) What is the ***vector angular momentum*** L_i of the particle? Use the provided coordinate system to give the direction.

c) Show that the magnitude of the force (the tension in the string) that must be exerted to keep the particle moving in this circle is:

$$F = \frac{L_i^2}{mr_i^3}$$

Note that ***this is a general result*** for a particle moving in a circle and in no way depends on the fact that the force is being exerted by a string in particular.

d) Show that the kinetic energy of the particle in terms of its angular momentum is:

$$K_i = \frac{L_i^2}{2mr_i^2}$$

From under the table, the string is *slowly* pulled down (so that the puck is always moving in an approximately circular trajectory and the tension in the string remains radial) to where the particle is moving in a circle of radius r_f.

e) If the tension in the string remains radial, what quantity ought to be conserved?

f) Find its velocity v_f using conservation of that quantity.

g) Compute the work done by the force from part c) above and identify the answer *as* the work-kinetic energy theorem. Use this principle *instead* to to find the velocity v_f. You should get the same answer!

Solution:

a) $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times (-T)\hat{\boldsymbol{r}} = 0 = \dfrac{d\vec{\boldsymbol{L}}}{dt}$

b) $L_i = |\vec{\boldsymbol{r}}_i \times m\vec{\boldsymbol{v}}_i| = mv_i r_i$. The direction, given by the right hand rule, is $\hat{\boldsymbol{x}}$ (or "up", although using coordinates is better).

c) Using Newton's Second Law and centripetal acceleration:

$$F(=T) = \frac{mv_i^2}{r_i} = \frac{mv_i^2}{r_i} \times \frac{mr_i^2}{mr_i^2} = \frac{(mv_i r_i)^2}{mr_i^3} = \frac{L_i^2}{mr_i^3}$$

Note that we multipled by "1" in a convenient form in the middle.

d) We'll start with the standard "easy" version of the kinetic energy:

$$K_i = \frac{1}{2}mv_i^2 = \frac{1}{2}mv_i^2 \times \frac{mr_i^2}{mr_i^2} = \frac{(mv_i r_i)^2}{2mr_i^2} = \frac{L_i^2}{2mr_i^2}$$

Note that we multipled ***again*** by "1" in that ***same*** convenient form in the middle.

e) Because the force exerted by the tension is almost always perpendicular to the velocity of the particle, the torque exerted by the tension remains (almost exactly) zero. We therefore *expect* the angular momentum to be (almost exactly) conserved.

f) This permits us to use angular momentum conservation to find v_f given (in the problem) r_f:

$$L_i = mv_i r_i = mv_f r_f = L_f \quad \Rightarrow \quad v_f = \frac{r_i}{r_f} v_i$$

g) We have to be very careful here. The tension T is *also* nearly perpendicular to $\vec{\boldsymbol{v}}$ throughout the motion, so one might conclude that no work is done and energy is also conserved. However, *one's hand, pulling down on the string, absolutely* does work and that work is not dissipated by any non-conservative forces, so it must appear as a change in kinetic energy! Expressing F in terms of (constant) $L = mv_i r_i$:

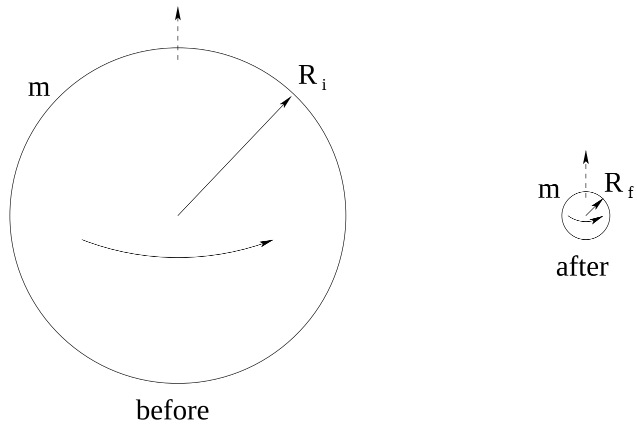
$$W = \int_{r_i}^{r_f} F dr = -\int_{r_i}^{r_f} \frac{L^2}{m} r^{-3} dr = \left. \frac{L^2}{2mr^2} \right|_{r_i}^{r_f} = fracL^2 2mr_f^2 - fracL^2 2mr_i^2 = \Delta K$$

which is the WKE theorem. The final step is easy – note that:

$$K_f = \frac{\cancel{1}}{\cancel{2}}\cancel{m}v_f^2 = \frac{(\cancel{m}v_i r_i)^2}{\cancel{2m}r_f^2} \quad \Rightarrow \quad v_f = \frac{r_i}{r_f} v_i$$

as before. Pay careful attention to this problem, as it is a conceptual key to steps involved in solving *several* problems in this course as well as deriving things like the "angular momentum barrier" in the chapter on gravitation.

Problem 376. problems-1/angular-momentum-pr-collapse-of-sun.tex



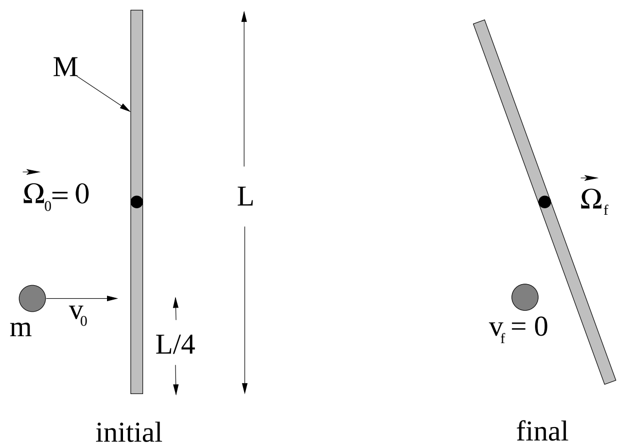
The sun reaches the end of its life and gravitationally collapses quite suddenly, forming a white dwarf. Before it collapses, it has a mass m, a radius R_i, and a period of rotation T_i. After it collapses, its radius is $R_f \ll R_i$ and we will assume that its mass is unchanged. We will also assume that before and after the moment of inertia of the sun is given by $I = \beta m R^2$ where R is the appropriate radius.

a) What is its final period of rotation T_f after the collapse?

b) Evaluate the escape velocity from the surface of the sun before and after its collapse.

For 2 points of extra credit, evaluate the numbers associated with these expressions given $\beta = 0.25$, $m = 2 \times 10^{30}$ kg, $R_i = 5 \times 10^5$ km, $R_f = 100$ km, and $T_i = 108,000$ seconds. These numbers are actually quite interesting in cosmology, as the escape velocity from the surface of the white dwarf approaches the speed of light...

Problem 377. problems-1/angular-momentum-pr-disk-collides-with-pivoted-rod.tex



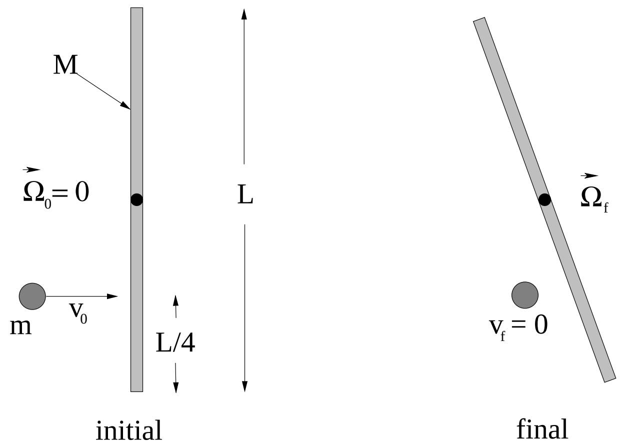
A steel rod of mass M and length L with a frictionless pivot in the center and moment of inertia $\frac{1}{12}ML^2$ sits on a frictionless table at rest. The pivot is attached to the table. A steel disk of mass m approaches with velocity v_0 from the left and strikes the rod a distance $L/4$ from the lower end as shown. This ***elastic*** collision instantly brings the disk to rest and causes the rod to rotate with angular velocity $\vec{\Omega}_f$ (rotating counterclockwise as drawn).

a) What quantities are conserved in this collision?

b) Find the angular velocity $\vec{\Omega}_f$ of the rod about the pivot after the collision (don't forget direction).

c) Find the ratio m/M such that the collision occurs ***elastically***, as described.

Problem 378. problems-1/angular-momentum-pr-disk-collides-with-pivoted-rod-soln.tex



A steel rod of mass M and length L with a frictionless pivot in the center and moment of inertia $\frac{1}{12}ML^2$ sits on a frictionless table at rest. The pivot is attached to the table. A steel disk of mass m approaches with velocity v_0 from the left and strikes the rod a distance $L/4$ from the lower end as shown. This ***elastic*** collision instantly brings the disk to rest and causes the rod to rotate with angular velocity $\vec{\Omega_f}$ (rotating counterclockwise as drawn).

a) What quantities are conserved in this collision?

b) Find the angular velocity $\vec{\Omega_f}$ of the rod about the pivot after the collision (don't forget direction).

c) Find the ratio m/M such that the collision occurs ***elastically***, as described.

Solution: a) **Energy** (because ***the problem states that the collision is elastic!***); **Angular momentum** (because the frictionless pivot exerts no torque); but **Not** linear momentum. The pivot **can** (and in this case obviously does, see below) exert an external impulse force on the rod+disk system during the collision!

b) *Using* angular momentum conservation (with $\vec{\boldsymbol{L}}$ out of the page before and after from the RHR):

$$L = L_i = mv_0\frac{\ell}{4} = \frac{1}{12}M\ell^2\Omega_f = L_f \quad \Rightarrow \quad \boxed{\Omega_f = 3\frac{m}{M}\frac{v_0}{\ell}}$$

c) We need $\Delta K = K_f - K_i = 0$ for the collision to be elastic as ***given***. Using $K_f = L^2/2I_f$:

$$\Delta K = \frac{m^2 v_0^2 \ell^2}{32\frac{1}{12}M\ell^2} - \frac{1}{2}mv_0^2 = \left(\frac{3}{4}\frac{m}{M} - 1\right)\frac{1}{2}mv_0^2$$

This is clearly zero when:

$$\boxed{\frac{m}{M} = \frac{4}{3}}$$

Note: Consider the initial momentum, $\vec{\boldsymbol{p}}_i = mv_0\hat{\boldsymbol{x}} \neq 0$ (with $\hat{\boldsymbol{x}}$ to the right as usual). The *final* momentum is clearly ***zero!*** The bullet is at rest and the center of mass of the rod is not moving as the rod rotates! The change in the momentum of the system is thus:

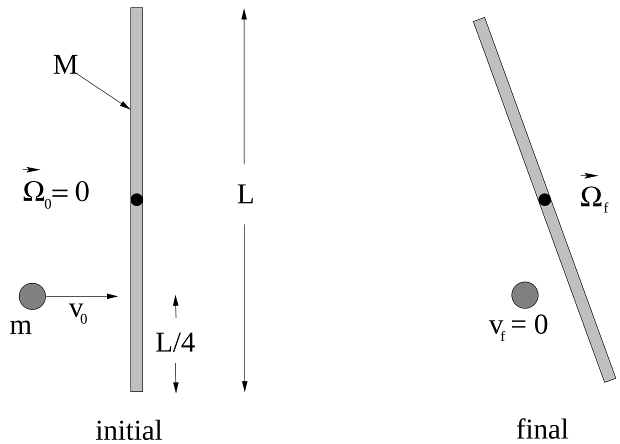
$$\Delta\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}_f - \vec{\boldsymbol{p}}_i = -mv_0\hat{\boldsymbol{x}}$$

If we were given, or could estimate, the time of actual contact in the collision as (say) $\Delta t = t_c$, we could evaluate the ***average force exerted by the pivot during the collision*** as:

$$\vec{\boldsymbol{F}}_{\rm avg} = \frac{\Delta\vec{\boldsymbol{p}}}{\Delta t} = \frac{-mv_0}{t_c}\hat{\boldsymbol{x}} \neq 0$$

Momentum is clearly not conserved, and the average force exerted by the pivot during the collision is *not negligible*.

Problem 379. problems-1/angular-momentum-pr-disk-collides-with-pivoted-rod-soln.tex



A steel rod of mass M and length L with a frictionless pivot in the center and moment of inertia $\frac{1}{12}ML^2$ sits on a frictionless table at rest. The pivot is attached to the table. A steel disk of mass m approaches with velocity v_0 from the left and strikes the rod a distance $L/4$ from the lower end as shown. This ***elastic*** collision instantly brings the disk to rest and causes the rod to rotate with angular velocity $\vec{\Omega_f}$ (rotating counterclockwise as drawn).

a) What quantities are conserved in this collision?

b) Find the angular velocity $\vec{\Omega_f}$ of the rod about the pivot after the collision (don't forget direction).

c) Find the ratio m/M such that the collision occurs ***elastically***, as described.

Solution: a) **Energy** (because ***the problem states that the collision is elastic!***); **Angular momentum** (because the frictionless pivot exerts no torque); but **Not** linear momentum. The pivot **can** (and in this case obviously does, see below) exert an external impulse force on the rod+disk system during the collision!

b) *Using* angular momentum conservation (with $\vec{L}$ out of the page before and after from the RHR):

$$L = L_i = mv_0 \frac{\ell}{4} = \frac{1}{12} M\ell^2 \Omega_f = L_f \quad \Rightarrow \quad \boxed{\Omega_f = 3\frac{m}{M}\frac{v_0}{\ell}}$$

c) We need $\Delta K = K_f - K_i = 0$ for the collision to be elastic as ***given***. Using $K_f = L^2/2I_f$:

$$\Delta K = \frac{m^2 v_0^2 \ell^2}{32 \frac{1}{12} M\ell^2} - \frac{1}{2}mv_0^2 = \left(\frac{3}{4}\frac{m}{M} - 1\right)\frac{1}{2}mv_0^2$$

This is clearly zero when:

$$\boxed{\frac{m}{M} = \frac{4}{3}}$$

Note: Consider the initial momentum, $\vec{\boldsymbol{p}}_i = mv_0\hat{\boldsymbol{x}} \neq 0$ (with $\hat{\boldsymbol{x}}$ to the right as usual). The *final* momentum is clearly ***zero!*** The bullet is at rest and the center of mass of the rod is not moving as the rod rotates! The change in the momentum of the system is thus:

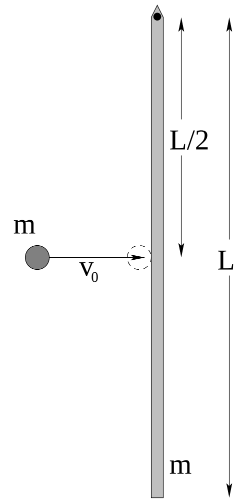
$$\Delta\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}_f - \vec{\boldsymbol{p}}_i = -mv_0\hat{\boldsymbol{x}}$$

If we were given, or could estimate, the time of actual contact in the collision as (say) $\Delta t = t_c$, we could evaluate the ***average force exerted by the pivot during the collision*** as:

$$\vec{\boldsymbol{F}}_{\rm avg} = \frac{\Delta\vec{\boldsymbol{p}}}{\Delta t} = \frac{-mv_0}{t_c}\hat{\boldsymbol{x}} \neq 0$$

Momentum is clearly not conserved, and the average force exerted by the pivot during the collision is *not negligible.*

Problem 380. problems-1/angular-momentum-pr-marble-and-rod.tex



In the figure above, a marble with mass m travelling to the right at speed v_0 collides with a rigid rod of length L pivoted about one end, also of mass m, . The marble strikes the rod $L/2$ down from the pivot and comes **precisely to rest** in the collision. Ignore gravity, drag forces, and any friction in the pivot.

a) What is the ***rotational velocity*** $\boldsymbol{\Omega_f}$ of the rod after the collision?

b) What is the change in ***linear momentum in the x direction*** $\boldsymbol{\Delta p_x}$ (to the right) during this collision?

c) What is the ***change in kinetic energy*** $\boldsymbol{\Delta K}$ in this collision? The sign of your answer should indicate whether energy was gained or lost.

Problem 381. problems-1/angular-momentum-pr-marble-and-rod-soln.tex

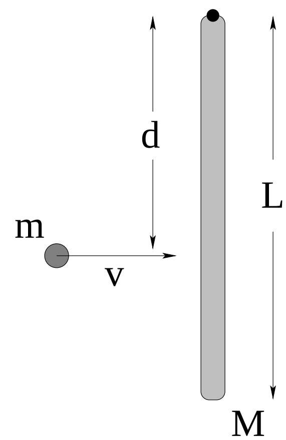
a) Angular momentum is conserved, so $L = L_i = mv_0L/2 = I_{\rm rod}\Omega = \frac{1}{3}mL^2\Omega = L_f$. Hence

$$\Omega = \frac{3mv_0}{2mL} = \boxed{\frac{3v_0}{2L}}$$

b) $p_{xi} = mv_0$ initially. $p_{xf} = mv_{\rm cm} = m\Omega L/2$ finally. So use answer to a) and form $\Delta p_x = p_{xf} - p_{xi}$.

c) Easiest to use $K_i = \frac{1}{2}mv_0^2$, $K_f = \frac{L^2}{2I_{\rm rod}}$, and subtract.

Problem 382. problems-1/angular-momentum-pr-putty-sticks-to-pivoted-rod-gravity.tex



A rod of mass M and length L is hanging vertically from a frictionless pivot (where gravity is "down"). A blob of putty of mass m approaches with velocity v from the left and strikes the rod a distance d from its center of mass as shown, sticking to the rod.

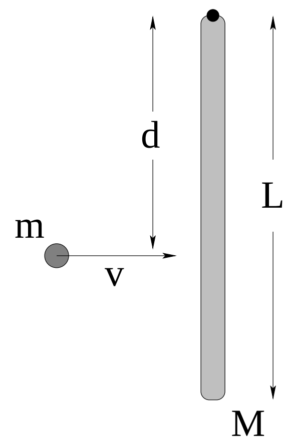
a) Find the angular velocity ω_f of the system about the pivot (at the top of the rod) after the collision.

b) Find the distance $x_{\rm cm}$ from the pivot of the center of mass of the rod-putty system immediately after the collision.

c) After the collision, the rod swings up to a maximum angle $\theta_{\rm max}$ and then comes momentarily to rest. Find $\theta_{\rm max}$.

All answers should be in terms of M, m, L, v, g and d as needed. The moment of inertia of a rod pivoted about one end is $I = \frac{1}{3}ML^2$, in case you need it.

Problem 383. problems-1/angular-momentum-pr-putty-sticks-to-pivoted-rod.tex



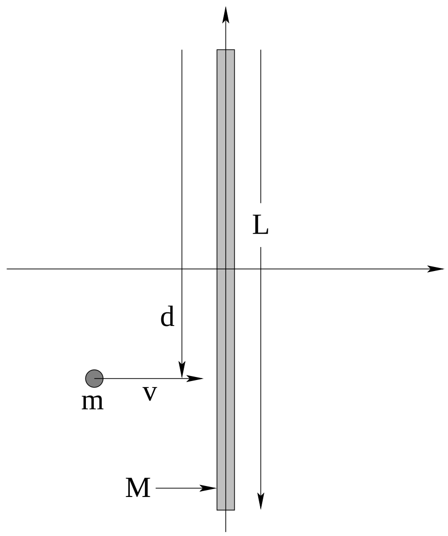
A rod of mass M and length L rests on a frictionless table and is pivoted on a frictionless nail at one end as shown. A blob of putty of mass m approaches with velocity v from the left and strikes the rod a distance d from the end as shown, sticking to the rod.

a) Find the angular velocity ω of the system about the nail after the collision.

b) Is the linear momentum of the rod/blob system conserved in this collision for a general value of d? If not, why not?

c) Is there a value of d for which it *is* conserved? If there were such a value, it would be called the *center of percussion* for the rod for this sort of collision.

Problem 384. problems-1/angular-momentum-pr-putty-sticks-to-unpivoted-rod.tex



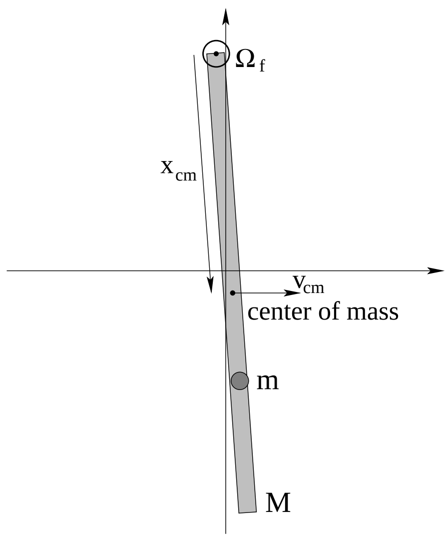
A rod of mass M and length L rests on a ***frictionless table***. A blob of putty of mass m approaches with velocity v from the left and strikes the rod a distance d from the end as shown, ***sticking to the rod***.

a) Find the angular velocity Ω_f of the system after the collision. Note that the rod and putty will be rotating about the center of mass of the ***system***, not the center of mass of the *rod by itself*!

b) Is the linear momentum of the rod/blob system conserved in this collision for a general value of d? If not, why not?

All answers should be in terms of M, m, L, v and d as needed.

Problem 385. problems-1/angular-momentum-pr-putty-sticks-to-unpivoted-rod-soln.tex



A rod of mass M and length L rests on a ***frictionless table***. A blob of putty of mass m approaches with velocity v from the left and strikes the rod a distance d from the ***center of the rod*** as shown, ***sticking to the rod***.

a) Find the angular velocity Ω_f of the system after the collision. Note that the rod and putty will be rotating about the center of mass of the ***system***, not the center of mass of the *rod by itself*!

b) Is the linear momentum of the rod/blob system conserved in this collision for a general value of d? If not, why not?

All answers should be in terms of M, m, L, v and d as needed.

Solution: In this problem ***there are no meaningful external forces acting on the system!*** Gravity is canceled by the (frictionless) normal force of the table. Consequently we expect ***linear momentum to be conserved in the collision***. However, there are also no external *torques* acting, so we expect ***angular momentum to be conserved as well!*** Which one should we use to answer the questions? What ***coordinate system*** should we use to answer the questions?

If we just consider momentum conservation:

$$p_i = mv = (m+M)v_c m = p_f$$

(to the right, say $\hat{x}$). This would make it very easy to find:

$$v_{\rm cm} = \frac{mv}{m+M}$$

as usual, but doesn't help us find Ω_f. It seems that angular momentum conservation is our best bet here. The problem remaining is *choosing a good pivot!*. After the collision, the center of mass will move in a straight line to the right in a predictable way, but every other point in the system will be undergoing somewhat complicated motion *around* the center of mass as it simultaneously moves. It therefore makes sense for us to ***use the center of mass as our pivot*** for conservation of angular momentum. This in turn is made simple by using the center of the rod as the origin of coordinates:

The steps:

$$x_{\rm cm} = \frac{md + M(0)}{m+M} = \frac{md}{m+M} \quad \text{and} \quad d - x_{\rm cm} = \frac{Md}{m+M}$$

(radii of circles of motion of the putty and rod centers of mass around the center of mass of the system) so that:

$$L_i = m(d - x_{\rm cm})v = \left\{ m(d - x_{\rm cm})^2 + \left(\frac{1}{12} M L^2 + M x_{\rm cm}^2 \right) \right\} \Omega_f$$

$$\frac{mM}{m+M} vd = \left\{ \frac{M^2}{(m+M)^2} m d^2 + \frac{1}{12} M L^2 \right\} \Omega_f = \frac{mM}{m+M} d^2 \left\{ \frac{M}{m+M} + \frac{1}{12} \frac{m+M}{m} \frac{L^2}{d^2} \right\} \Omega_f$$

Note that we used the parallel axis theorem to find the moment of inertia of the rod rotating around the new center of mass. Now we just solve for:

$$\boxed{\Omega_f = \frac{1}{\left\{ \frac{M}{m+M} + \frac{1}{12} \frac{m+M}{m} \frac{L^2}{d^2} \right\}} \left(\frac{v}{d} \right)}$$

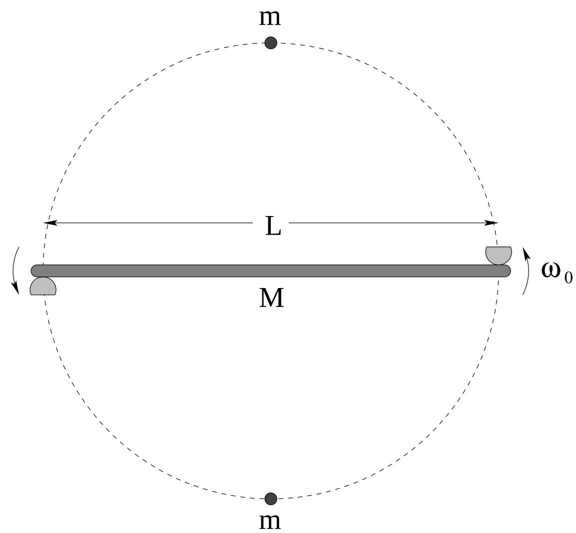
where the final direction is (obviously, RHR) out of the page.

This can be simplified to:

$$\boxed{\Omega_f = \frac{12m(m+M)d^2}{12mMd^2 + (m+M)^2 L^2} \left(\frac{v}{d} \right)}$$

which obviously has the correct dimensions as the entire fraction on the left is dimensionless. If $M \gg m$, $\Omega_f \to 0$ as we might expect as well. It could be wrong – a lot of algebra in there, and I make algebra errors as easily as the next person – but *it isn't crazy!*

Problem 386. problems-1/angular-momentum-pr-spinning-cups-catch-balls.tex

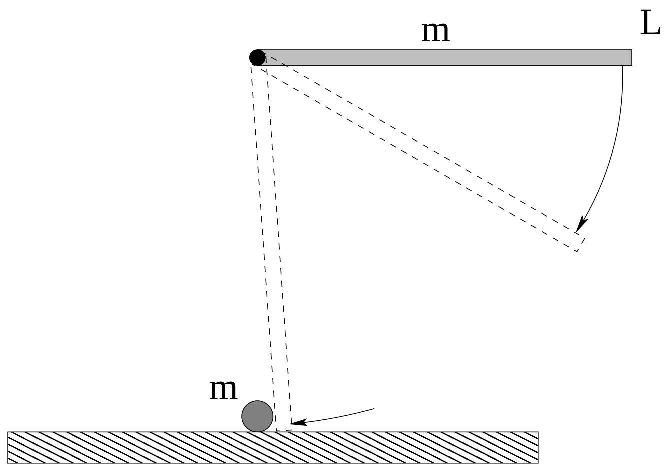


In the figure above, a bar of length L with two cups at the ends is freely rotating (in space – ignore gravity and friction or drag forces) about its center of mass with angular velocity ω_0. The bar and cups together have a mass M and a moment of intertia of $I = \beta M L^2$. When the bar reaches the vertical position, the cups catch two small balls of mass m that are at rest, which stick in the cups. The balls have a negligible moment of inertia about their own center of mass – you may think of them as particles.

a) What is the velocity of the center of mass of the system after the collision?

b) What is the angular velocity of the bar after it has caught the two balls in its cups? Is kinetic energy gained or lost in this process?

Problem 387. problems-1/angular-momentum-pr-swinging-rod-strikes-putty.tex



A uniform rod of mass m and length L swings about a frictionless peg through its ***end***. The rod is held horizontally and released from rest as shown in the figure. At the bottom of its swing the rod strikes a ball of putty of mass m that sits at rest on a frictionless table. In answering the questions take the magnitude of acceleration due to gravity to be g and assume that gravity acts downward (in the usual way). The questions below should be answered in terms of the given quantities.

a) What is the angular speed Ω_i of the rod just before it hits the putty?

b) If the putty ***sticks to the rod***, what is the angular speed Ω_f of the rod-putty system immediately after the collision?

c) What is ΔE, the mechanical energy change of the system in this collision (be sure to specify its sign).

Problem 388. problems-1/angular-momentum-pr-swinging-rod-strikes-putty-soln.tex

a) Energy conservation, with $I = \frac{1}{3}mL^2$:

$$mgL/2 = \frac{1}{2}I\Omega_i^2 \rightarrow \Omega_i = \sqrt{\frac{3g}{L}}$$

b) Angular momentum conservation:

$$L = L_i = I\Omega_i = \frac{1}{3}mL^2\sqrt{3g}L = \frac{4}{3}mL^2\Omega_f = L_f$$

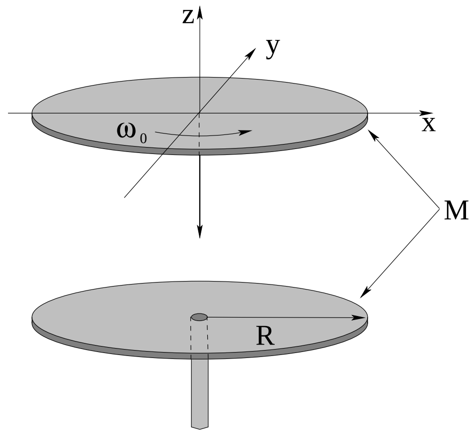
so

$$\Omega_f = \frac{1}{4}\Omega_i = \frac{1}{4}\sqrt{\frac{3g}{L}}$$

c) Subtract initial energy from energy ***after*** collision:

$$\Delta E = \frac{L^2}{2I_f} - \frac{mgL}{2} = -\frac{3mgL}{8}$$

Problem 389. problems-1/angular-momentum-pr-two-circular-plates-collide.tex



A **disk** of mass M and radius R sits at rest on a turntable that permits it to rotate freely. A second identical disk, this one rotating around their mutual axis at an angular speed ω_0, is dropped gently onto it so that (after sliding for an instant) they rotate together. In terms of the givens M, R, ω_0 and known constants:

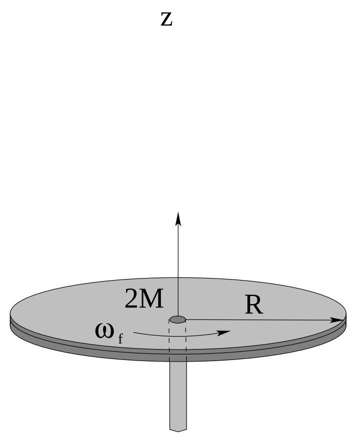
a) Find the ***final*** angular speed ω_f of the two disks moving together ***after*** the collision:

$\omega_f =$ $\boxed{}$

b) What ***fraction*** of the original kinetic energy of the system K_0 is gained (+) or lost (-) in this rotational collision?

$\Delta K =$ $\boxed{}$ $\times K_0$

Problem 390. problems-1/angular-momentum-pr-two-circular-plates-collide-soln.tex

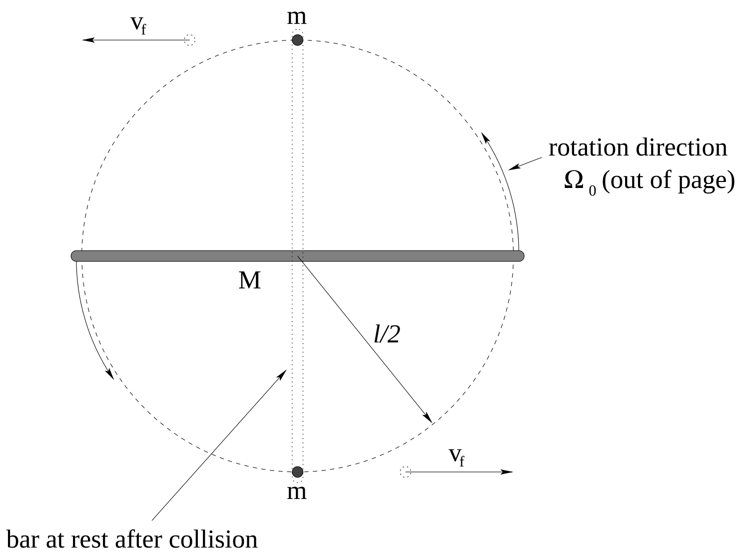


This is a fully inelastic rotational collision. There are no external torques about the axis of rotation, so angular momentum in this direction (relative to a pivot at the orgin of the bottom disk say) is conserved. In this (z) direction:

a)
$$L = L_i = I\omega_0 = 2I\omega_f = L_f$$
where $I = \frac{1}{2}MR^2$ for both disks. Hence:
$$\boxed{\omega_f = \omega_0/2}$$

b) There are many ways to get this, but the easiest (since angular momentum is conserved) is to write:
$$K_0 = \frac{L^2}{2I} \qquad K_f = \frac{L^2}{2(2I)} = \frac{K_0}{2}$$
or
$$\boxed{\Delta K = K_f - K_0 = -\frac{1}{2}} \times K_0$$
and energy is $\boxed{\textbf{\textit{lost}}}$ in the collision, as expected.

Problem 391. problems-1/angular-momentum-pr-rotating-bar-elastic-collision-balls.tex

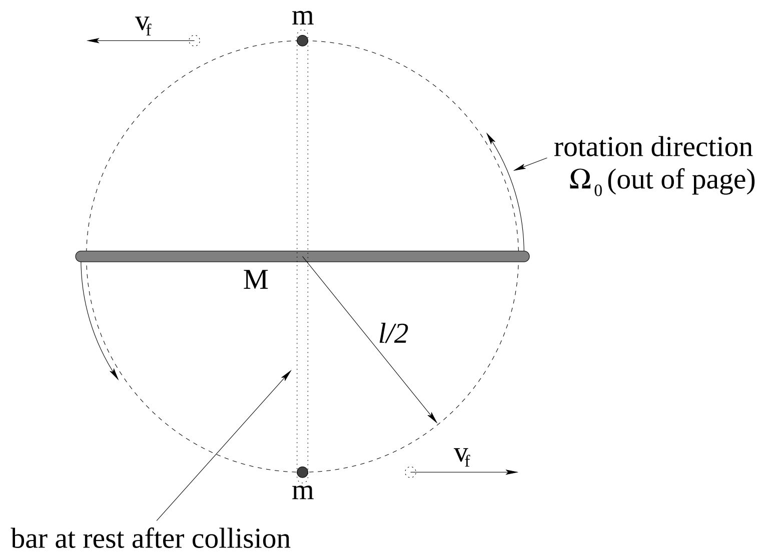


In the figure above, a unpivoted solid rod of length ℓ and mass M is rotating around its center of mass with an angular velocity Ω_0 out of the page. It simultaneously strikes two hard balls of mass m sitting at rest a distance $\ell/2$ from the center of rotation as shown, causing them to recoil to the left and right respectively. After the collision ***the rod is at rest***.

a) Is momentum conserved in this collision? b) Find the final ***speed*** of either ball, v_f.

c) Find the ratio of masses m/M such that the collision as described is ***elastic***.

Problem 392. problems-1/angular-momentum-pr-rotating-bar-elastic-collision-balls-soln.tex



In the figure above, a unpivoted solid rod of length ℓ and mass M is rotating around its center of mass with an angular velocity Ω_0 out of the page. It simultaneously strikes two hard balls of mass m sitting at rest a distance $\ell/2$ from the center of rotation as shown, causing them to recoil to the left and right respectively. After the collision ***the rod is at rest***.

a) Is momentum conserved in this collision? b) Find the final ***speed*** of either ball, v_f.

c) Find the ratio of masses m/M such that the collision as described is ***elastic***.

Solution: a) As it happens, the answer is $\boxed{\textbf{yes}}$, momentum is conserved. Before the center of mass is at rest, and afterwards (from symmetry) it is *still* at rest. But this doesn't really help us solve the problem.

b) To find the speed of the balls, we need to use conservation of ***angular*** momentum.

$$L = L_{i,rod} + L_{i,balls} = \frac{1}{12}M\ell^2\Omega_0 \ (+\ 0) = (0\ +)\ \cancel{2}\left(m\frac{\ell}{\cancel{2}}v_f\right) = L_{f,rod} + L_{f,balls}$$

(out of the page, RHR) using $L = mv_f r_\perp = mv_f\ell/2$ for the magnitude of the angular momentum of the two balls, each, as well as $I_{rod} = \frac{1}{12}M\ell^2$ for the moment of inertia of a rod pivoted in the middle. Thus:

$$\boxed{v_f = \frac{M}{12m}\Omega_0\ell}$$

c) We can use $K = L^2/2I$ for both initial and final kinetic energies (with $I_f = 2 \times m(\ell/2)^2 = m\ell^2/2$ for the two balls after the collision), and take their ratio:

$$\frac{K_f}{K_i} = \frac{L^2/2I_f}{L^2/2I_i} = \frac{I_i}{I_f} = \frac{\frac{1}{2}m\ell^2}{\frac{1}{12}M\ell^2} = \frac{6m}{M} = 1$$

or

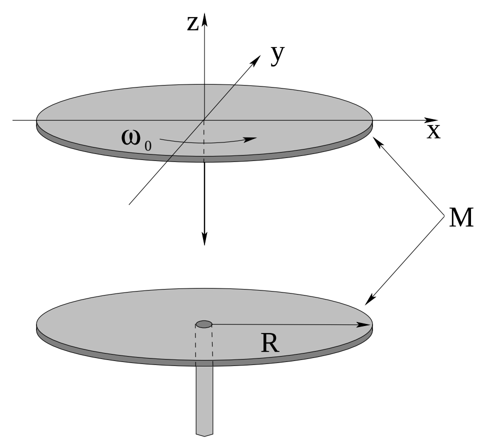
$$\boxed{\frac{m}{M} = \frac{1}{6}}$$

for the collision to be elastic.

8.2 Vector Torque and Precession

8.2.1 Multiple Choice Problems

Problem 393. problems-1/torque-vector-mc-rotational-collision-two-disks.tex



Two identical disks with mass M and radius R have a common axis and frictionless bearing. Initially, one disk is spinning with some angular velocity ω_0 and the other is rest. The two disks are brought together quickly so that they stick and rotate as one without the application of any external torque. Circle the true statement below:

a) The total kinetic energy and the total angular momentum are unchanged.

b) The total kinetic energy and total angular momentum are both reduced to half their original values.

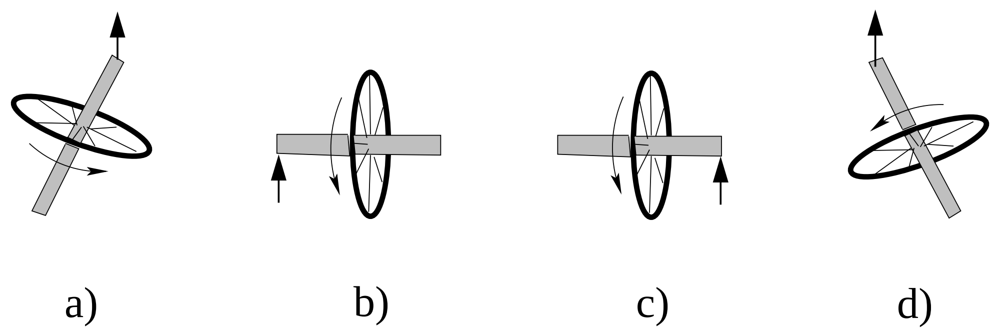
c) The total kinetic energy is unchanged, but the total angular momentum is reduced to half of its original value.

d) The total angular momentum is unchanged, but the total kinetic energy is reduced to half of its original value.

e) We cannot tell what happens to the angular momentum and kinetic energy from the information given.

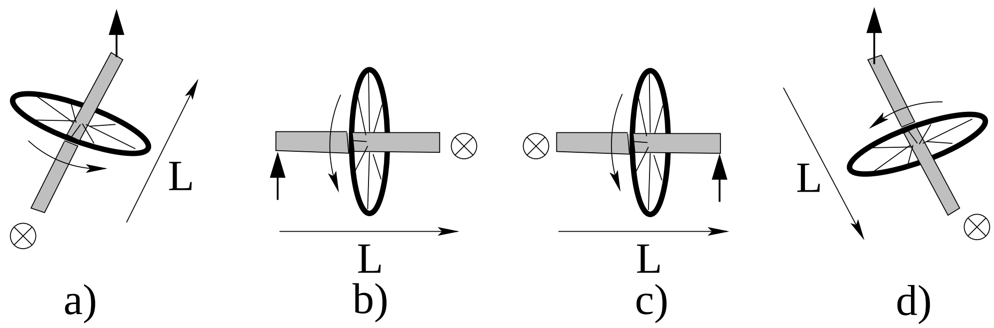
8.2.2 Short Answer Problems

Problem 394. problems-1/torque-vector-sa-direction-of-precession-1.tex

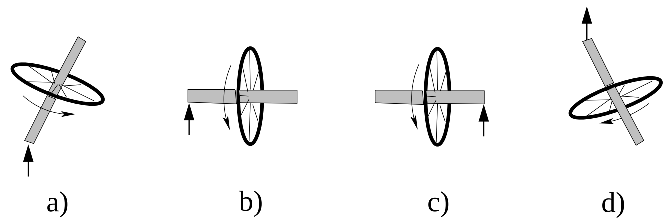


In the figure above four symmetric gyroscopes are portrayed. Each gyroscope is spinning very rapidly in the direction shown, and is suspended/pivoted from one end as shown at the big arrow (gravity points ***down***). For each figure a-d indicate whether the gyroscope will precess ***in*** or ***out*** of the page at the *other* (non-pivoted, free) *end* at the instant shown.

Problem 395. problems-1/torque-vector-sa-direction-of-precession-1-soln.tex

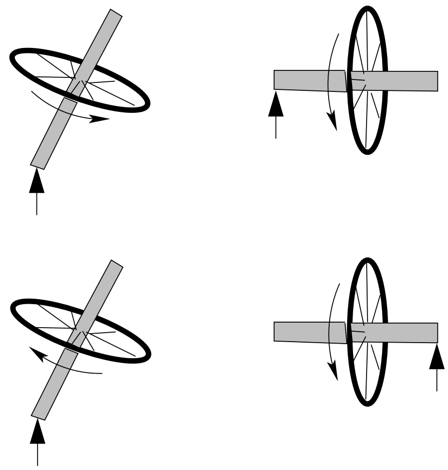


Problem 396. problems-1/torque-vector-sa-direction-of-precession-2.tex



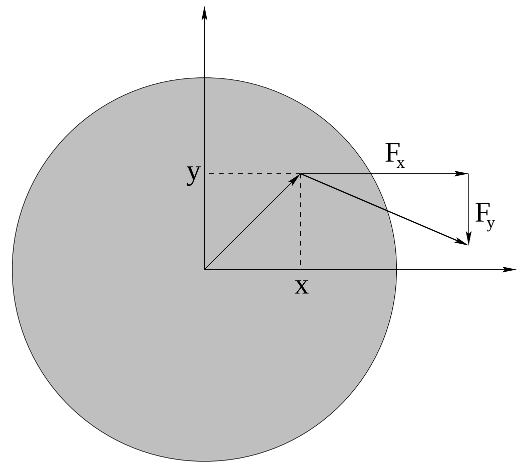
In the figure above four symmetric gyroscopes are portrayed. Each gyroscope is spinning very rapidly in the direction shown, and is suspended/pivoted from one end as shown at the big arrow (gravity points ***down***). For each figure a-d indicate whether the gyroscope will precess ***in*** or ***out*** of the page at the *other* (non-pivoted, free) *end* at the instant shown.

Problem 397. problems-1/torque-vector-sa-direction-of-precession.tex



In the figure above four symmetric gyroscopes are portrayed. Each gyroscope is spinning very rapidly in the direction shown, and is suspended from one end as shown (at the big arrow). For each figure indicate whether the gyroscope will precess **in** or **out** of the page at the *other* (free) *end* at the instant shown.

Problem 398. problems-1/torque-vector-sa-evaluate-the-torque-1.tex



In the figure above, a force

$$\vec{\boldsymbol{F}} = 2\hat{\boldsymbol{x}} - 1\hat{\boldsymbol{y}}$$

Newtons is applied to a disk at the point

$$\vec{\boldsymbol{r}} = 2\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}}$$

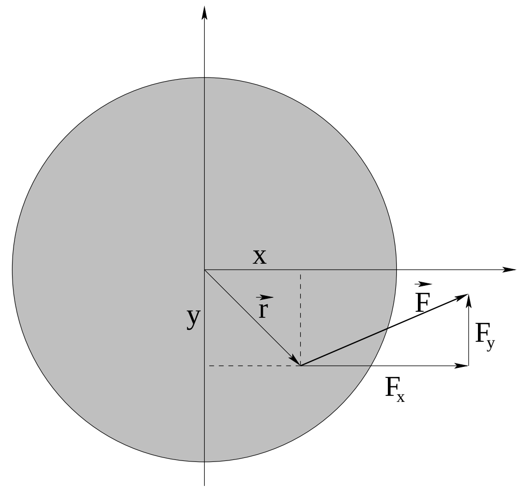
as shown. (That is, $F_x = 2$ N, $F_y = -1$ N, $x = 2$ m, $y = 2$ m). Find the *total torque* about a pivot *at the origin.*

Don't forget that torque is a vector, so either give the answer in cartesian coordinates or otherwise specify its direction!

Problem 399. problems-1/torque-vector-sa-evaluate-the-torque-1-soln.tex

$$\begin{aligned}
\vec{\boldsymbol{\tau}} &= \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}} \\
&= (r_x F_y - r_y F_x)\hat{\boldsymbol{z}} \\
&= (2 * (-1) - 2 * 2)\hat{\boldsymbol{z}} \\
&= -6\hat{\boldsymbol{z}} \quad \mathrm{N-m}
\end{aligned}$$

Problem 400. problems-1/torque-vector-sa-evaluate-the-torque-2.tex



In the figure above, a force

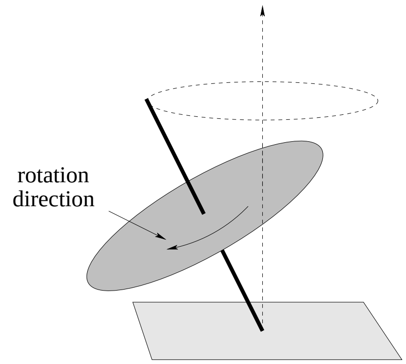
$$\vec{\boldsymbol{F}} = 2\hat{\boldsymbol{x}} + 1\hat{\boldsymbol{y}}$$

Newtons is applied to a disk at the point

$$\vec{\boldsymbol{r}} = 2\hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}}$$

as shown. (That is, $F_x = 2$ N, $F_y = 1$ N, $x = 2$ m, $y = -2$ m). Find the *total torque* about a pivot *at the origin*. Don't forget that torque is a ***vector***, so specify its direction as well as its magnitude (or give the answer as a cartesian vector)! Show your work!

Problem 401. problems-1/torque-vector-sa-precession-of-top-1.tex

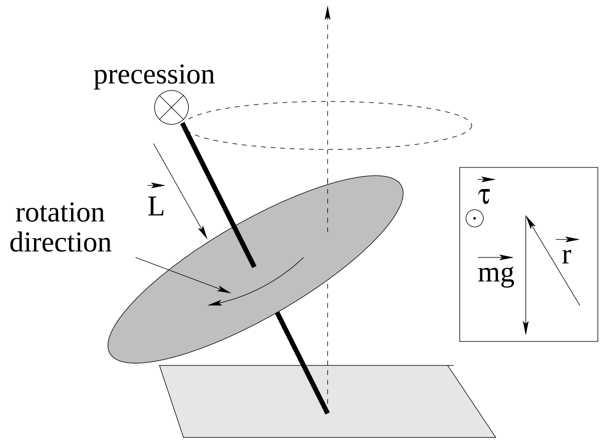


Draw the direction of $\vec{L}$ onto the spinning top in the figure above, and ***circle*** the direction that the ***upper tip of the top*** will precess:

in **out**

of the page. **Draw this direction onto the figure as well.**

Problem 402. problems-1/torque-vector-sa-precession-of-top-1-soln.tex

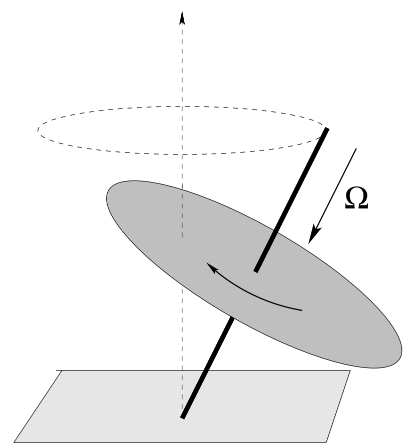


Draw the direction of $\vec{L}$ onto the spinning top in the figure above, and ***circle*** the direction that the ***upper tip of the top*** will precess:

$\boxed{\textbf{in}}$ **out**

Solution: Curl fingers of the right hand (mentally) around the axis of rotation in the direction of rotation; the right thumb points ***down and to the right*** along the axis as the direction of $\vec{L}$ as drawn. Again, the RHR indicates that the gravitational torque on the top is ***out of the page***. In order for $\Delta\vec{L}$ to point out of the page, the upper tip of the top has to go ***the other way*** and precess ***into the page***.

Problem 403. problems-1/torque-vector-sa-precession-of-top-2.tex



A top tipped at some angle θ is spinning with an ***angular velocity*** directed ***towards the point where it rests on the ground, as shown***.

The torque due to gravity about a pivot at the point where the top rests on the ground is:

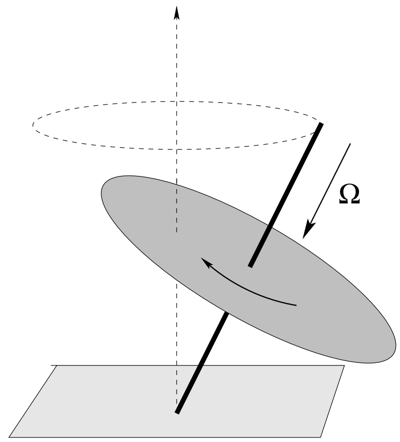
☐ Into the page ☐ Out of the page

The top will precess:

☐ Into the page ☐ Out of the page

at the instant drawn above.

Problem 404. problems-1/torque-vector-sa-precession-of-top-2-soln.tex



A top tipped at some angle θ is spinning with an ***angular velocity*** directed ***towards the point where it rests on the ground, as shown***.

The torque due to gravity about a pivot at the point where the top rests on the ground is:

☒ Into the page ☐ Out of the page

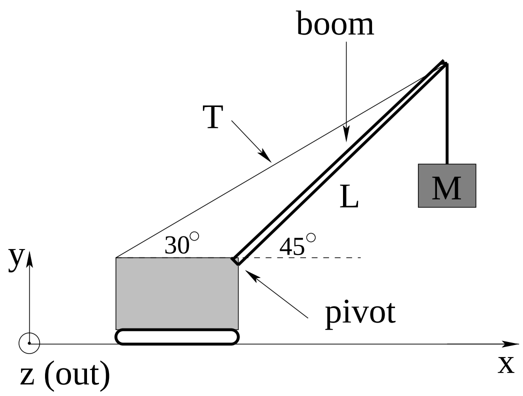
The top will precess:

☐ Into the page ☒ Out of the page

at the instant drawn above.

8.2.3 Regular Problems

Problem 405. problems-1/torque-vector-pr-crane-boom.tex



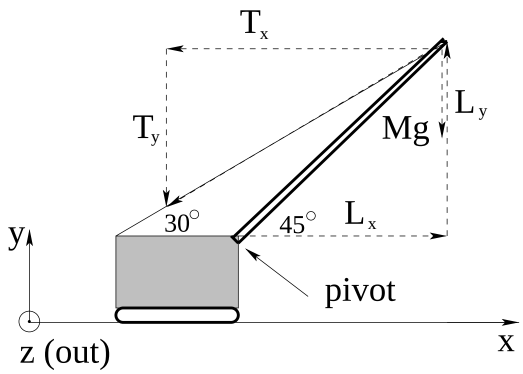
$$\sin(30^\circ) = \cos(60^\circ) = \frac{1}{2}$$
$$\cos(30^\circ) = \sin(60^\circ) = \frac{\sqrt{3}}{2}$$
$$\sin(45^\circ) = \cos(45^\circ) = \frac{\sqrt{2}}{2}$$

A crane with a "massless" boom (the long support between the body and the load) of length L holds a mass M suspended as shown. Note that the wire with the tension T is **fixed** to the top of the boom, not run over a pulley to the mass M.

a) Find the torque (magnitude and direction) exerted by the tension in the wire on the boom, relative to a pivot at the base of the boom.

b) Find the torque (magnitude and direction) exerted by the hanging mass, relative to a pivot at the base of the boom.

Problem 406. problems-1/torque-vector-pr-crane-boom-soln.tex



$$\sin(30°) = \cos(60°) = \frac{1}{2}$$
$$\cos(30°) = \sin(60°) = \frac{\sqrt{3}}{2}$$
$$\sin(45°) = \cos(45°) = \frac{\sqrt{2}}{2}$$

This is ***by far*** easiest to do using the cartesian form for the torques. Note that $L_x = L_y = L\sqrt{2}/2$. $T_x = -\sqrt{3}/2T$. $T_y = -T/2$. The weight pulls down with a force $F_y = -Mg$, all as drawn above. No fair using calculators!

a) Find the torque (magnitude and direction) exerted by the tension in the wire on the boom, relative to a pivot at the base of the boom.

$$\tau_{z,T} = L_xT_y - L_yT_x = LT\left(-\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\right) = LT\frac{\sqrt{2}(\sqrt{3}-1)}{4}$$

If you want to cause yourself more pain, you can instead try to figure out $\sin(15°) = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$ using all three triangles and the law of sines... it takes me about ten minutes and some careful pictures and reasoning.

It's a bit faster if you use the following trig identity (one that you probably don't remember but that is fairly easy to derive using complex exponentials):

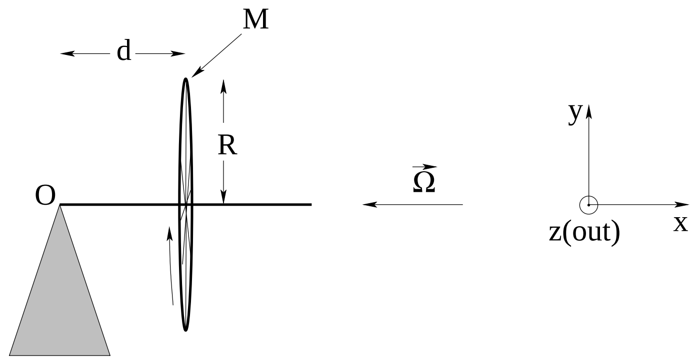
$$\sin(A + B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$

Let $A = 45°$ and $B = -30°$ and there you have it in one step (don't forget, the sine function is odd). Using the Cartesian form is really the simplest approach though, and is a useful thing to remember.

b) Find the torque (magnitude and direction) exerted by the hanging mass, relative to a pivot at the base of the boom. Here the easy way and cartesian form are identical:

$$\tau_{z,Mg} = L_xF_y = -MgL\frac{\sqrt{2}}{2}$$

Problem 407. problems-1/torque-vector-pr-precessing-bicycle-wheel.tex



A bicycle wheel (basically a ring) of mass M and radius R has massless spokes and a massless axle of length d. The other end O of the axle rests on a conical support as shown. The axle is held in a horizontal position and the wheel is spun with a large angular velocity $\vec{\Omega}$ that points ***towards O***, and then released so that the wheel precesses about O.

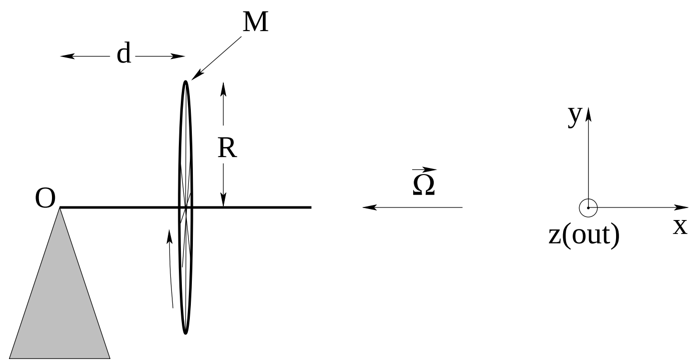
(Note: To specify the direction of vectors you may use *up*, *down*, *towards O*, *away from O*, *into the page*, *out of the page* as shown.)

a) What is the angular momentum $\vec{L}$ of the wheel about its center of mass?

b) What is (find, with any argument) the ***angular frequency of precession*** $\vec{\omega}_p$ of the wheel? Don't forget to give the direction!

c) What is the kinetic energy K of the wheel in the frame of O (i.e., the lab frame) ***including the contribution from the motion of the center of mass as it precesses!*** This is one of the factors we ignored in our elementary treatment in class.

Problem 408. problems-1/torque-vector-pr-precessing-bicycle-wheel-soln.tex



A bicycle wheel (basically a ring) of mass M and radius R has massless spokes and a massless axle of length d. The other end O of the axle rests on a conical support as shown. The axle is held in a horizontal position and the wheel is spun with a large angular velocity $\vec{\Omega}$ that points ***towards O***, and then released so that the wheel precesses about O.

(Note: To specify the direction of vectors you may use *up*, *down*, *towards O*, *away from O*, *into the page*, *out of the page* as shown.)

a) What is the angular momentum $\vec{L}$ of the wheel about its center of mass?

b) What is (find, with any argument) the ***angular frequency of precession*** $\vec{\Omega}_p$ of the wheel? Don't forget to give the direction!

c) What is the kinetic energy K of the wheel in the frame of O (i.e., the lab frame) ***including the contribution from the motion of the center of mass as it precesses!*** This is one of the factors we ignored in our elementary treatment in class.

Solution: a) is simple:

$$\vec{\boldsymbol{L}} = I\vec{\boldsymbol{\Omega}} = -MR^2\Omega\,\hat{\boldsymbol{x}} \text{ (or } MR^2\Omega \text{ "to the left" or "in the direction of } \vec{\boldsymbol{\Omega}}\text{" etc.)}$$

There are three ways to show b). All start by noting that:

$$\tau = Mgd \text{ (in)}$$

at the position drawn.

The worst (but adequate) way is to note that in one period of precession T_p, the angular momentum sweeps out a circle with (angular momentum radius) $L = MR^2\Omega$, so that:

$$\tau = Mgd = \frac{\Delta L}{\Delta t} = \frac{2\pi MR^2\Omega}{T_p} \quad \Rightarrow \quad \boxed{\Omega_p = \frac{2\pi}{T_p} = \frac{gd}{R^2\Omega}}$$

A slightly better treatment considers only the differential precession in a very short time Δt. In that time, the angular momentum will change only by:

$$\Delta L = L\Delta\phi$$

where $\Delta\phi$ is the small angle of the change in direction (but not magnitude) of the angular momentum vector. Then:

$$\tau = Mgd = \frac{\Delta L}{\Delta t} = \frac{MR^2\Omega\Delta\phi}{\Delta t} = MR^2\Omega\Omega_p \quad \Rightarrow \quad \boxed{\Omega_p = \frac{2\pi}{T_p} = \frac{gd}{R^2\Omega}}$$

The final, best way to derive it uses the full vector form:

$$\vec{\boldsymbol{\tau}} = \frac{d\vec{\boldsymbol{L}}}{dt}\vec{\boldsymbol{d}} \times (-mg)\hat{\boldsymbol{z}}$$

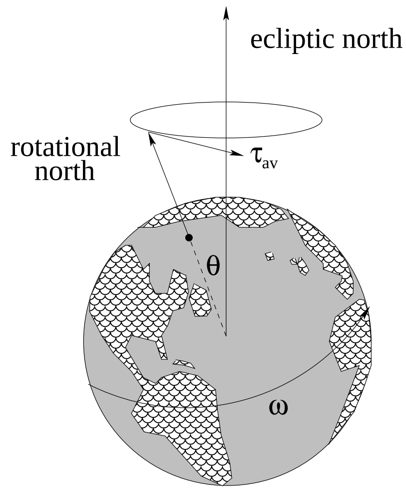
and is beyond the scope of the course at this point (as we have not yet covered the simple harmonic oscillator equation) but if a student uses it and gets the right magnitude and direction of precession, so much the better.

In all cases, the free end will precess out of the page, opposite to the direction of the torque!

For c), we have to add the "orbital" angular energy of the wheel (neglecting axle etc) to the "spin" angular energy. Using the form $K = \frac{1}{2}I\Omega^2$ for both:

$$K_{\text{tot}} = \frac{1}{2}(MR^2)\Omega^2 + \frac{1}{2}(Md^2)\Omega_p^2$$

Problem 409. problems-1/torque-vector-pr-precession-of-equinoxes.tex



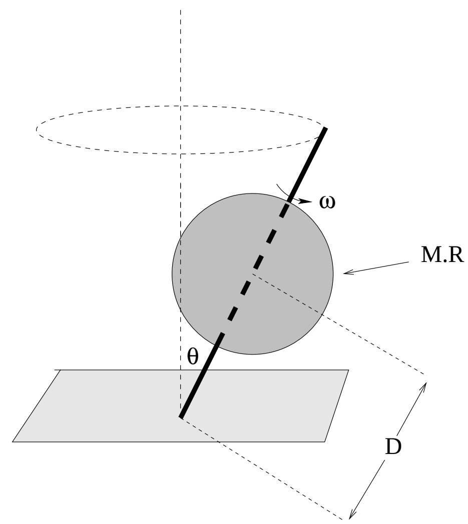
The Earth revolves on its axis. Its north (right handed) axis is significantly tipped relative to the "ecliptic" pole of the Earth's revolution around the Sun. It is currently aligned with Polaris, the pole star, but because the Sun exerts a small *torque* on it due to tides acting on its slightly oblate spheroidal shape, it also *precesses* around the ecliptic north once every (approximately) 26,000 years!

a) Assuming that the average torque on the earth over the course of any given year remains perpendicular to its angular momentum in the direction/handedness shown, derive an algebraic expression for the angular frequency of precession in terms of the magnitude of the torque. You may use I as the moment of inertia for the earth about its rotational axis as that quantity is given below.

b) Given the data that the moment of inertia of the Earth about its axis of rotation is roughly 8×10^{37} kg-m^2, that its axis is tipped at roughly 20 degrees relative to the ecliptic and that its period of revolution about its own axis is one day, estimate the approximate magnitude of the average torque exerted by the Sun on the Earth over the course of a year.

(You may find it useful to know that 1 day = 86400 seconds, and 1 year = 3.15×10^7 seconds – you can remember the latter as approximately $\pi \times 10^7$ seconds.)

Problem 410. problems-1/torque-vector-pr-precession-of-spherical-top.tex



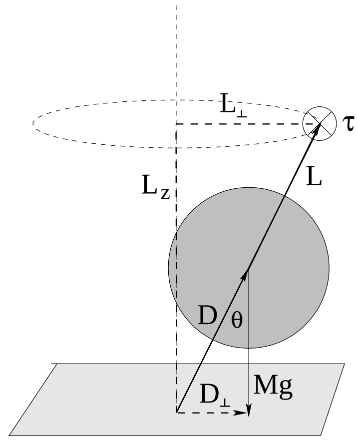
A top is made from a ball of radius R and mass M with a very thin, light nail ($r \ll R$ and $m \ll M$) for a spindle so that the center of tha ball is a distance D from the tip. The top is spun with a large angular velocity Ω, and has a moment of inertia $I = \frac{2}{5}MR^2$.

a) What is the angular momentum of the spinning ball? Indicate its (vector) direction with an arrow on the figure.

b) When the top is spinning at a small angle θ with the vertical (as shown) what is the angular speed Ω_p of the top's precession?

c) Does the top precess ***into*** or ***out of*** the page at the instant shown?

Problem 411. problems-1/torque-vector-pr-precession-of-spherical-top-soln.tex



a) Magnitude:

$$L = I\Omega = \frac{2}{5}MR^2\Omega$$

b) When the top is spinning at a small angle θ with the vertical (as shown) what is the angular frequency Ω_p of the top's precession?

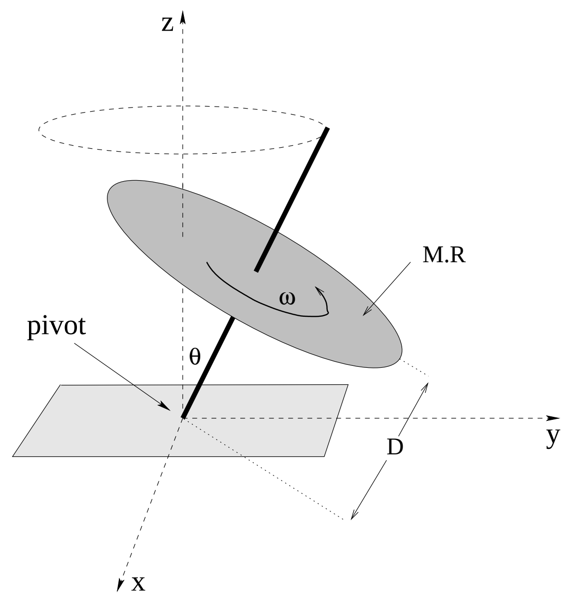
$$\tau = |\vec{\boldsymbol{D}} \times (-Mg)\hat{\boldsymbol{z}}| = MgD_\perp = MgD\sin(\theta) \quad \text{(in)} \quad = L_\perp\Omega_p = L\sin(\theta)\Omega_p$$

or

$$\Omega_p = \frac{MgD}{L} = \frac{5gD}{2R^2\Omega}$$

c) In (in direction of torque, given direction of $\vec{\boldsymbol{L}}$).

Problem 412. problems-1/torque-vector-pr-precession-of-top-3-parts.tex



A top is made of a uniform ***disk*** of radius R and mass M with a very thin, light (assume massless) nail for a spindle so that the center of the disk is a distance D from the tip. The top is spun with a large angular velocity ω with the nail vertically above the y-axis as shown above.

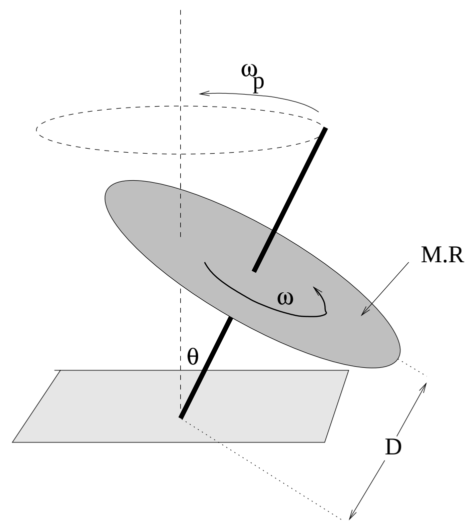
a) Find the ***vector*** torque $\vec{\tau}$ exerted about the pivot at the instant shown in the figure. You may express the vector however you wish (e.g. magnitude and direction, cartesian components).

b) What is the axis of precession?

c) Derive the precession frequency ω_p. Any of the derivations used in class or discussed in the textbook are acceptable.

Express all answers in terms of M, R, g, D, and θ as needed.

Problem 413. problems-1/torque-vector-pr-precession-of-top.tex



This problem will help you learn required concepts such as:

- Vector Torque
- Vector Angular Momentum
- Geometry of Precession

so please review them before you begin.

A top is made of a disk of radius R and mass M with a very thin, light nail ($r \ll R$ and $m \ll M$) for a spindle so that the disk is a distance D from the tip. The top is spun with a large angular velocity ω. When the top is spinning at a small angle θ with the vertical (as shown) what is the angular frequency ω_p of the top's precession?

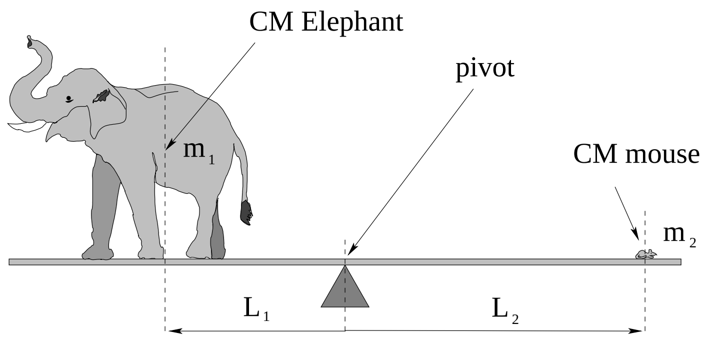
Chapter 9

Static Equilibrium

9.1 Static Equilibrium

9.1.1 Multiple Choice Problems

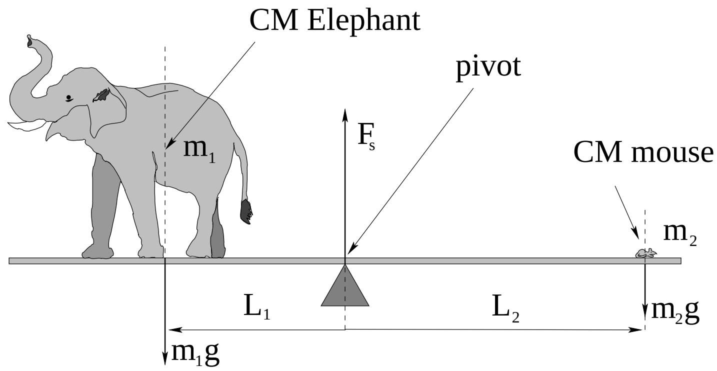
Problem 414. problems-1/statics-mc-elephant-mouse.tex



An elephant and a mouse sit at either end of a really long, really strong see-saw. The elephant, whose mass is m_1, sits so that its center of mass is a distance L_1 from the pivot. The mouse, whose mass is m_2, sits at L_2. The see-saw is ***balanced*** so the mouse and elephant are not moving up or down. Which is the following must be true:

- ☐ $m_2 = m_1$
- ☐ $m_2 = m_1(L_2/L_1)$
- ☐ $m_1 = m_2(L_2/L_1)$
- ☐ $m_1 = m_2(L_2/L_1)^2$
- ☐ The mouse can never balance the elephant!

Problem 415. problems-1/statics-mc-elephant-mouse-soln.tex



An elephant and a mouse sit at either end of a really long, really strong see-saw. The elephant, whose mass is m_1, sits so that its center of mass is a distance L_1 from the pivot. The mouse, whose mass is m_2, sits at L_2. The see-saw is ***balanced*** so the mouse and elephant are not moving up or down. Which is the following must be true:

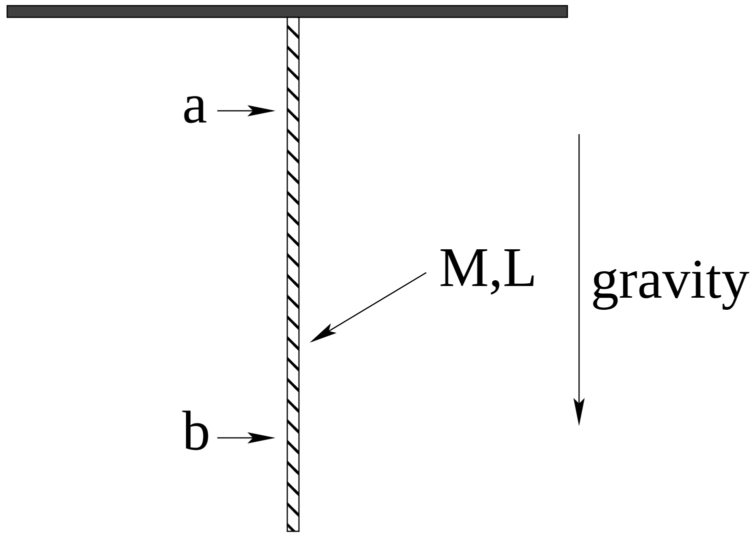
- ☐ $m_2 = m_1$
- ☐ $m_2 = m_1(L_2/L_1)$
- ☒ $m_1 = m_2(L_2/L_1)$
- ☐ $m_1 = m_2(L_2/L_1)^2$
- ☐ The mouse can never balance the elephant!

Solution: All we need do is consider the torque around the labelled pivot. Algebraically:

$$\tau_{out} = m_1\cancel{g}L_1 - m_2\cancel{g}L_2 = 0 \qquad \Rightarrow \qquad \boxed{m_2 = m_1\frac{L_1}{L_2} \qquad \Leftrightarrow \qquad m_1 = m_2\frac{L_2}{L_1}}$$

All that is left is to search the provided answers for the match, shown above.

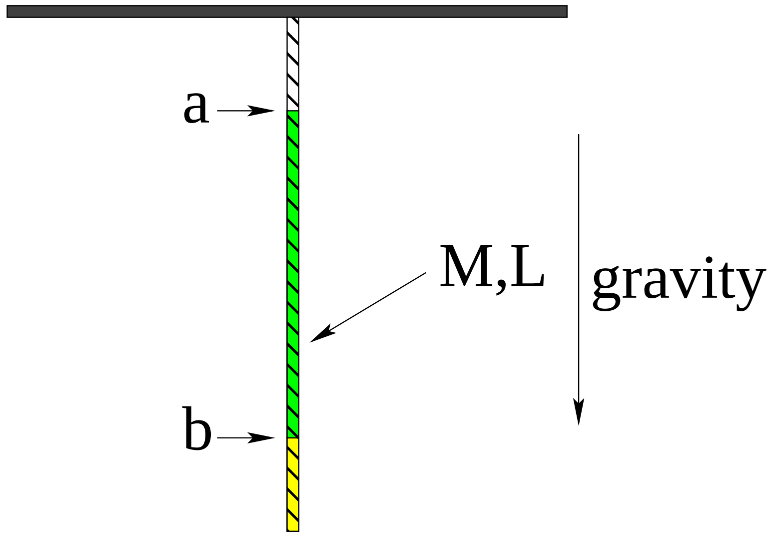
Problem 416. problems-1/statics-mc-hanging-rope.tex



In the figure above, a rope of mass M, length L is hanging from the ceiling in static equilibrium. Select the correct rank order of the tension in the rope at the points a and b:

- ☐ $T_a < T_b$
- ☐ $T_a > T_b$
- ☐ $T_a = T_b$
- ☐ Insufficient information given to determine the answer.

Problem 417. problems-1/statics-mc-hanging-rope-soln.tex



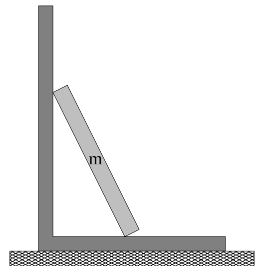
In the figure above, a rope of mass M, length L is hanging from the ceiling in static equilibrium. Select the correct rank order of the tension in the rope at the points a and b:

- ☐ $T_a < T_b$
- ☒ $T_a > T_b$
- ☐ $T_a = T_b$
- ☐ Insufficient information given to determine the answer.

Solution: The tension at point b only has to support the weight of the lower segment (colored yellow above) of rope. The tension at point a has to support ***both*** the yellow ***and*** the green colored segments – which is simply more rope! Hence

$$\boxed{T_a > T_b}$$

Problem 418. problems-1/statics-mc-leaning-bar-reaction-pairs.tex



In the figure above, a board is sitting on a rough floor and leaning against a wall. Circle **three** action-reaction pairs in the list below:

a) The ladder top pushes against the wall; the wall pushes back against the ladder top.

b) The floor pushes up on the ladder base; gravity pulls the ladder base down towards the floor.

c) Static friction from the floor pushes the ladder base towards the wall; the wall pushes back on the ladder.

d) The floor pushes down on the ground; the ground pushes back on the floor.

e) The Earth pulls down on the ladder via gravity; the ladder pulls up on the Earth via gravity.

Problem 419. problems-1/statics-mc-leaning-bar-reaction-pairs-soln.tex

Newton's Third Law (describing action-reaction pairs) is simple:

> **If object A exerts a named force $\vec{F}_{BA}$ on object B, object B exerts an equal and opposite named force, $\vec{F}_{AB} = -\vec{F}_{BA}$, on object A.**

Note that ***A and B must be the same in both directions***. Note also that ***the force $\vec{F}$*** must be the same – have the same origin, have the same name, be the same interaction. Thus:

a) The ladder top pushes against the wall; the wall pushes back against the ladder top.

   A = ladder; B = wall; Force = Normal force. **This statement describes a Newton's Third Law pair of forces.**

b) The floor pushes up on the ladder base; gravity pulls the ladder base down towards the floor.

   A = floor surface; B = ladder base; Force on ladder base is Normal force; Force on ladder is gravity. **This statement is not! It fails on multiple counts.**

c) Static friction from the floor pushes the ladder base towards the wall; the wall pushes back on the ladder.

   A = floor; B ladder; C = wall. $\boldsymbol{F_{BA}}$ is static friction. $\boldsymbol{F_{BD}}$ is normal force. **This statement is not! It fails on multiple counts.**

d) The floor pushes down on the ground; the ground pushes back on the floor.

   A = floor (of building); B = ground (under building); Force is Normal force. **This statement describes a Newton's Third Law pair of forces.**

e) The Earth pulls down on the ladder via gravity; the ladder pulls up on the Earth via gravity.

   A = the Earth; B = ladder; Force = gravity. **This statement describes a Newton's Third Law pair of forces.**

Problem 420. problems-1/statics-mc-pick-reaction-pairs-2.tex

Which of the following list are *not* action-reaction force pairs? (More than one answer is possible.)

a) A hydraulic piston pushes on the fluid in its cylinder; the fluid pushes back on the hydraulic piston.

b) The earth's gravity pulls a pendulum bob at rest down; the string pulls it up.

c) My finger pushes down against a grape I'm squeezing; my thumb pushes up against the grape.

d) My hammer pushes on a nail as it hits it; the nail pushes back on the hammer.

e) A bathroom scale pushes up on my feet as I stand on it; my feet push down on the scale.

Problem 421. problems-1/statics-mc-pick-reaction-pairs-2-soln.tex

Which of the following list are *not* action-reaction force pairs?

The rules are that ***the two objects exerting force and reaction force must be the same (reversed)*** and ***the interaction force must be the same***. See notes below in boldface.

(More than one answer is possible.)

a) A hydraulic piston pushes on the fluid in its cylinder; the fluid pushes back on the hydraulic piston.

b) The earth's gravity pulls a pendulum bob at rest down; the string pulls it up. **earth $\neq$ string, gravity $\neq$ tension.**

c) My finger pushes down against a grape I'm squeezing; my thumb pushes up against the grape. **finger $\neq$ thumb, the normal forces are exerted at two different places.**

d). My hammer pushes on a nail as it hits it; the nail pushes back on the hammer.

e). A bathroom scale pushes up on my feet as I stand on it; my feet push down on the scale.

Problem 422. problems-1/statics-mc-pick-reaction-pairs.tex

(3 points) Which of the following list are *not* action-reaction force pairs? (More than one answer is possible.)

a) The earth's gravity pulls down on an apple; the stem of the apple holds it up.

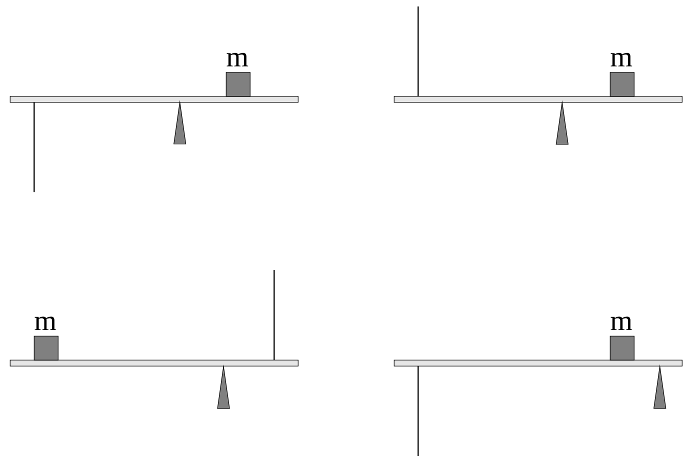
b) Water pressure pushes out against a glass, the glass holds in the water.

c) I push forward on a bow; the bowstring pulls forward on me (as I draw an arrow).

d) I lean my head on the wall; the wall pushes back on my head.

e) I pull down on the rope with my hand; the rope pulls up on my hand.

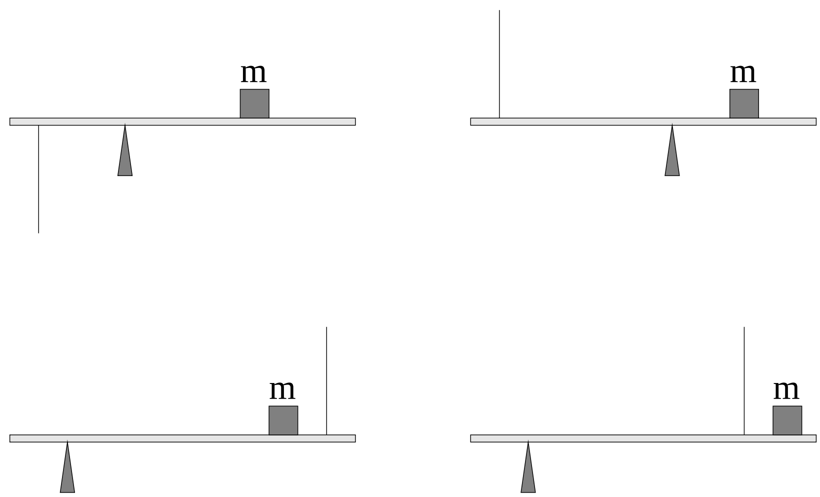
Problem 423. problems-1/statics-mc-plank-mass-rod.tex



(3 points) In the figure above, a very light (approximately massless) plank supports a mass m. The plank is resting on (not attached to) a sawhorse that can support as much weight as you like, and a rod is attached to the plank as shown (where the other end is firmly attached to the ceiling or floor as the case may be). The rod, however, will *break* if it is compressed or stretched with a force $F_b = mg$, the weight of the mass.

Circle all of the configurations where the plank and mass will not move **and** the rod will not break.

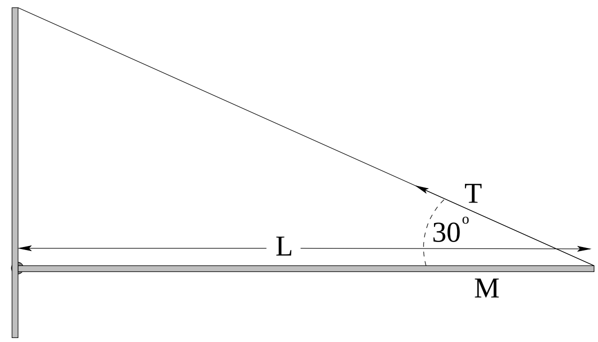
Problem 424. problems-1/statics-mc-plank-mass-string.tex



In the figure above, a very light (approximately massless) plank supports a mass m. The plank is resting on (not attached to) a sawhorse/pivot that can support as much weight as you like, and a massless ***string*** is attached to the plank as shown (the other end is tied to the ceiling or floor as the case may be). The string, however, will ***break*** at a force $F_b = mg$, the weight of the mass.

Circle all of the configurations where the plank and mass will not move ***and*** the string will not break.

Problem 425. problems-1/statics-mc-string-and-bar.tex



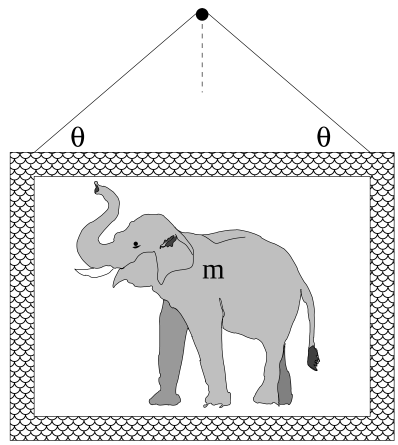
A bar of mass M and length L is pivoted by a hinge on the left and is supported on the right by a string attached to the wall and the right hand end of the bar. The angle made by the string with the bar is $\theta = 30°$. Select the true statement from the list below.

- ☐ $T = Mg/2$
- ☐ $T = Mg$
- ☐ $T = \frac{\sqrt{3}}{2}Mg$
- ☐ $T = 2Mg$
- ☐ There is not enough information to determine T.

Problem 426. problems-1/statics-mc-string-and-bar-soln.tex

b) $T = Mg$. (This balances $\tau_{\text{out}} = TL\sin(30^\circ) - MgL/2 = 0$.)

Problem 427. problems-1/statics-mc-support-the-picture.tex



A picture of mass m has been hung by a piece of thread as shown. The thread will break at a tension of mg. Find the smallest angle theta such that the thread will not break. FYI: $\sin(30°) = \cos(60°) = 1/2$, $\cos(30°) = \sin(60°) = \sqrt{3}/2$, $\sin(45°) = \cos(45°) = \sqrt{2}/2$, $\sin(90°) = \cos(0°) = 1$.

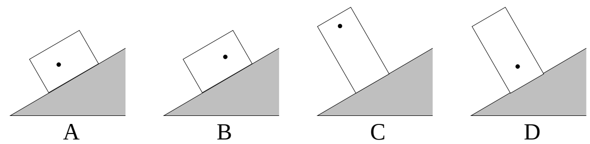
a) 30°

b) 45°

c) 60°

d) 90°

Problem 428. problems-1/statics-mc-tipping-blocks.tex



In the figure, four blocks are placed on an inclined plane that has sufficient static friction that the blocks will not slip. The dots in the figures indicate the center of mass of each block. Which of the following is/are true?

a) A and D will tip.

b) A B and D will not tip.

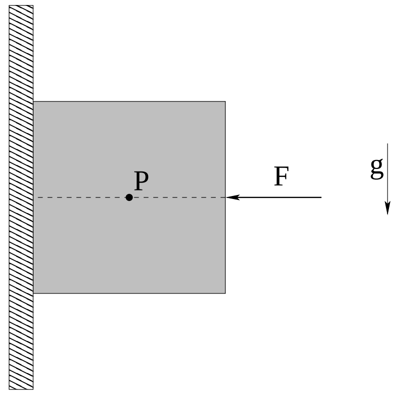
c) B and C will tip.

d) C and D will tip.

Problem 429. problems-1/statics-mc-tipping-blocks-soln.tex

b) A B and D will not tip. That is, ***only C will tip!*** Only in case C is the center of mass/center of gravity to the left of the lower left corner pivot.

Problem 430. problems-1/statics-mc-torque-direction-block-held-to-wall-friction.tex



A cube of mass M is held at rest against a vertical ***rough wall*** by applying a perfectly horizontal force $\vec{\boldsymbol{F}}$ as shown. Gravity is down as usual as shown. What is the ***direction*** of the torque ***about the point P*** due to the ***force of friction*** exerted by the wall on the block?

a) Left.

b) Right.

c) Up.

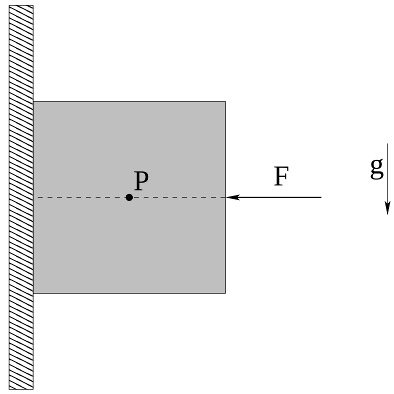
d) Down.

e) Into the plane of the figure.

f) Out of the plane of the figure.

g) The torque is zero, so the direction is undefined.

Problem 431. problems-1/statics-mc-torque-direction-block-held-to-wall-normal.tex



A cube of mass M is held at rest against a vertical ***rough wall*** by applying a perfectly horizontal force $\vec{F}$ as shown. Gravity is down as usual as shown. What is the ***direction*** of the torque ***about the point P*** due to the ***normal force*** exerted by the wall on the block?

a) Left.

b) Right.

c) Up.

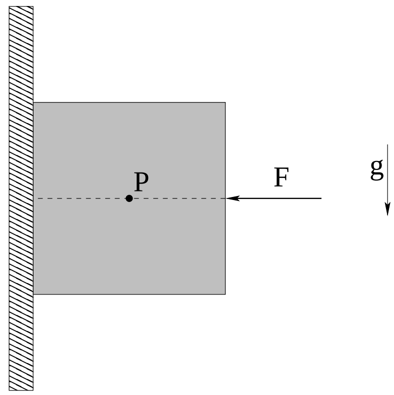
d) Down.

e) Into the plane of the figure.

f) Out of the plane of the figure.

g) The torque is zero, so the direction is undefined.

Problem 432. problems-1/statics-mc-torque-direction-block-held-to-wall.tex



A cube of mass M is held at rest against a vertical ***rough wall*** by applying a perfectly horizontal force $\vec{\boldsymbol{F}}$ as shown. Gravity is down as usual as shown. What is the ***direction*** of the torque ***about the point P*** due to the ***force of friction*** exerted by the wall on the block?

a) Left.

b) Right.

c) Up.

d) Down.

e) Into the plane of the figure.

f) Out of the plane of the figure.

g) The torque is zero, so the direction is undefined.

Now, what is the ***direction*** of the torque about the point P due to the ***normal force*** exerted by the wall on the block?

a) Left.

b) Right.

c) Up.

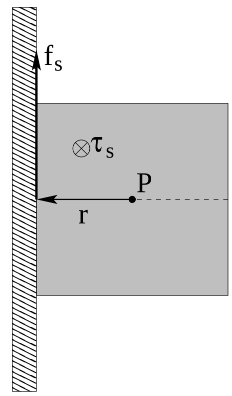
d) Down.

e) Into the plane of the figure.

f) Out of the plane of the figure.

g) The torque is zero, so the direction is undefined.

Problem 433. problems-1/statics-mc-torque-direction-block-held-to-wall-soln.tex



e) Into the plane of the figure.

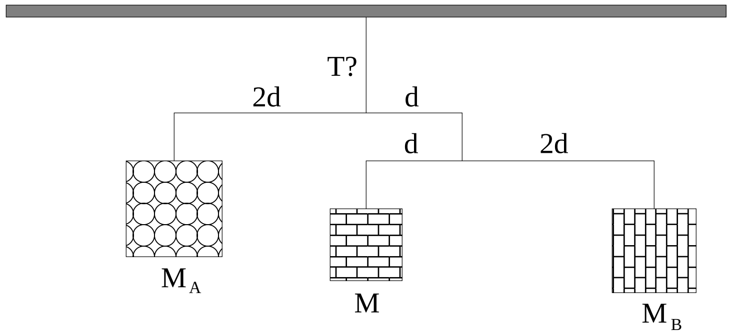
Gravity exerts no torque about P. $\vec{F}$ exerts no torque about P. Friction (as shown above) is required to oppose gravity and exerts a torque into the page. Consequently, the normal force ***must*** exert a torque:

f) Out of the plane of the figure.

This basically means that the normal force must be larger near the bottom corner than it is near the top corner.

9.1.2 Short Answer Problems

Problem 434. problems-1/statics-sa-balance-the-mobile-1.tex



A static mobile suspends three beautifully patterned blocks over a baby's bed. The lengths of the supporting rigid rods (of negligible mass) are given in the figure above, as is the mass of the central block, M. You must find M_A and M_B (in terms of/units of M as shown) so that the mobile perfectly balances, and you must also make sure that the string you are using to hang the mobile is strong enough to support its weight. Note well that the unknown blocks are ***not necessarily drawn to scale!***

a) $\dfrac{M_A}{M} =$ $\boxed{}$

b) $\dfrac{M_B}{M} =$ $\boxed{}$

c) What is the total tension T in the top supporting string when the mobile perfectly balances?

$T =$ $\boxed{}$

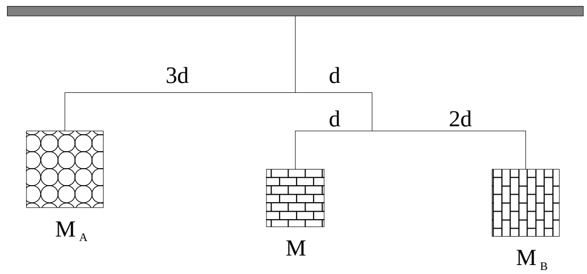
Problem 435. problems-1/statics-sa-balance-the-mobile-1-soln.tex

a) $\dfrac{M_A}{M} = 3/4$

b) $\dfrac{M_B}{M} = 1/2$

c) What is the total tension T in the top supporting string when the mobile perfectly balances?
$T = \dfrac{9}{4} Mg$

Problem 436. problems-1/statics-sa-balance-the-mobile-2.tex

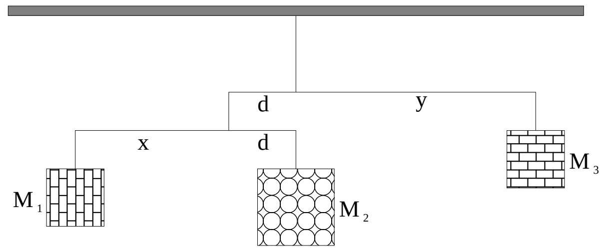


A static mobile suspends three patterned blocks over a baby's bed. The lengths of the supporting rigid rods (of negligible mass) are given in the figure above, as is the mass of the central block, M. Find M_A and M_B in terms of M so that the mobile perfectly balances. Note well that the unknown blocks are ***not necessarily drawn to scale!***

a) $M_A =$

b) $M_B =$

Problem 437. problems-1/statics-sa-balance-the-mobile-reversed.tex



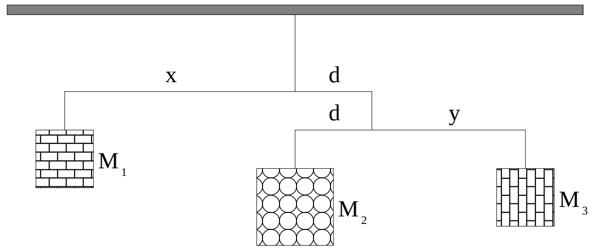
A static mobile suspends three patterned blocks over a baby's bed. The masses of the blocks and the lengths of the supporting rigid rods (of negligible mass) are given in the figure above (although the relative distances may not be correctly to scale). Find x and y in terms of d so that the mobile perfectly balances when:

$M_1 = 1$ kg, $M_2 = 3$ kg, $M_3 = 1$ kg

x =

y =

Problem 438. problems-1/statics-sa-balance-the-mobile.tex

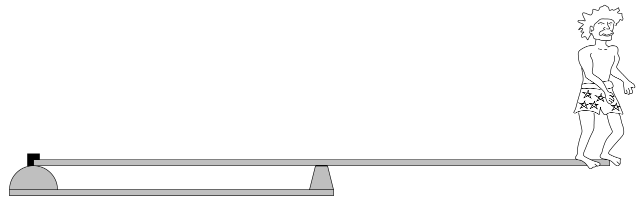


A static mobile suspends three patterned blocks over a baby's bed. The masses of the blocks and the lengths of the supporting rigid rods (of negligible mass) are given in the figure above. Find x and y in terms of d so that the mobile perfectly balances when $\boldsymbol{M_1 = 1}$ kg, $\boldsymbol{M_2 = 4}$ kg, $\boldsymbol{M_3 = 1}$ kg.

$x =$ $\boxed{}$

$y =$ $\boxed{}$

Problem 439. problems-1/statics-sa-diving-board.tex



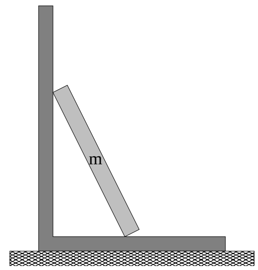
Albert tries to make a diving board for his backyard swimming pool by attaching the board firmly to two vertical supports. The perfectly rigid uniform board has a length of 6 m and a mass of 40 kg. The left hand support is attached to the left end, and the right hand support is attached 3 m to the right of the left support (at the center of the board).

a) What force (magnitude and direction) does the ***middle*** support exert on the board when Albert (whose mass is 80 kg) stands on the right hand end of the board as shown?

b) What force (magnitude and direction) does the ***left*** support exert on the board at this time?

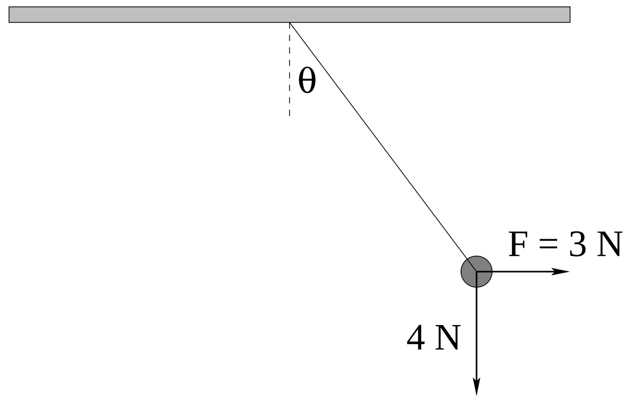
c) Which support needs to be bolted down?

Problem 440. problems-1/statics-sa-leaning-bar-reaction-pairs.tex



In the figure above, a board is sitting on a rough floor and leaning against a wall. Identify *three* action-reaction force pairs in the figure.

Problem 441. problems-1/statics-sa-pendulum-bob.tex



A 4 N pendulum bob supported by a massless string is held motionless at an angle θ from the vertical by a horizontal force $F = 3$ N as shown. The string used to hang the mass will break at any tension $T > T_c = 4\sqrt{2}$ N.

a) What is the angle θ (expression OK).

b) The force F is slowly increased (while keeping the force horizontal). At what value will the string break?

c) What is the angle θ at which the string breaks?

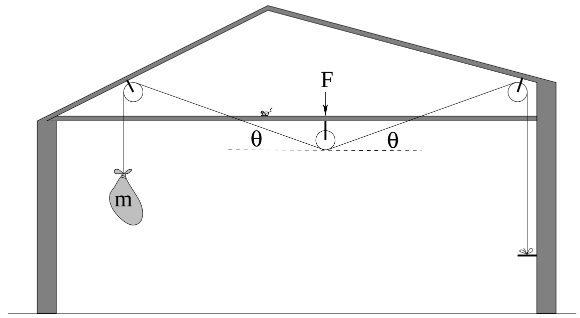
Problem 442. problems-1/statics-sa-pendulum-bob-soln.tex

a) $\theta = \tan^{-1}(3/4) = 37^\circ$.

b) $T = \sqrt{4^2 + F^2} = 4\sqrt{2} = T_c$. This implies $F = 4$ N.

c) $\theta = 45^\circ = \pi/4$.

Problem 443. problems-1/statics-sa-suspended-food-bag.tex



A gold prospector living in a rustic cabin mounts a sturdy wooden peg and three (approximately massless and frictionless) pulleys in fixed positions on the wall and rafters as shown in the diagram so he can suspend his food bag up off the floor and away from mice. He hangs a bag of food of mass m so that the rope makes an angle θ with the central pulley as shown.

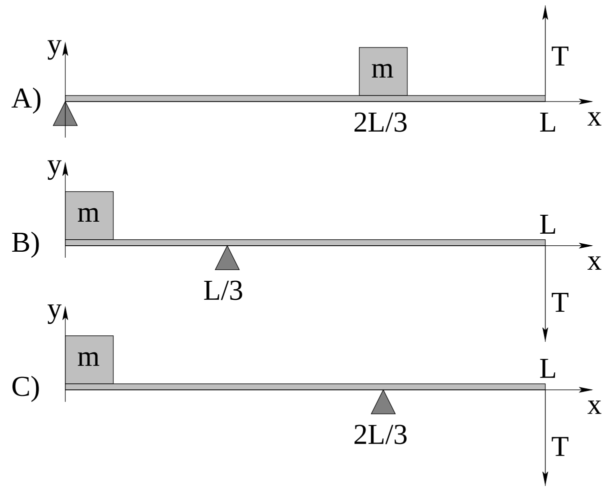
Help him find the magnitude of the force F that his rafter must exert downward on the pulley when he has hung his bag of food.

Problem 444. problems-1/statics-sa-suspended-food-bag-soln.tex

No torques! Force balance on the bag on the left tells you T, force balance on the picture in the middle tells you that:

$F = 2T\sin(\theta) = 2mg\sin(\theta)$

Problem 445. problems-1/statics-sa-which-mass-breaks-string.tex

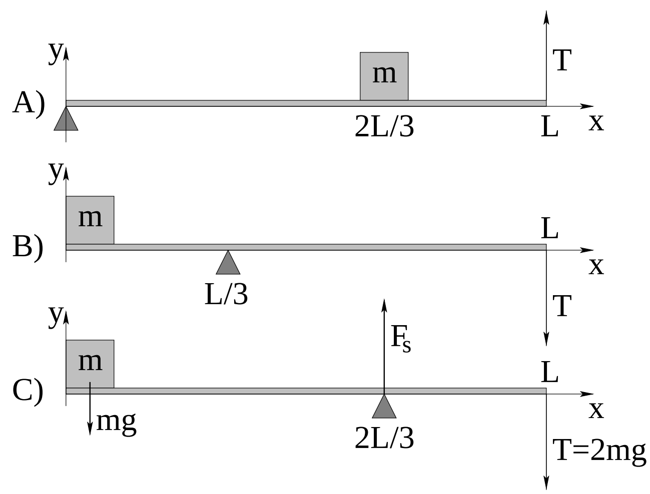


In the figure above, a massless plank supports a massive block m placed at the locations shown. The plank is supported by a wedge shaped support and a string that will break at the same tension $T_{\rm max}$ (in all three cases) positioned as shown.

a) Suppose the mass m is gradually increased (in all three figures). In which configuration (**A**, **B**, or **C**) will the string break ***first***?

b) For ***that configuration*** (that you picked in part a), what is the value of the upward support force F_s exerted by the wedge right as (just before) the string breaks?

Problem 446. problems-1/statics-sa-which-mass-breaks-string-soln.tex



In the figure above, a massless plank supports a massive block m placed at the locations shown. The plank is supported by a wedge shaped support and a string that will break at the same tension $T_{\max}$ (in all three cases) positioned as shown.

a) Suppose the mass m is gradually increased (in all three figures). In which configuration (**A**, **B**, or **C**) will the string break ***first***?

b) For ***that configuration*** (that you picked in part a), what is the value of the upward support force F_s exerted by the wedge right as (just before) the string breaks?

Solution: The easiest way to answer this is to consider the ***torques around the support/pivot only!*** The support can either *add to* or *subtract from* the force required to balance the mass.

By inspection, then, in the first figure the weight is *shared* between the support and T, with 2/3 of it supported by T and 1/3 by the support force. In the second figure, we can see that the moment arm of T is twice that of the mass m, so we expect to balance its weight mg with half its weight $mg/2$ – again less than the weight itself.

In case C), however, this is reversed! It therefore breaks first, with the explicit calculations:

$$\tau_{out} = \frac{2mg\cancel{L}}{\cancel{3}} - \frac{T\cancel{L}}{\cancel{3}} = 0 \qquad \Rightarrow \qquad \boxed{T = 2mg}$$

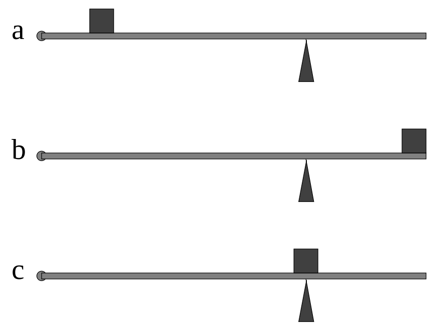
and

$$F_y = F_s - mg - T = 0 \qquad \Rightarrow \qquad \boxed{F_s = 3mg}$$

Note that while I show all of the details, here, one ***should*** be able to solve the entire problem with your eyeballs only if you understand the relationship between forces, moment arms, and torque. All of the ratios are 2:1, so it is really pretty easy!

9.1.3 Ranking Problems

Problem 447. problems-1/statics-ra-board-and-pivot-force.tex

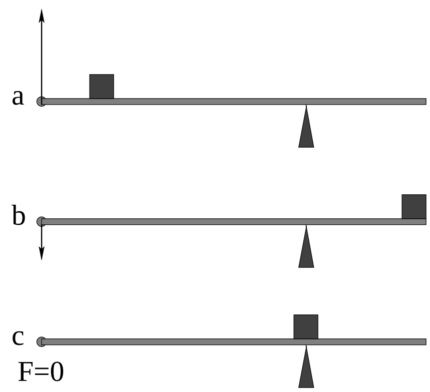


In the three figures above, a ***massless*** board is held in ***static equilibrium*** by a hinge at the left end and a trestle. A mass M is placed on the board at the three places shown. For each figure:

a) Draw an arrow at the ***hinge*** indicating the ***direction*** of the force (if any) exerted by the hinge for all three figures. If the force is zero please indicate this.

b) Rank the three figures in the order of the ***magnitude*** of the force exerted on the board ***by the trestle***, from least to greatest.

Problem 448. problems-1/statics-ra-board-and-pivot-force-soln.tex

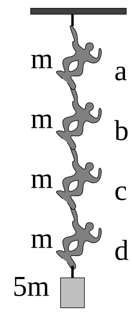


a)

b) For the trestle force magnitude F_i:

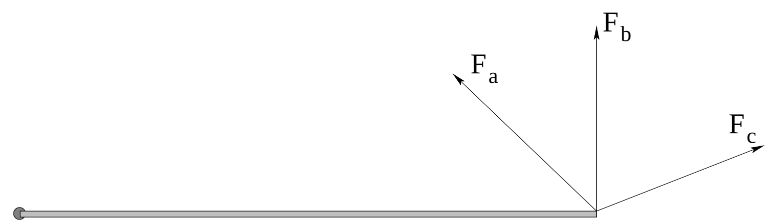
$F_a < F_c < F_b$

Problem 449. problems-1/statics-ra-chain-of-hanging-monkeys.tex



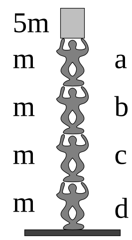
In the figure above four monkeys, each of mass m, are shown holding very still as they hang from a pole at the top of a circus tent. The top monkey (a) is holding a strap attached to the pole above, and the bottom monkey (d) is holding a mass $5m$ above with his foot. Which monkey (a-d) is pulling ***up with the largest force*** with its feet?

Problem 450. problems-1/statics-ra-holding-up-the-bar.tex



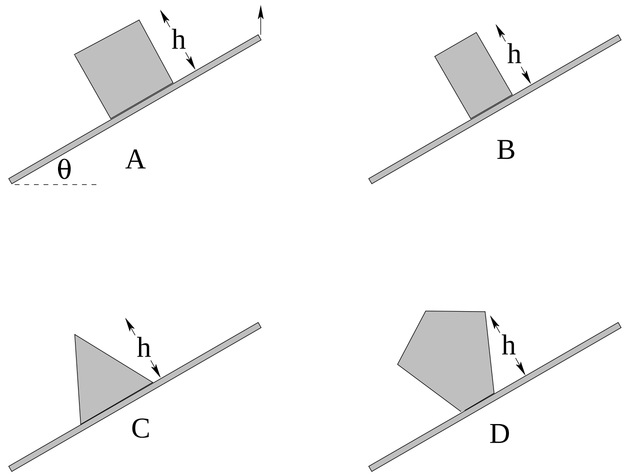
A bar of mass M is pivoted by a hinge on the left and has a wire attached to the right as shown. The wire can be attached to the ceiling on eyebolts on any one of the three angles shown to suspend the rod so that it is in static equilibrium. Rank the force F exerted on the rod ***by the wire*** when the wire comes off in the a, b, c directions (where equality is a possibility). That is, your answer might look like $F_a < F_b = F_c$ (but don't count on this being the answer). Note well: The arrows in the figure above are **not proportional** to the forces, they indicate **only** the directions.

Problem 451. problems-1/statics-ra-stack-of-standing-monkeys.tex



(3 points) In the figure above four monkeys, each of mass m, are shown holding very still in a tower they've made at the circus. The bottom monkey (d) is standing on the floor, the top monkey (a) is holding a mass $5m$ above his head. Which monkey is pushing *up* with the largest force with its arms?

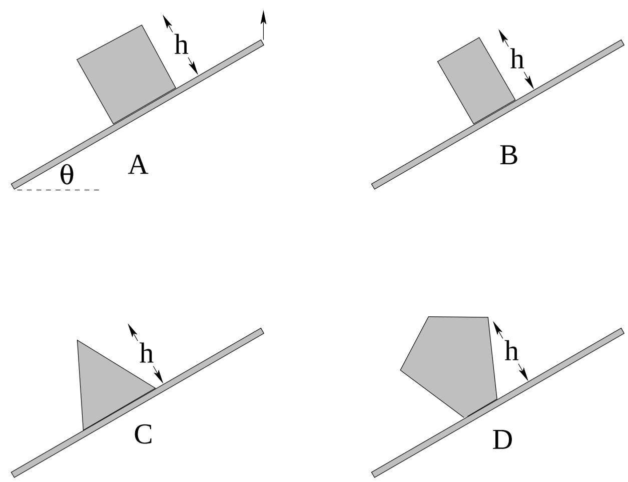
Problem 452. problems-1/statics-ra-tipping-shapes-2.tex



In the four figures above, the coefficient of static friction is high enough that the uniform objects shown will not slip before they tpp. Rank the angles at which each mass will tip over as the right end of the plank they sit on is raised, from ***smallest*** (the block that tips *first*) tipping angle θ ***to the largest*** (the block that tips *last*). Your answer will be some permutation of A,B,C,D.

$\boxed{}$, $\boxed{}$, $\boxed{}$, $\boxed{}$

Problem 453. problems-1/statics-ra-tipping-shapes-2-soln.tex



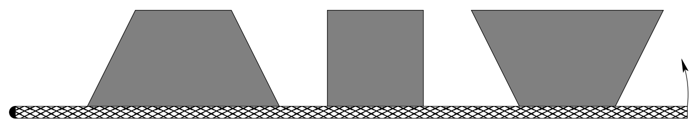
In the four figures above, the coefficient of static friction is high enough that the uniform objects shown will not slip before they tpp. Rank the angles at which each mass will tip over as the right end of the plank they sit on is raised, from ***smallest*** (the block that tips *first*) tipping angle θ ***to the largest*** (the block that tips *last*). Your answer will be some permutation of A,B,C,D.

Solution:

$$\boxed{\text{D}} \quad , \quad \boxed{\text{B}} \quad , \quad \boxed{\text{A}} \quad , \quad \boxed{\text{C}}$$

where D should already have tipped, B is almost "at" the tipping point, A is fairly stable, and C is extremely stable. The trick to it is simply deciding which blocks will have their center of mass come over the lower right corner first, second, etc.

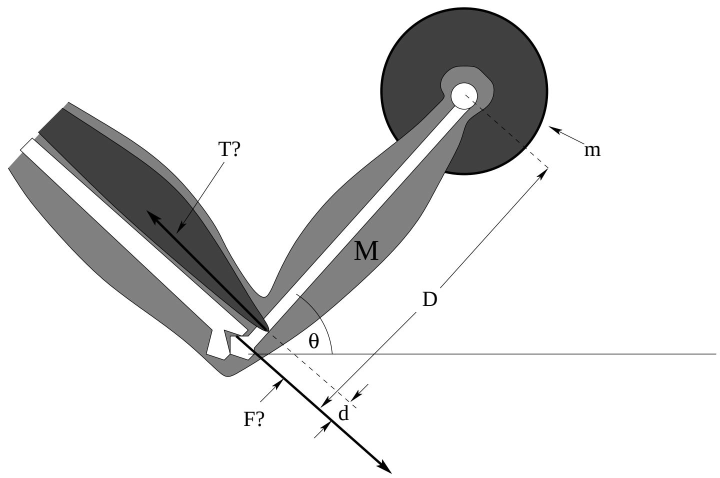
Problem 454. problems-1/statics-ra-tipping-shapes.tex



In the figure above, three shapes (with uniform mass distribution and thickness) are drawn sitting on a plane that can be tipped up gradually. Assuming that static friction is great enough that all of these shapes will tip over before they slide, rank them in the order they will tip over as the angle of the board they are sitting on is increased. Be sure to indicate any ties.

9.1.4 Regular Problems

Problem 455. problems-1/statics-pr-arm-with-barbell.tex

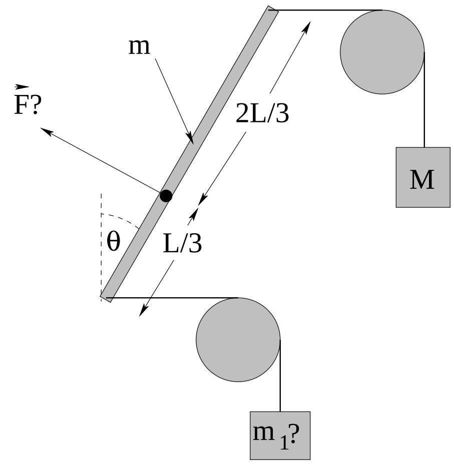


An exercising human person holds their arm of mass M and a barbell of mass m at rest at an angle θ with respect to the horizontal in an isometric curl as shown. The muscle that supports the suspended weight is connected a short distance d up from the elbow joint. The bone that supports the weight has length D.

a) Find the tension T in the muscle, assuming for the moment that the center of mass of the forearm is in the middle at $D/2$. Note that it is *much larger* than the weight of the arm and barbell combined, assuming a reasonable ratio of $D/d \approx 25$ or thereabouts.

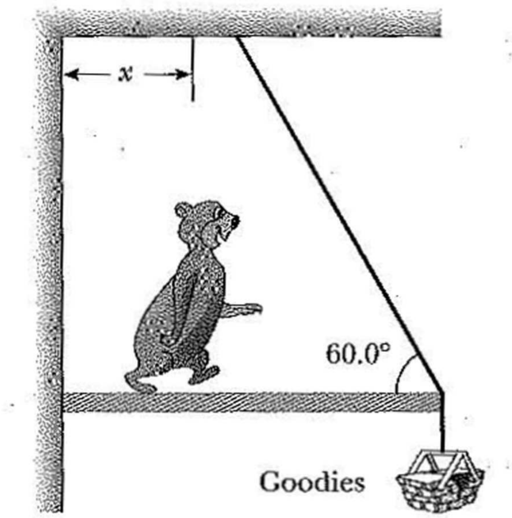
b) Find the force $\vec{F}$ (magnitude *and* direction) exerted on the supporting bone by the elbow joint. Again, note that it is much larger than "just" the weight being supported.

Problem 456. problems-1/statics-pr-bar-and-pulleys.tex



Find the components of the pivot force $\vec{\boldsymbol{F}} = (F_x, F_y)$ and find m_1 **in terms of** M **and** m **as givens** in the figure above, if the bar of mass m is in static equilibrium.

Problem 457. problems-1/statics-pr-bear-seeking-goodies.tex



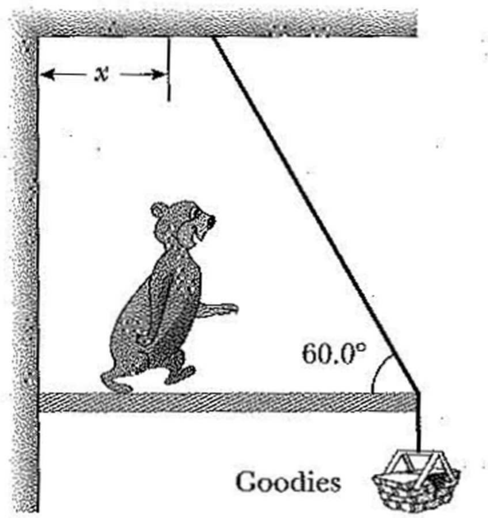
A bear of mass M_B walks out on a beam of mass m_b to get a basket of food of mass of mass m_f. The beam has length L, and is supported by a wire at an angle of 60 degrees, as in the sketch.

a) Find the ***vector*** force that the wall exerts on the left end of the beam when the bear is a distance x from the wall.

b) Also find the tension in the wire.

c) Suppose that the bear is too heavy to reach the basket without breaking the wire. If the maximum tension that the wire can support without breaking is T_{max}, find an expression for the largest distance from the wall x_{max} that the bear can walk without breaking the wire.

Problem 458. problems-1/statics-pr-bear-seeking-goodies-soln.tex



A bear of mass M_B walks out on a beam of mass m_b to get a basket of food of mass of mass m_f. The beam has length L, and is supported by a wire at an angle of 60 degrees, as in the sketch.

a) Find the ***vector*** force that the wall exerts on the left end of the beam when the bear is a distance x from the wall.

b) Also find the tension in the wire.

c) Suppose that the bear is too heavy to reach the basket without breaking the wire. If the maximum tension that the wire can support without breaking is $T_{\max}$, find an expression for the largest distance from the wall $x_{\max}$ that the bear can walk without breaking the wire.

$$\tau_{\text{out}} = LT\sin 60^\circ - \frac{L}{2}m_b g - Lm_f g - xM_b g = 0 \Rightarrow \boxed{T = \frac{\sqrt{3}}{3}m_b g + \frac{2\sqrt{3}}{3}m_f g + \frac{2\sqrt{3}}{3}\frac{x}{L}M_b g}$$

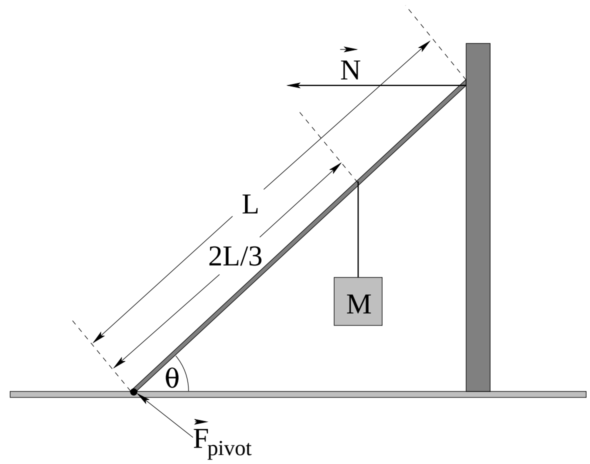
$$F_x - T\cos 30^\circ = 0 \Rightarrow \boxed{F_x = \frac{\sqrt{3}}{6}m_b g + \frac{\sqrt{3}}{3}m_f g + \frac{\sqrt{3}}{3}\frac{x}{L}M_b g}$$

$$F_y + T\cos 60^\circ - (m_b + m_f + M_b)g = 0 \Rightarrow F_y = (m_b + m_f + M_b)g - \frac{m_b g}{2} - m_f g - \frac{x}{L}M_b g$$

Or:

$$\boxed{F_y = \frac{m_b g}{2} + \left(\frac{L-x}{L}\right)M_b g}$$

Problem 459. problems-1/statics-pr-brace-against-house.tex

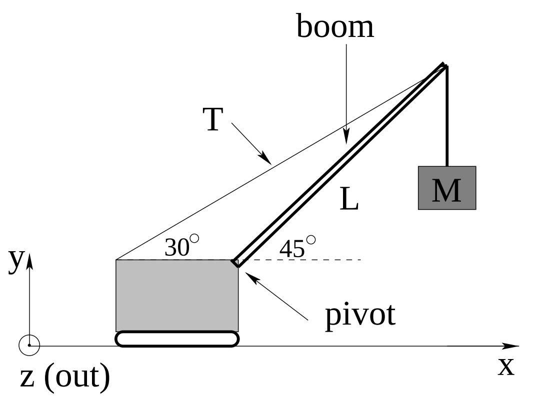


In the figure above, a "massless" rigid beam of length L that makes an angle of θ with the ground is leaned against a frictionless wall at the upper end, which exerts a ***normal force only*** N as shown on the beam. A mass M is suspended vertically from a point 2/3 of the way from the pivot attached to the ground. Find:

a) The *magnitude* of the normal force N exerted ***by the wall on the beam*** when the entire beam is in static equilibrium.

b) The *vector force* $\vec{\boldsymbol{F}}_p$ exerted by the pivot on the ground ***on the beam*** to hold the beam in place. It is probably easiest to express this answer as F_{px} and F_{py}.

Problem 460. problems-1/statics-pr-crane-boom.tex



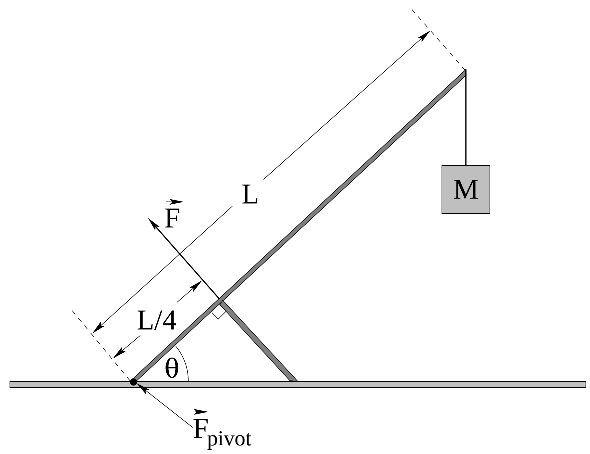
$$\sin(30^\circ) = \cos(60^\circ) = \frac{1}{2}$$
$$\cos(30^\circ) = \sin(60^\circ) = \frac{\sqrt{3}}{2}$$
$$\sin(45^\circ) = \cos(45^\circ) = \frac{\sqrt{2}}{2}$$

A crane with a boom (the long support between the body and the load) of mass m and length L holds a mass M suspended as shown. Assume that the center of mass of the boom is at $L/2$. Note that the wire with the tension T is **fixed** to the top of the boom, not run over a pulley to the mass M.

a) Find the tension in the wire.

b) Find the force exerted on the boom by the crane body.

Problem 461. problems-1/statics-pr-crane-vertical-support.tex

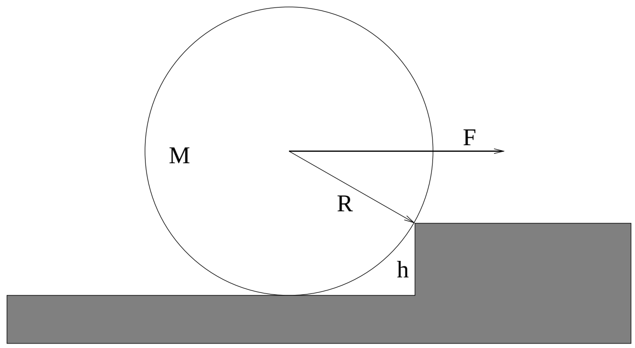


In the figure above, a "massless" rigid beam of length L that makes an angle of θ with the ground is braced with a piece of wood a distance $L/4$ from the end on the ground. This piece of wood is attached at right angles to the beam as shown. At the upper end of the beam a mass M is suspended. Find:

a) The *magnitude* F of the force exerted ***by the support bar*** when the entire beam is in static equilibrium.

b) The *vector force* $\vec{F}_p$ exerted by the pivot on the ground ***on the beam*** (**not** the support bar) to hold the beam in place. It is probably easiest to express this answer as F_{px} and F_{py}.

Problem 462. problems-1/statics-pr-cylinder-and-corner-2.tex

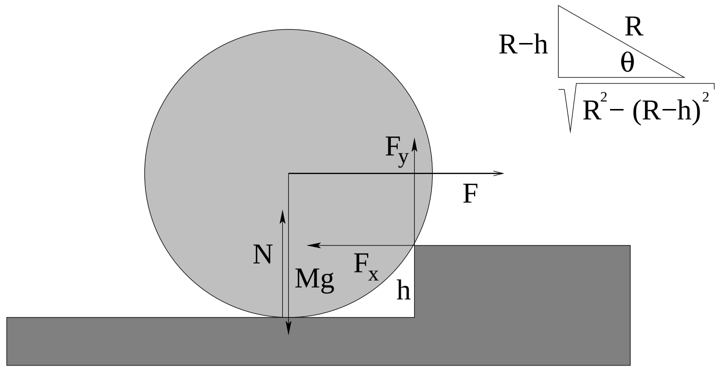


A cylinder of mass M and radius R sits against a step of height $h = R/2$ as shown above. A force $\vec{F}$ is applied parallel to the ground as shown. All answers should be in terms of M, R, g.

a) Find the minimum value of $|\vec{F}|$ that will roll the cylinder over the step if the cylinder does not slide on the corner.

b) What is the force exerted by the corner (magnitude and direction) when that force $\vec{F}$ is being exerted on the center?

Problem 463. problems-1/statics-pr-cylinder-and-corner-2-soln.tex



Use the force diagram above and the associated triangle diagram to figure out the geometry that describes the stuff below. Note that if $h = R/2$, $\theta = 30^\circ = \pi/6$ and the two sides are $R/2$ and $\sqrt{3}R/2$, but the solution I give below is more general than this specific case!

a) At the critical/minimum value, the normal force, and hence the torque into the page by the normal force (exerted by the step above) "barely" goes to zero. If we choose the corner (where two unknown force components act) as the pivot, we can write the total torque equation *including* the torque due to N as:

$$\tau_z = \sqrt{R^2 - (R-h)^2}Mg - \sqrt{R^2 - (R-h)^2}N - F(R-h) = 0$$

where positive z is out of the page. In the "critical" $N \to 0$ case this then becomes:

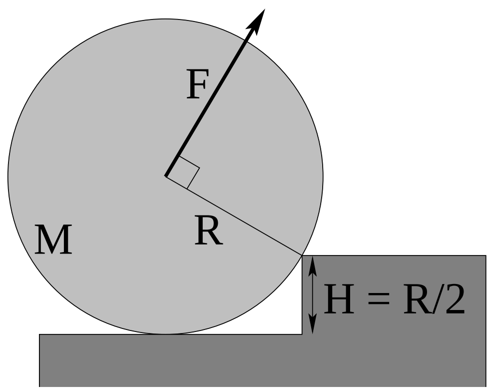
$$F_{\rm min} = \frac{\sqrt{R^2 - (R-h)^2}}{R-h}Mg = \sqrt{3}Mg$$

(at $h = R/2$). Any $F > F_{\rm min}$ will cause the cylinder to rotate up and over the corner.

b) Here we need to use the two remaining non-trivial equations for equilibrium, where I've used the fact that F_x will certainly point in the negative x direction already (note the minus sign). F_x from this equation will be a positive number, but represents a negative x force component.

$$\begin{aligned} F - F_x &= 0 \quad\Longrightarrow\quad F_x = \sqrt{3}Mg \\ F_y - Mg\ + (N \to 0) &= 0 \quad\Longrightarrow\quad F_y = Mg \end{aligned}$$

Problem 464. problems-1/statics-pr-cylinder-and-corner.tex

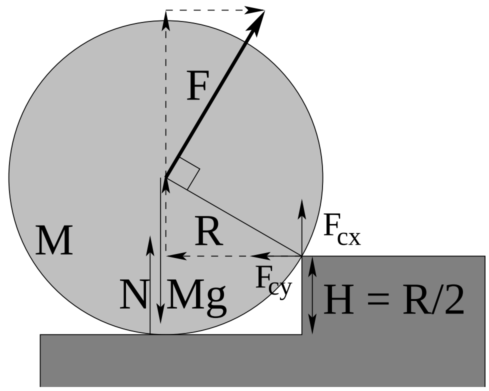


A cylinder of mass M and radius R sits against a step of height $H = R/2$ as shown above. A force $\vec{F}$ is applied ***at right angles*** to the line connecting the corner of the step and the center of the cylinder as shown to the left. All answers should be in terms of M, R, g.

a) Find the minimum value of $|\vec{F}|$ that will roll the cylinder over the step if the cylinder does not slide on the corner.

b) What is the force exerted by the corner (magnitude and direction) when that force $\vec{F}$ is being exerted on the center?

Problem 465. problems-1/statics-pr-cylinder-and-corner-soln.tex



A cylinder of mass M and radius R sits against a step of height $H = R/2$ as shown above. A force $\vec{F}$ is applied ***at right angles*** to the line connecting the corner of the step and the center of the cylinder as shown to the left. All answers should be in terms of M, R, g.

a) Static equilibrium is $\sum \vec{\boldsymbol{F}} = 0$ and $\sum \vec{\boldsymbol{\tau}} = 0$. We start with the torque and balance ***it in the limit that $\boldsymbol{N \to 0}$*** to get the critical value of $F = |\vec{\boldsymbol{F}}|$, noting that the components of F are the sides of a 30,60,90 triangle (as are the components of R):

$$RF - mg(\sqrt{3}/2)R = 0 \Longrightarrow \boxed{F_{\min} = mg\frac{\sqrt{3}}{2}}$$

b) Then balance the forces:

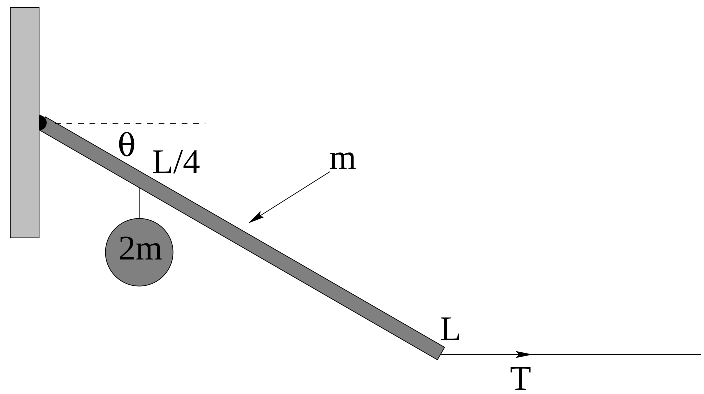
$$\frac{F}{2} - F_{cx} = 0 \qquad F_{cy} + F\sqrt{\frac{3}{2}} - mg = 0$$

or

$$F_{cx} = \boxed{mg\frac{\sqrt{3}}{4}} \qquad F_{cy} = mg - mg\frac{3}{4} = \boxed{mg\frac{1}{4}}$$

Note that these are *also* components of a 30,60,90 triangle!

Problem 466. problems-1/statics-pr-dangling-bar.tex

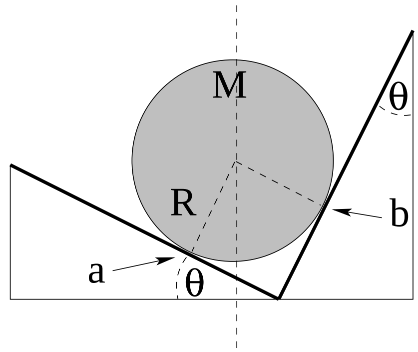


In the figure above, a rod of length L with mass m is suspended by a hinge on the left and a horizontal string on the right. A second mass $2m$ is suspended from the rod a distance $L/4$ from the hinge end. Find:

a) The tension T in the horizontal string.

b) The vector force $\vec{\boldsymbol{F}}$ exerted by the hinge, in any of the acceptable forms we use to completely specify a vector.

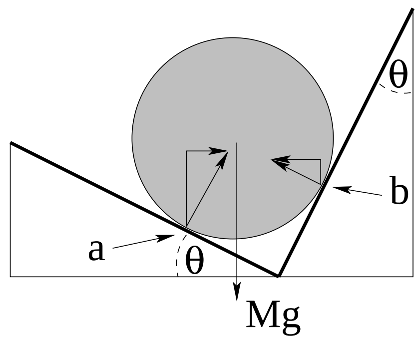
Problem 467. problems-1/statics-pr-disk-in-corner.tex



Find the magnitude of the normal forces N_a and N_b exerted by the two walls on the disk of mass M and radius R at the points a and b such that it sits in static equilibrium in the picture above:

- $N_a =$
- $N_b =$

Problem 468. problems-1/statics-pr-disk-in-corner-soln.tex



No torque! N_a and N_b ***along*** the radii as drawn above. Decompose them both into components and write force balance in x and y:

$$\begin{aligned} N_a \sin(\theta) - N_b \cos(\theta) &= 0 \\ N_a \cos(\theta) + N_b \sin(\theta) - Mg &= 0 \end{aligned}$$

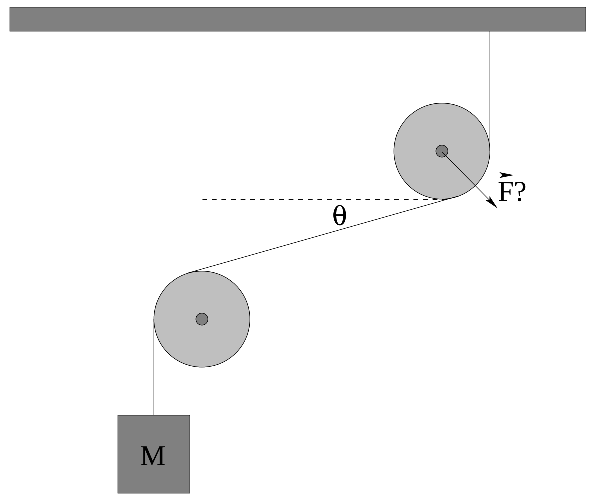
For example, solve the first equation for $N_b = N_a \sin(\theta)/\cos(\theta)$, substitute this into the second equation:

$$N_a \cos(\theta) + N_a \frac{\sin^2(\theta)}{\cos(\theta)} = Mg$$

and multiply both sides by $\cos(theta))$ and back substitute to get:

$$\begin{aligned} N_a &= N_a(\cos^2(\theta) + \sin^2(\theta)) = Mg\cos(\theta) \\ N_b &= Mg\sin(\theta) \end{aligned}$$

Problem 469. problems-1/statics-pr-double-diagonal-pulleys.tex



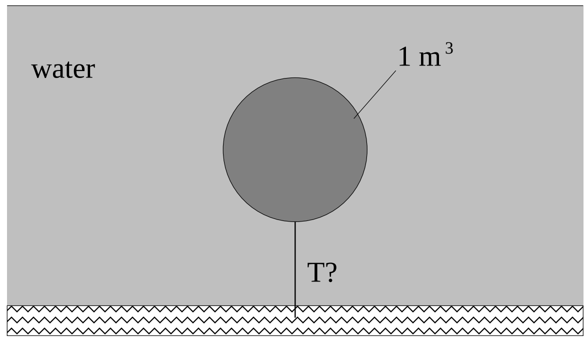
In the figure above, two massless pulleys and a massless unstretchable string support a mass M in static equilibrium as shown. The pulleys are fixed on unmoveable frictionless axles.

a) (3 points) Draw a force diagram for the mass M *and both pulleys*.

b) (5 points) Find the *vector* force $\vec{F}$ exerted by the axle of the upper pulley at equilbrium.

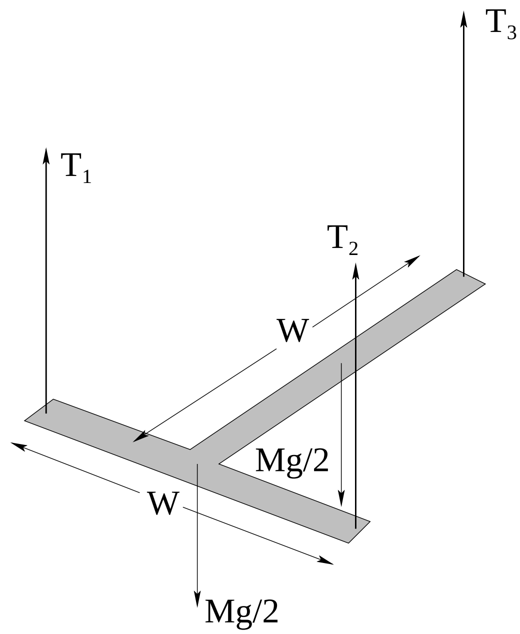
c) (1 point) If the angle θ is increased (by lowering the lower pulley, for example) is there *more* or *less* force exerted by the upper axle to keep the pulley in place?

Problem 470. problems-1/statics-pr-floating-buoy.tex



A round buoy at the beach floats in fresh water when it is exactly half submerged. Its spherical volume is 1 cubic meter. If it is pulled all the way underwater and suspended from the bottom by means of an anchored rope, what is the tension in the rope?

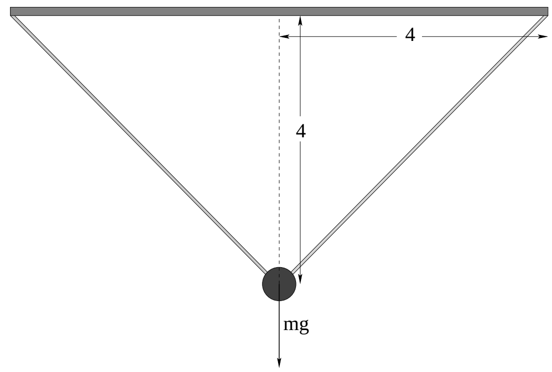
Problem 471. problems-1/statics-pr-hang-a-stable-T.tex



The "**T**" shaped object above has mass M, and has both a height and width of W. Assume that this mass is uniformly distributed in the long arm and the crossbar, that is, that the center of mass of the long arm is at $W/2$ and the center of mass of the crossbar is also at $W/2$ and that the long arm and crossbar each has mass $M/2$ (and hence gravity exerts a downward force at their centers of mass of $Mg/2$ as shown).

Find the tension $T_{1,2,3}$ in ***each of the three ropes*** that support the **T** above. Note that the ropes all pull straight up (they are vertical) and the **T** is completely horizontal.

Problem 472. problems-1/statics-pr-hanging-ball-2.tex



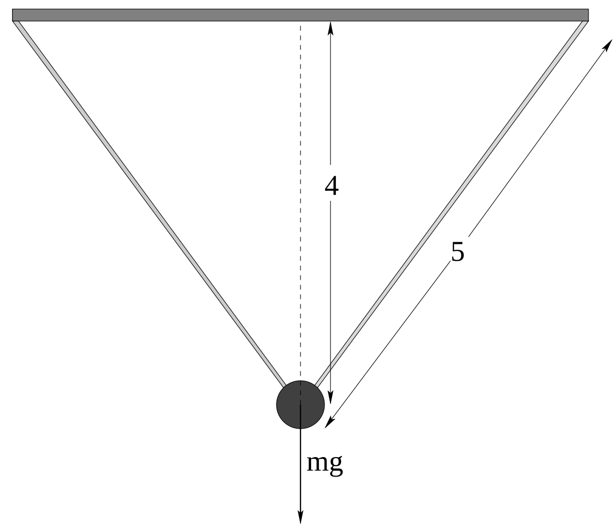
(9 points total) In the figure above, a mass m is hanging from two massless, unstretchable ropes. Gravity pulls straight down on the mass with a force of magnitude mg. Assume that the tension in both ropes has the equal magnitude T. The mass is hanging 4 meters beneath the ceiling, and each rope is fastened to the ceiling offset by 4 meters from where the mass hangs as shown.

a) (3 points) Draw a coordinate system and free body diagram representing all the forces acting on the hanging mass. Label any angles that might be of use to you.

b) (3 points) Write the *algebraic* equations for the total force in the x and y directions that are the conditions for static equilibrium.

c) (3 points) Find the tension T in terms of mg.

Problem 473. problems-1/statics-pr-hanging-ball.tex



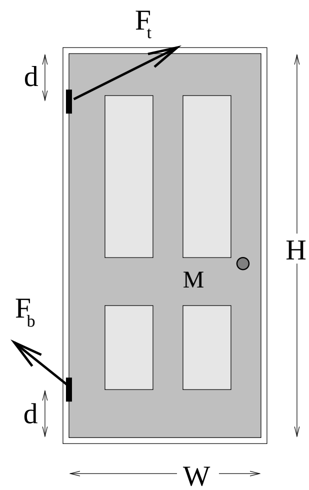
In the figure above, a mass m is hanging from two massless, unstretchable ropes. Gravity pulls straight down on the mass with a force of magnitude mg. Assume that the tension in both ropes has the equal magnitude T. The length of the each rope is 5 meters, and the mass is hanging 4 meters beneath the ceiling as shown

a) Draw a coordinate system and free body diagram representing all the forces acting on the hanging mass. Label any angles that might be of use to you.

b) Write the *algebraic* equations for the total force in the x and y directions that are the conditions for static equilibrium.

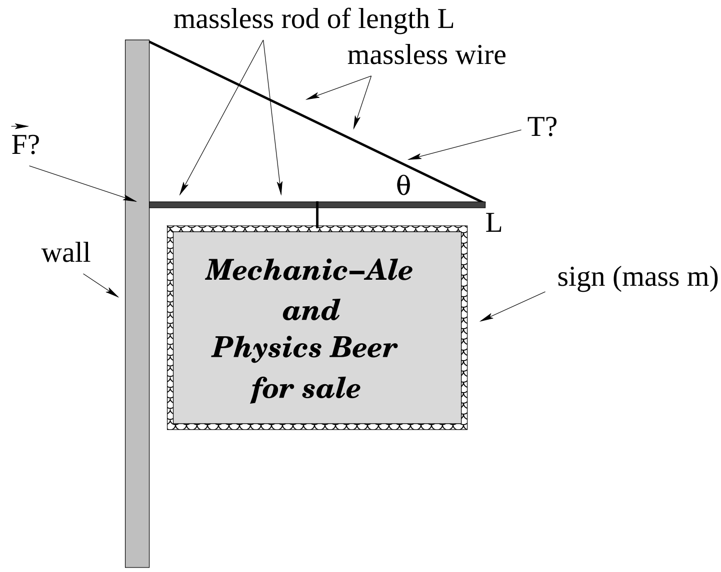
c) Find the tension T in terms of mg.

Problem 474. problems-1/statics-pr-hanging-door.tex



A door of mass M that has height H and width W is hung from two hinges located a distance d from the top and bottom, respectively. Assuming that the weight of the door is equally distributed between the two hinges, find the total force (magnitude and direction) exerted by each hinge. (Neglect the mass of the doorknob. The force directions drawn for you are **NOT** likely to be correct or even close.)

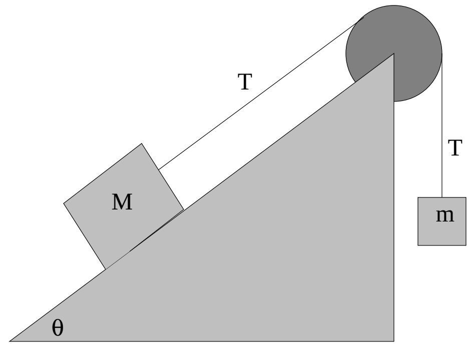
Problem 475. problems-1/statics-pr-hanging-tavern-sign.tex



In the figure above, a tavern sign belonging to a certain home-brewing physics professor is shown suspended from the ***middle*** of a ***massless*** supporting rod of length L (at $L/2$). Find the ***tension*** in the (massless) wire, $\boldsymbol{T}$, and the ***total force*** exerted on the suspending rod by ***the wall***, $\vec{\boldsymbol{F}}$, in terms of m, g, L, and θ.

Please indicate the coordinate system you are using on the figure and the location of the pivot point used, if any.

Problem 476. problems-1/statics-pr-inclined-plane.tex



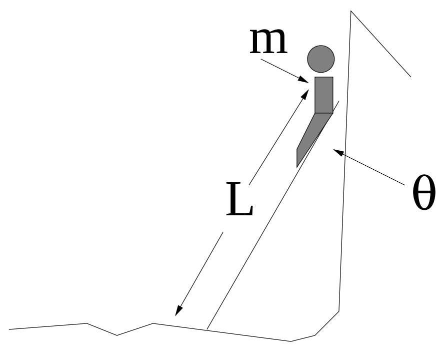
This problem will help you learn required concepts such as:

- Newton's Third Law
- Momentum Conservation
- Fully Inelastic Collisions

so please review them before you begin.

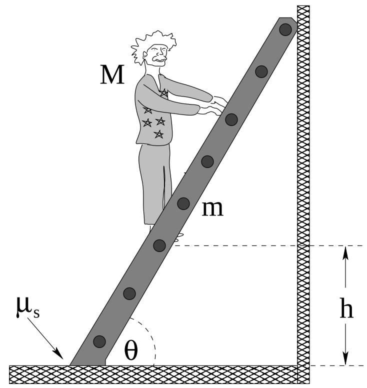
In the inclined plane problem above all masses are at rest and the pulley and string are both massless. Find the normal force exerted by the inclined plane on the mass M and the mass m required to keep the system in static balance in terms of M and θ.

Problem 477. problems-1/statics-pr-ladder-on-glacier.tex



An ultralight (assume massless) ladder of length L rests against a vertical block of (frictionless) ice during a hazardous ascent of a glacier at an angle $\theta = 30^\circ$ as drawn. A mountaineer of mass m climbs the ladder. When the mountaineer is standing *at rest* at the very top of the ladder and about to reach over the cliff edge, what is the net force exerted on the *base* of the ladder by the glacier?

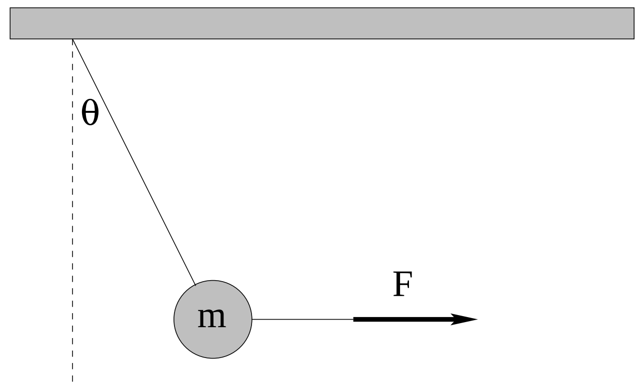
Problem 478. problems-1/statics-pr-ladder-on-wall.tex



In the figure above, a ladder of mass m and length L is leaning against a wall at an angle θ. A person of mass M begins to climb the ladder. The ladder sits on the ground with a coefficient of static friction μ_s between the ground and the ladder. The wall is frictionless – it exerts only a normal force on the ladder.

If the person climbs the ladder, find the height h where the ladder slips.

Problem 479. problems-1/statics-pr-mass-two-strings.tex



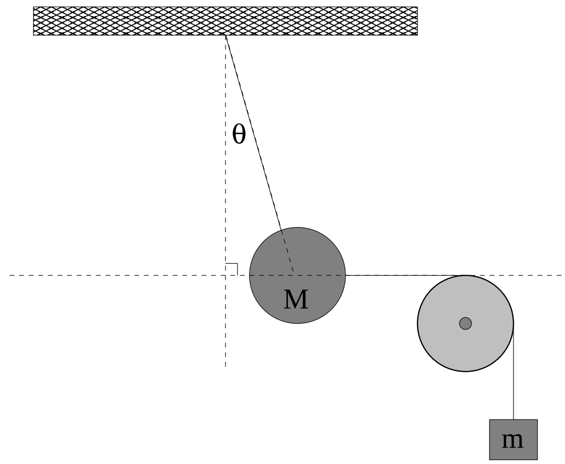
A ball of mass m hangs from the ceiling on a massless string. A second massless string is attached to the ball and a force $\vec{\boldsymbol{F}}$ is applied to it in the horizontal direction so that the system remains in static equilibrium in the position shown, where θ is the angle between the first string and the vertical. Gravity acts down as usual. Each string can support a ***maximum*** tension $T_{\rm max} = 2mg$ without breaking.

a) If $\vec{\boldsymbol{F}}$ is slowly increased while keeping its direction horizontal, which string will break first? Explain your reasoning.

b) Find the maximum value $\theta_{\rm max}$ that the hanging string can have when the system is in static equilibrium with both strings unbroken. (You may express this angle as an inverse sine, cosine, or tangent if you wish – you do not need a calculator.)

c) Find the force magnitude $F_{\rm max}$ that produces the maximum angle $\theta_{\rm max}$ in static equilibrium. Express this answer in terms of m and g.

Problem 480. problems-1/statics-pr-pendulum-bob-pulley.tex



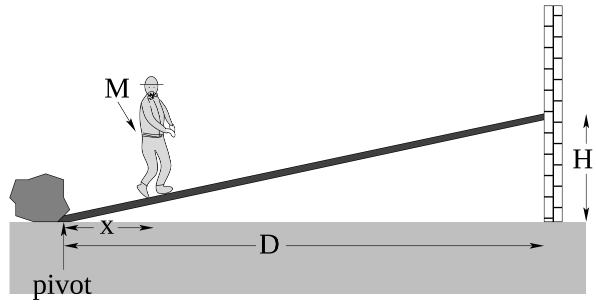
A pendulum bob of mass m is attached both to the ceiling and to a mass M hanging over a pulley by unstretchable massless strings as shown. The pulley is fixed on an unmoveable frictionless axle.

a) (3 points) Draw free body diagrams for *both* mass m *and* mass M.

b) (3 points) Find an expression for the angle θ at which the system is in static equilibrium.

c) (3 points) Find the total tension T in the string connecting the pendulum bob to the ceiling.

Problem 481. problems-1/statics-pr-pushing-down-wall.tex



Tom is a hefty construction worker (mass $M = 100$ kilograms) with a good sense of balance who wants to push down a brick wall. The wall, however, is strong enough to withstand any horizontal push up to 2000 N and Tom can only exert a sideways equal to his weight with his muscles.

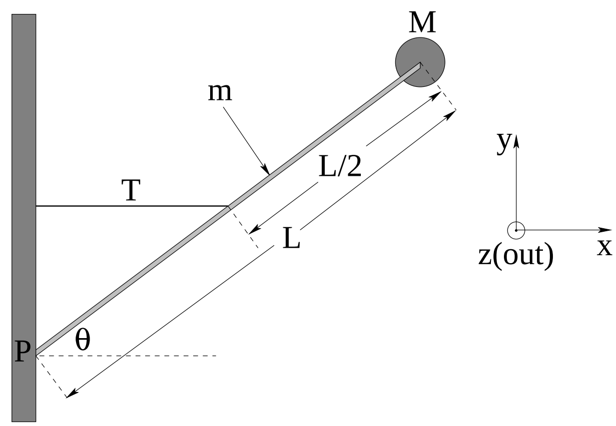
Fortunately, Tom has a perfectly rigid 4×4 beam (of negligible mass), and there is a solid rock (that can withstand essentially any push) a distance $D = 5$ meters from the wall to brace it on. Even more fortuitously, Tom has taken introductory physics! He therefore cuts the beam to lean against the wall a height H as shown and proceeds to walk up the beam towards the wall..

a) Assuming that the beam is frictionless where it presses against the wall what is the largest value of H that will permit him to knock down the wall if he walks to the end of the beam so that his horizontal distance $x = D$?

b) Suppose that he has cut the beam so that it rests a height $H = 1$ meter above the ground against the wall. What is his horizontal position x when the beam knocks down the wall (if it does at all)?

c) Of course the beam is ***not*** frictionless where it rests against the wall. Does this fact mean that, for any given value of H, the wall is easier to knock down (happens when he has walked a smaller horizontal distance x toward the wall), harder to knock down (happens when he has walked a greater horizontal distance x), or just the same (it falls at the same horizontal distance x) as it is without friction?

Problem 482. problems-1/statics-pr-rod-mass-on-hinge.tex

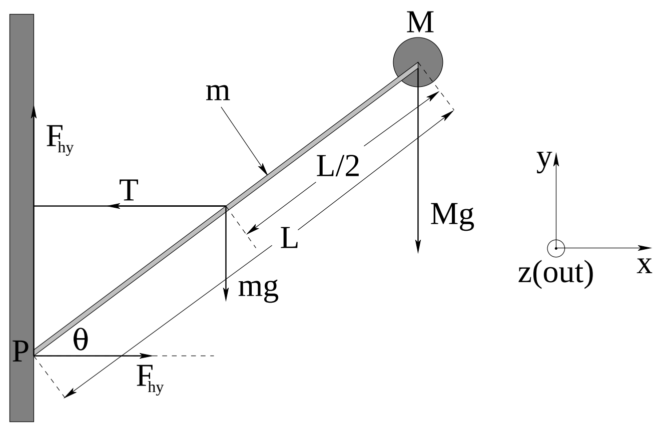


A small round mass M sits on the end of a rod of length L and mass m that is attached to a wall with a hinge at point P. The rod is kept from falling by a thin (massless) string attached horizontally between the midpoint of the rod ($L/2$ from either end) and the wall. The rod makes an angle θ with the ground. Find:

a) the tension T in the string;

b) the ***vector*** force $\vec{\boldsymbol{F}}_h$ exerted by the hinge on the rod.

Problem 483. problems-1/statics-pr-rod-mass-on-hinge-soln.tex



A small round mass M sits on the end of a rod of length L and mass m that is attached to a wall with a hinge at point P. The rod is kept from falling by a thin (massless) string attached horizontally between the midpoint of the rod ($L/2$ from either end) and the wall. The rod makes an angle θ with the ground. Find:

a) the tension T in the string;

b) the ***vector*** force $\vec{\boldsymbol{F}}_h$ exerted by the hinge on the rod.

Solution: This is a classic **two force, one torque** problem, solving for T, F_{hx}, and F_{hy} as shown above.

$$F_x = F_{hx} - T = 0 \quad \Rightarrow \quad F_{hx} = T$$

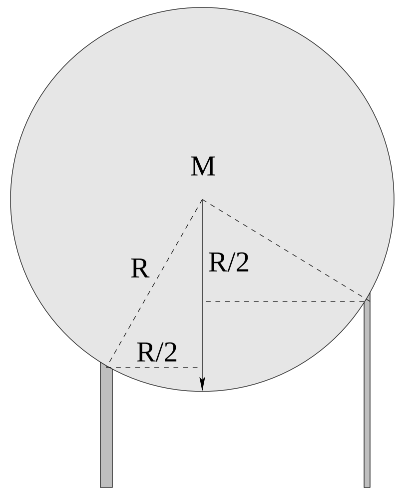
$$F_y = F_{hy} - mg - Mg \quad \Rightarrow \quad \boxed{F_{yh} = (m+M)g}$$

$$\tau_z = T\frac{\cancel{L}}{2}\sin\theta - mg\frac{\cancel{L}}{2}\cos\theta - Mg\cancel{L}\cos\theta = 0 \quad \Rightarrow \quad \boxed{T = F_{hx} = (mg + 2Mg)\cot\theta}$$

It is not really necessary to explicitly write out the vector, but here it is anyway:

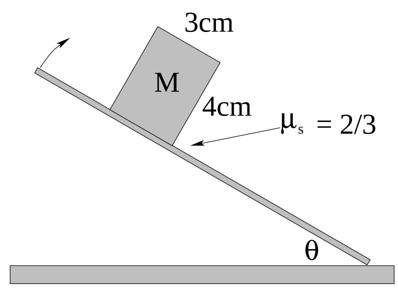
$$\boxed{\vec{\boldsymbol{F}}_h = \{(mg + 2Mg)\cot\theta\}\,\hat{\boldsymbol{x}} + \{(m+M)g\}\,\hat{\boldsymbol{y}}}$$

Problem 484. problems-1/statics-pr-supporting-a-disk.tex



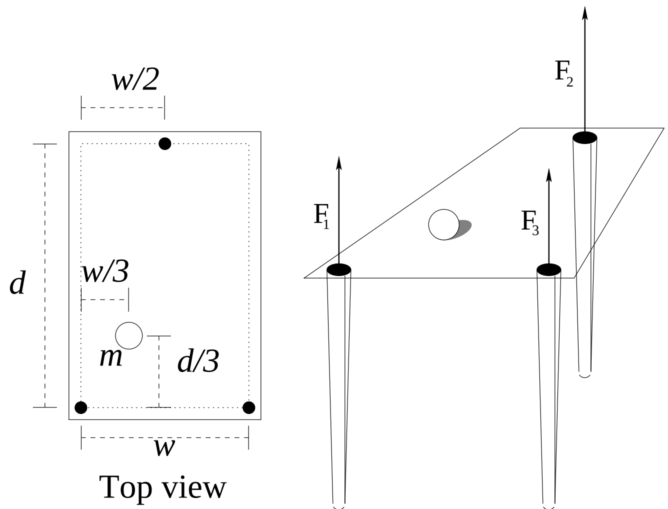
Find the force exerted by each of the two rods supporting the disk of mass M and radius R as shown. Note that the two triangles shown are both 30-60-90 triangles with side opposite the small angle of $R/2$.

Problem 485. problems-1/statics-pr-tipping-vs-slipping.tex



A block of mass M with width 3 cm and height 4 cm sits on a rough plank. The coefficient of static friction between the plank and the block is $\mu_s = 2/3$. The plank is slowly tipped up. Does the block slip first, or tip first?

Problem 486. problems-1/statics-pr-vector-torque-plexiglass-table.tex



The figure below shows a mass m placed on a table consisting of three narrow cylindrical legs at the positions shown with a light (presume massless) sheet of Plexiglas placed on top. Find the vertical forces F_1, F_2, F_3 exerted on the Plexiglas by each leg when the mass is at rest in the position shown.

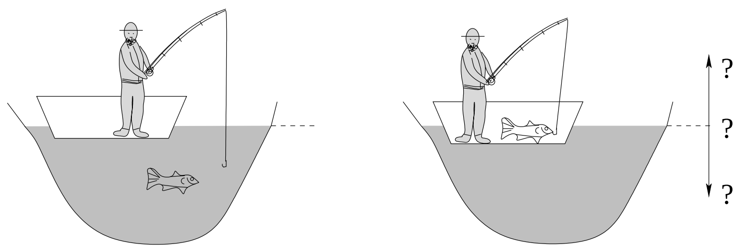
Chapter 10

Fluids

10.1 Fluids

10.1.1 Multiple Choice Problems

Problem 487. problems-1/fluids-mc-boat-catches-fish.tex



I go fishing in a pond where there is a big, fat fish perfectly suspended by buoyant forces in the water under the boat. I catch him and reel him in up into the boat. As I do so, the level of the water in the pond will:

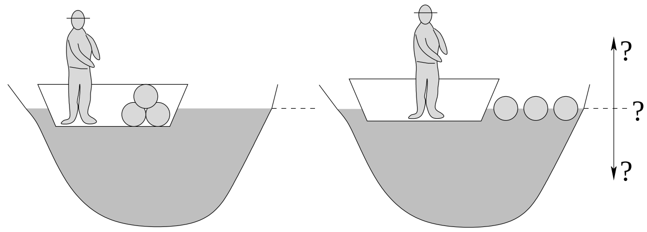
a) Rise a bit.

b) Fall a bit.

c) Remain unchanged.

d) Can't tell from the information given (it depends, for example, on the kind of fish...).

Problem 488. problems-1/fluids-mc-boat-floats-wood.tex



A person stands in a boat floating on a pond and containing several pieces of wood. He throws the wood out of the boat so that it floats on the surface of the pond. The water level of the pond will:

a) Rise a bit.

b) Fall a bit.

c) Remain unchanged.

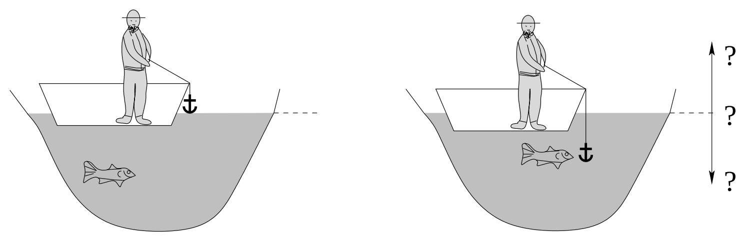
d) Can't tell from the information given (it depends on, for example, the shape of the boat, the mass of the person, whether the pond is located on the Earth or on Mars...).

Problem 489. problems-1/fluids-mc-boat-floats-wood-soln.tex

The wood is “floating” in both cases, so the total displacement of the water level:

c) Remain unchanged.

Problem 490. problems-1/fluids-mc-boat-lowers-anchor.tex



I go fishing in a pond and spot a big, fat fish in the water under the boat and decide to anchor for a bit to try to catch it. As I lower the anchor into the water (so that it ***hangs suspended*** under the boat as shown) level of the water in the pond will:

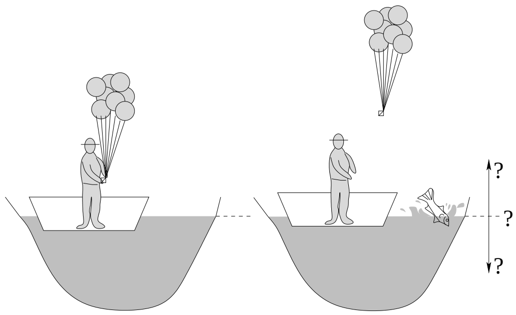
a) Rise a bit.

b) Fall a bit.

c) Remain unchanged.

d) Can't tell from the information given (it depends, for example, on whether the anchor is made of iron or lead...).

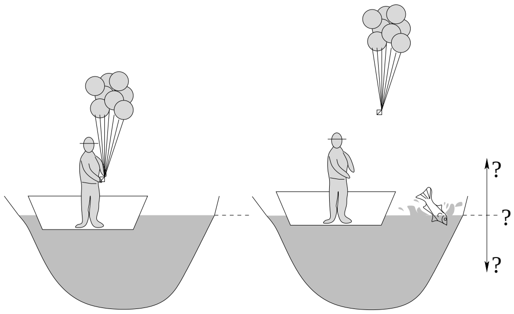
Problem 491. problems-1/fluids-mc-boat-releases-balloons.tex



The fish aren't biting, so a person standing in a boat floating on a pond and inflates a bunch of helium balloons instead. Then an enormous fish jumps nearby and he is so startled that he accidentally releases the balloons. As he does so, the water level of the pond will:

- ☐ Rise a bit.
- ☐ Fall a bit.
- ☐ Remain unchanged.
- ☐ Can't tell from the information given.

Problem 492. problems-1/fluids-mc-boat-releases-balloons-soln.tex

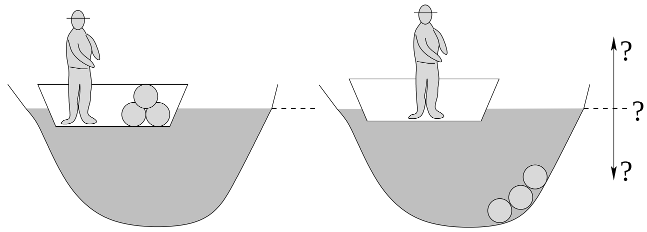


The fish aren't biting, so a person standing in a boat floating on a pond and inflates a bunch of helium balloons instead. Then an enormous fish jumps nearby and he is so startled that he accidentally releases the balloons. As he does so, the water level of the pond will:

- ☒ Rise a bit.
- ☐ Fall a bit.
- ☐ Remain unchanged.
- ☐ Can't tell from the information given.

Explanation: The balloons support part of the weight of the person, so the boat needs to displace less water while he's holding them to support the weight of person and boat.

Problem 493. problems-1/fluids-mc-boat-sinks-rocks.tex



A person stands in a boat floating on a pond and containing several large, round, rocks. He throws the rocks out of the boat so that they sink to the bottom of the pond. The water level of the pond will:

a) Rise a bit.

b) Fall a bit.

c) Remain unchanged.

d) Can't tell from the information given (it depends on, for example, the shape of the boat, the mass of the person, whether the pond is located on the Earth or on Mars...).

Problem 494. problems-1/fluids-mc-boat-sinks-rocks-soln.tex

Before, the boat displaces a volume of water whose weight ***equals the weight of the rocks***. After, the rocks themselves displace a volume of water ***equal to the volume of the rocks***. This is surely much less than the volume of the water with the same weight as the rocks! Hence:

b) Fall a bit.

Problem 495. problems-1/fluids-mc-buoyant-boxes.tex

Two wooden boxes with the same shape but different density are held in the same orientation beneath the surface of a large container of water. Box A has a smaller average density than box B. When the boxes are released, they accelerate up towards the surface. Which box has the greater acceleration when they are initially released?

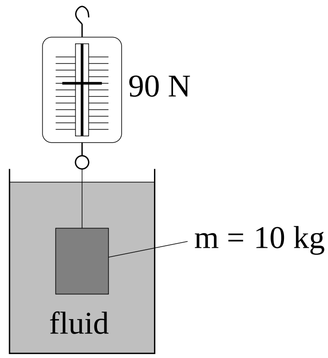
a) Box A.

b) Box B.

c) They are the same.

d) We cannot tell from the information given.

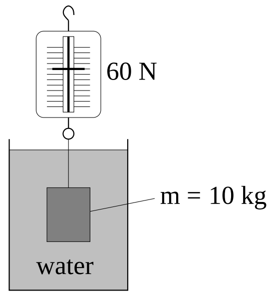
Problem 496. problems-1/fluids-mc-density-of-fluid-from-stone.tex



A block of lead has a mass of $m = 10$ kg (that weighs 100 Newtons in air) and a density of $\rho = 1.1 \times 10^4$ kg/m^3 is hung from a scale and immersed in an unknown fluid. The scale then reads 90 Newtons. What is the approximate ***density of the fluid***? (Use $g = 10.$ m/sec^2)

$\rho_f = \boxed{}$ kg/m^3

Problem 497. problems-1/fluids-mc-density-of-stone-underwater.tex



A stone of mass $m = 10$ kg (that weighs 100 Newtons in air) is hung from a scale and immersed in water. The scale reads 60 Newtons. What is the ***density*** of the stone? (Use $g = 10$ m/sec^2)

a) $\rho = 1000$ kg/m^3

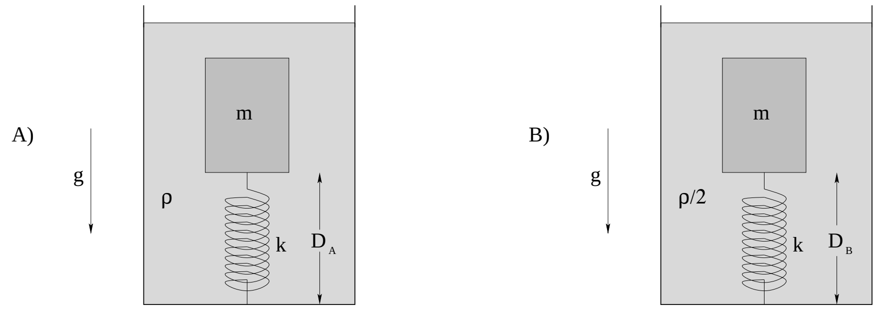
b) $\rho = 4000$ kg/m^3

c) $\rho = 6000$ kg/m^3

d) $\rho = 1667$ kg/m^3

e) $\rho = 2500$ kg/m^3

Problem 498. problems-1/fluids-mc-floating-mass-on-spring-different-rho.tex

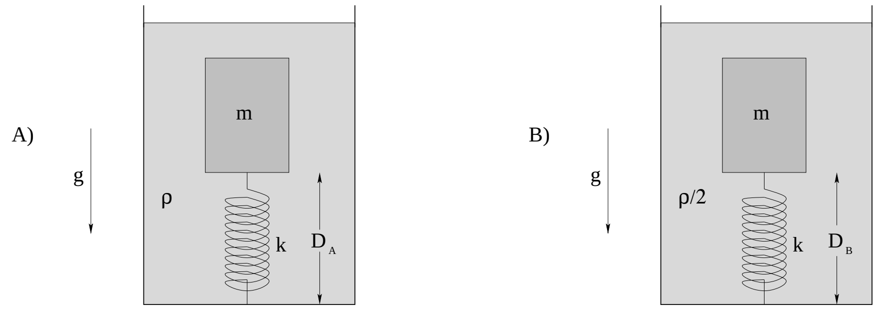


In the figures above, two ***identical springs*** (with spring constant k) are attached to the bottoms of two ***identical*** containers filled with two ***different*** fluids with densities (A) ρ and (B) $\rho/2$ respectively. ***Identical*** wooden blocks (that would ordinarily float in both fluids) are attached to these springs, which stretch out to total lengths D_A and D_B and suspend the blocks so that they are fully immersed as shown. Near-Earth gravity (g) is also the same for both scenarios.

Identify the true statement:

☐ $\boldsymbol{D_A > D_B}$ ☐ $\boldsymbol{D_A < D_B}$ ☐ $\boldsymbol{D_A = D_B}$

Problem 499. problems-1/fluids-mc-floating-mass-on-spring-different-rho-soln.tex



In the figures above, two ***identical springs*** (with spring constant k) are attached to the bottoms of two ***identical*** containers filled with two ***different*** fluids with densities (A) ρ and (B) $\rho/2$ respectively. ***Identical*** wooden blocks (that would ordinarily float in both fluids) are attached to these springs, which stretch out to total lengths D_A and D_B and suspend the blocks so that they are fully immersed as shown. Near-Earth gravity (g) is also the same for both scenarios.

Identify the true statement:

☒ $\boldsymbol{D_A > D_B}$ ☐ $\boldsymbol{D_A < D_B}$ ☐ $\boldsymbol{D_A = D_B}$

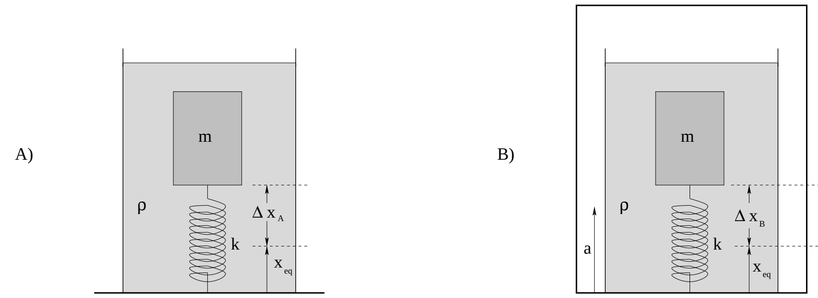
Solution: Use ***Force balance***, including the *buoyant force* acting on each block. Note that we don't need to know $x_{A,B}$, how much each spring stretches in absolute terms, we only need to know which one must exert the stronger downward force to keep the block immersed in equilibrium as it must stretch *more*. Note that the volume, weight, and so on of the blocks are all the same.

$$mg + kx_A - \rho V g = 0 \quad \Rightarrow \quad x_A = \frac{\rho V g}{mgk}$$

$$mg + kx_B - (\rho/2)Vg = 0 \quad \Rightarrow \quad x_B = \frac{\rho V g}{2mgk} < x_A \qquad \Rightarrow \qquad \boxed{D_A > D_B}$$

In words: "The buoyant force in A is clearly greater, so the spring has to pull down harder to keep it submerged, so its stretch must be larger." No real need for the algebraic argument if you think clearly.

Problem 500. problems-1/fluids-mc-floating-mass-on-spring-elevator.tex



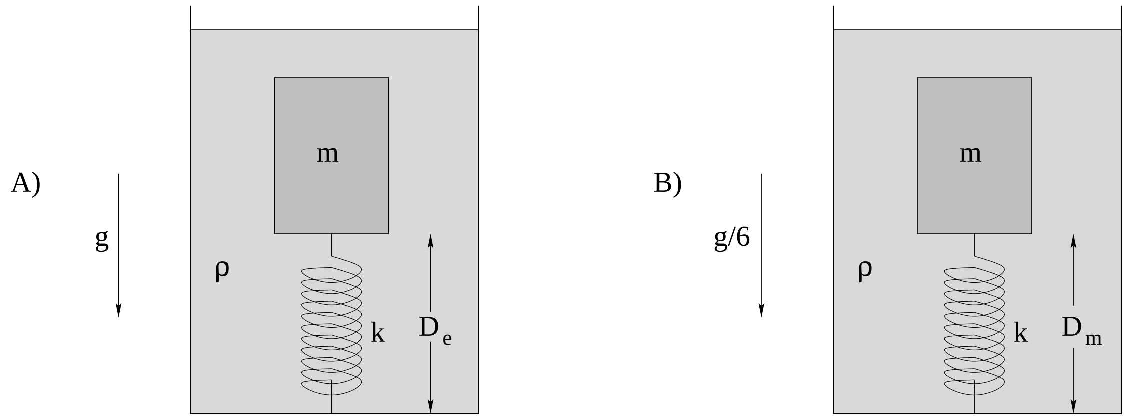
In the figures above, two identical springs (with spring constant k) are attached to the bottoms of two identical containers filled with water (density ρ). At the other end, the springs are attached to identical wooden blocks that would ordinarily ***float*** on the water so that they are completely submerged.

The container on the left (A) is located at rest on the ground, and Δx_A is the total distance that its spring is stretched from its equilibrium length when the block is stationary relative to container A. The container on the right (B) is located on the floor of an elevator accelerating upwards with an acceleration a, and Δx_B is the total length that its spring is stretched from its equilibrium length when the block is stationary relative to container B (accelerating upwards with the elevator).

Circle the true statement:

$\boldsymbol{\Delta x_A > \Delta x_B}$ $\qquad$ $\boldsymbol{\Delta x_A < \Delta x_b}$ $\qquad$ $\boldsymbol{\Delta x_A = \Delta x_B}$

Problem 501. problems-1/fluids-mc-floating-mass-on-spring-moon.tex



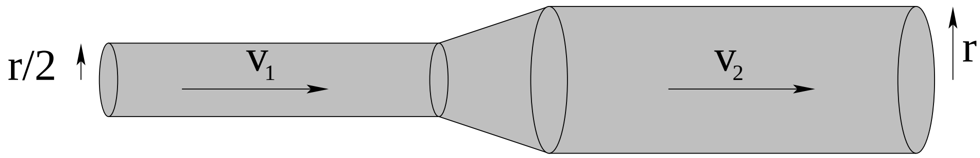
In the figures above, two identical springs (with spring constant k) are attached to the bottoms of two identical containers filled with water (density ρ). At the other end, the springs are attached to identical wooden blocks that would ordinarily ***float*** on the water so that they are completely submerged.

The apparatus on the left (A) is located on the Earth's surface, where the acceleration due to gravity is g. The apparatus on the right (B) is located on the moon, where the acceleration due to gravity is $g/6$. D_e is the total length of the stretched spring on the Earth, D_m is the total length of the stretched spring on the moon.

Circle the true statement:

$\boldsymbol{D_e > D_m}$ $\qquad$ $\boldsymbol{D_e < D_m}$ $\qquad$ $\boldsymbol{D_e = D_m}$

Problem 502. problems-1/fluids-mc-flow-constricted-pipe-reverse.tex



Water flows at speed v_1 in a pipe with radius $r/2$ and passes into a pipe with radius r through a smooth constriction as shown. Neglect viscosity. Select the statements that correctly describes v_2, the speed in the wider section of pipe, and the relative pressure in the narrower and wider segments of pipe

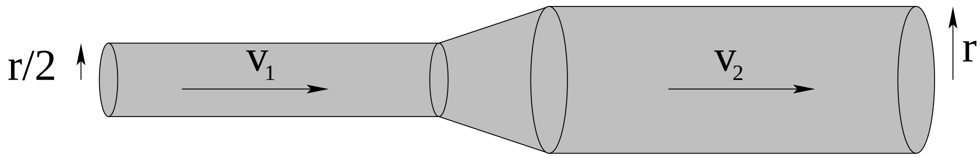
---

☐ $v_2 = 4v_1$ ☐ $v_2 = 2v_1$ ☐ $v_2 = v_1$ ☐ $v_2 = \frac{1}{2}v_1$ ☐ $v_2 = \frac{1}{4}v_1$

☐ $P_1 > P_2$ ☐ $P_1 = P_2$ ☐ $P_1 < P_2$

☐ We cannot tell from Bernoulli's equation without knowing the fluid's density ρ.

Problem 503. problems-1/fluids-mc-flow-constricted-pipe-reverse-soln.tex



Water flows at speed v_1 in a pipe with radius $r/2$ and passes into a pipe with radius r through a smooth constriction as shown. Neglect viscosity. Select the statements that correctly describes v_2, the speed in the wider section of pipe, and the relative pressure in the narrower and wider segments of pipe

---

☐ $v_2 = 4v_1$ ☐ $v_2 = 2v_1$ ☐ $v_2 = v_1$ ☐ $v_2 = \frac{1}{2}v_1$ ☒ $v_2 = \frac{1}{4}v_1$

☐ $P_1 > P_2$ ☐ $P_1 = P_2$ ☒ $P_1 < P_2$

☐ We cannot tell from Bernoulli's equation without knowing the fluid's density ρ.

Solution: Use:

$$A_1 v_1 = A_2 v_2$$

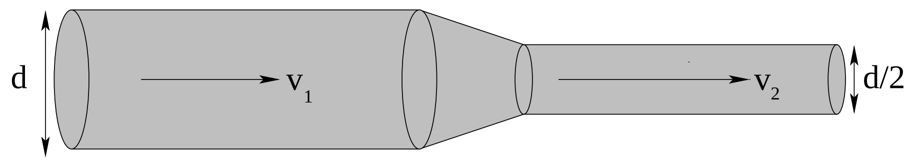
with $A_1 = \pi(r/2)^2 = \pi r^2/4 = A_2/4$ to conclude that $\boxed{\boldsymbol{v_2 = v_1/4}}$.

Bernoulli (at constant height) also tells us that:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Since $v_1 > v_2$, $\boxed{\boldsymbol{P_1 < P_2}}$. The pressure increases as the fluid slows down!

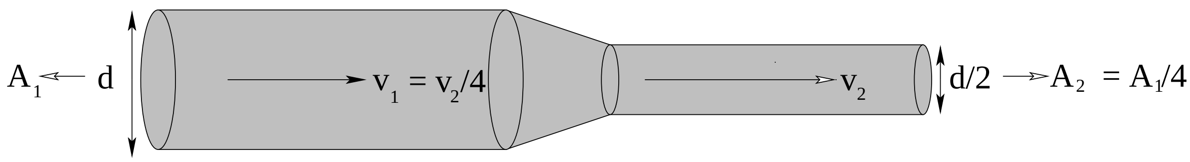
Problem 504. problems-1/fluids-mc-flow-constricted-pipe.tex



Water flows at speed v_1 in a pipe with diameter d and passes into a pipe with diameter $d/2$ through a smooth constriction as shown. Select the statement that correctly describes v_2, the speed in the narrower pipe.

☐ $v_2 = \frac{1}{2}v_1$ ☐ $v_2 = \frac{1}{4}v_1$ ☐ $v_2 = 2v_1$ ☐ $v_2 = 4v_1$ ☐ $v_2 = v_1$

Problem 505. problems-1/fluids-mc-flow-constricted-pipe-soln.tex



Water flows at speed v_1 in a pipe with diameter d and passes into a pipe with diameter $d/2$ through a smooth constriction as shown. Select the statement that correctly describes v_2, the speed in the narrower pipe.

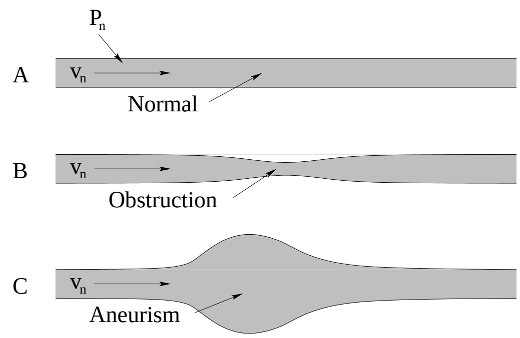
☐ $v_2 = \frac{1}{2}v_1$ ☐ $v_2 = \frac{1}{4}v_1$ ☐ $v_2 = 2v_1$ ☒ $v_2 = 4v_1$ ☐ $v_2 = v_1$

Solution: First note that (as indicated on the figure above) $A_2 = A_1/4$. Conservation of Flow then yields:

$$A_1 v_1 = A_2 v_2 = A_1 v_2/4 \quad \text{and (rearranging)} \qquad \Rightarrow \quad \boxed{v_2 = 4v_1}$$

10.1.2 Short Answer Problems

Problem 506. problems-1/fluids-sa-aneurism-pressure-flow.tex



Consider the models above of a normal blood vessel (A), an obstructed blood vessel (B) and an aneurism (C). In case (A) blood is flowing from left to right at a "normal" fluid velocity v_n and pressure P_n. Assume that the blood pressure right before and after the obstruction or aneurism is also P_n. Neglect viscosity while answering the following questions:

a) Is the blood pressure in the obstructed region P_o in (B) higher or lower than P_n?

b) Is the blood pressure in the aneurism P_a in (C) higher or lower than P_n?

Problem 507. problems-1/fluids-sa-aneurism-pressure-flow-soln.tex

Pressure in a confined, flowing fluid ***increases where the fluid speed decreases*** and ***decreases where the fluid speed increases***.

Hence $P_o < P_n$ and $P_a > P_n$.

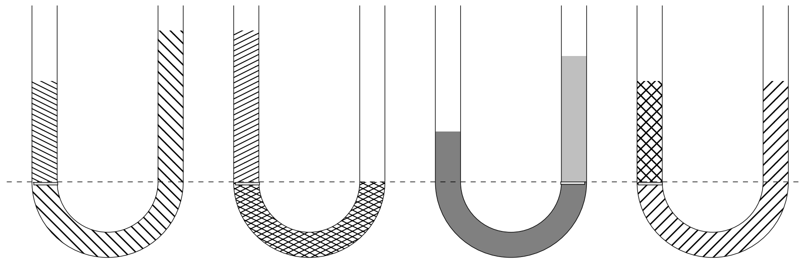
Problem 508. problems-1/fluids-sa-balloon-in-car.tex

A small boy is riding in a minivan with the windows closed, holding a helium balloon. The van goes around a corner to the left. Does the balloon swing to the left, still pull straight up, or swing to the right as the van swings around the corner?

Problem 509. problems-1/fluids-sa-breathing-underwater-through-a-tube.tex

In adventure movies, the hero is often being chased by the bad guys and escapes by hiding deep underwater and breathing through a tube of some sort. Assuming that you can barely manage to breathe if a 500 Newton person is standing directly on your chest while you are lying on the floor, ***estimate*** the maximum depth (of your chest) where one is likely to have the muscular strength to be able to breathe through a rigid tube extending to the surface. Your estimate should be quantitative and you should support it with both a very short piece of algebra and a picture clearly showing the forces you must work against to breathe underwater.

Problem 510. problems-1/fluids-sa-four-utubes-1.tex

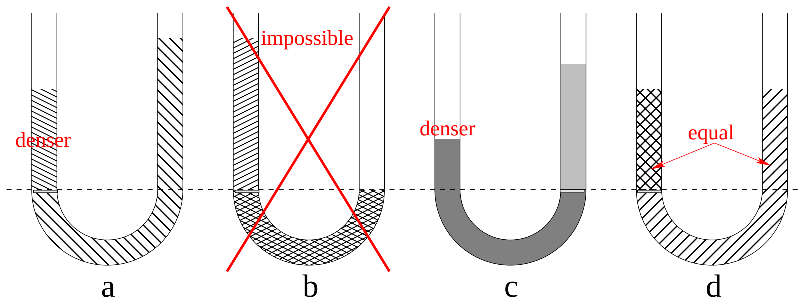


Two different incompressible fluids separated by a thin (massless, frictionless) piston so that they cannot mix are open to the atmosphere and are in static equilibrium in each of the four U-tubes pictured above.

a) One of the four U-tubes ***makes no sense*** (cannot be in equilibrium). Circle it and label it "impossible".

b) Underneath each u-tube that ***does*** make sense indicate whether the fluid at the top of the ***left-hand side*** of the "U" is ***denser than, less dense than, or the same density as*** the fluid at the top of the ***right***-hand side.

Problem 511. problems-1/fluids-sa-four-utubes-1-soln.tex



Two different incompressible fluids separated by a thin (massless, frictionless) piston so that they cannot mix are open to the atmosphere and are in static equilibrium in each of the four U-tubes pictured above.

a) One of the four U-tubes ***makes no sense*** (cannot be in equilibrium). Circle it and label it "impossible".

b) Underneath each u-tube that ***does*** make sense indicate whether the fluid at the top of the ***left-hand side*** of the "U" is ***denser than, less dense than, or the same density as*** the fluid at the top of the ***right***-hand side.

Solution: The pressure at the meniscus between the fluids must be equal descending from (equal) air pressure at the top of the fluid in the U-tube. The pressure at the depth equal to the height of the fluid column above the meniscus is:

$$P_m = P_a + \rho_{l,r} g H_{l,r}$$

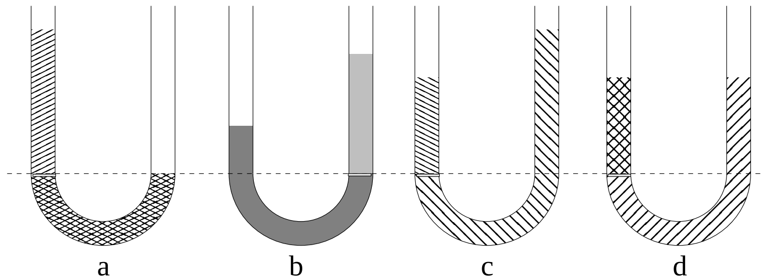
(for the l(eft) or r(ight) columns respectively). Equating the two:

$$\cancel{P_a} + \rho_l \cancel{g} H_l = \cancel{P_a} + \rho_r \cancel{g} H_r$$

and by inspection, the fluid with the greatest height above the meniscus must have the lowest density, and in d) the columns have equal heights and hence have the same density.

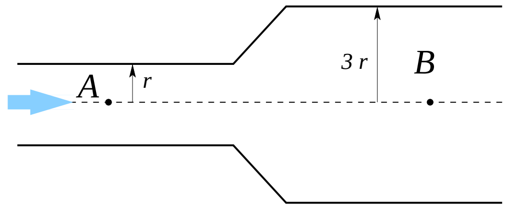
b) is impossible because (at least as far as I know) there are no fluids with a density ***equal to that of air!*** at standard temperature and pressure, and that's what the figure implies – that the fluid on the left has the same density as air.

Problem 512. problems-1/fluids-sa-four-utubes-2.tex



(6 points) Two different incompressible fluids separated by a thin (massless, frictionless) piston so that they cannot mix are open to the atmosphere and presumably in static equilibrium in each of the four u-tubes pictured above. One of the four u-tubes makes no sense (cannot be in equilibrium). Circle it. Underneath each u-tube that *does* make sense indicate whether the fluid at the top of the **left**-hand side of the "u" is denser than, less dense than, or the same density as the fluid at the top of the **right**-hand side. Briefly indicate your reasons.

Problem 513. problems-1/fluids-sa-horizontal-necked-pipe.tex



An incompressible, non-viscous fluid flows from the left to the right through a pipe of varying radius as shown in the figure. Let us compare the fluid at point A and the fluid at point B.

Answer questions (a) to (c) below by entering the letters "A", "B", or if the same magnitude, "=" in the provided boxes.

[*Showing your work is recommended, but not mandatory.*]

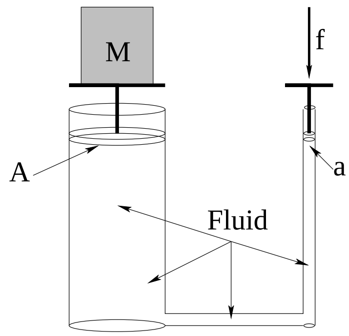
a) The speed of the fluid is larger at point $\square$;

b) Volume flow rate (Q) is higher at point $\square$;

c) Pressure is higher at point $\square$;

d) Write down the ratio of the flow speed at two points: $\dfrac{v_B}{v_A} = \square$.

Problem 514. problems-1/fluids-sa-hydraulic-lift-2.tex



A piston of small cross sectional area a is used in a hydraulic press to exert a force f on the enclosed liquid. A connecting pipe leads to the larger piston of cross sectional area A, so that $A > a$. The two pistons are at the same height. The weight $w = Mg$ that can be supported by the larger piston is

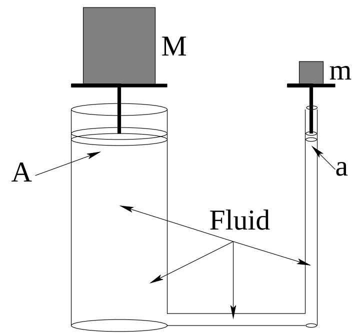
(a) $w > f$

(b) $w < f$

(c) $w = f$

(d) depends on whether the liquid is compressible or not.

Problem 515. problems-1/fluids-sa-hydraulic-lift.tex



The pair of coupled piston-and-cylinders shown above are sitting in air and filled with an incompressible fluid. The entire system is in static equilibrium (so nothing moves). The cross-sectional area of the large piston is A; the cross-sectional area of the small piston is a. In this case we know that:

a) $M = \frac{A}{a}m$

b) $M = \frac{a}{A}m$

c) $M = \sqrt{\frac{A}{a}}m$

d) $M = m$

e) We cannot tell what M is relative to m without more information.

Problem 516. problems-1/fluids-sa-hydraulic-lift-soln.tex

The pressure at the top (just under the pistons) must be the same on both sides as the stationary fluid is connected and at the same height. This pressure supports the weight on both sides. So for the left vertical force balance is:

$$F_{\rm left} = PA - Mg = 0$$

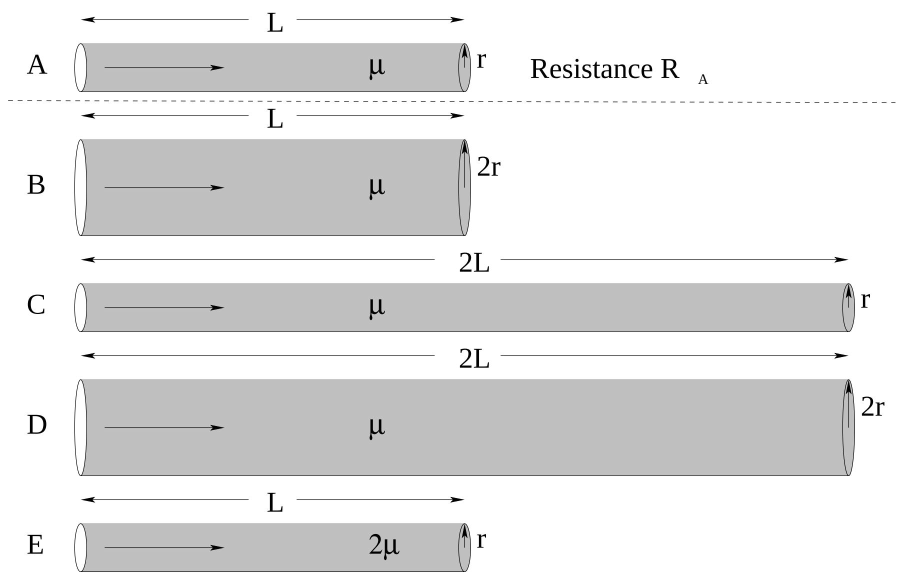
and on the right:

$$F_{\rm right} = Pa - mg = 0$$

Equating P and cancelling g, it is then easy to show that:

a) $M = \frac{A}{a} m$

Problem 517. problems-1/fluids-sa-poiseiulles-law-2.tex



In the figure above, fluids of the given viscosities flow through circular pipes A-E with the given dimensions. The ***resistance*** to fluid flow of circular pipe A is known to be R_A. What are the resistances of the other four pipes in terms of R_A?

$\frac{R_B}{R_A} =$ $\boxed{}$

$\frac{R_C}{R_A} =$ $\boxed{}$

$\frac{R_D}{R_A} =$ $\boxed{}$

$\frac{R_E}{R_A} =$ $\boxed{}$

Problem 518. problems-1/fluids-sa-poiseiulles-law-2-soln.tex

The (Poiseiulle) formula for resistance of a circular pipe (that is used in e.g. in $\Delta P = IR$) is:

$$R = \frac{8\mu L}{\pi r^4}$$

Hence doubling L doubles it, doubling η doubles it. Doubling r reduces it by 16 (etc).

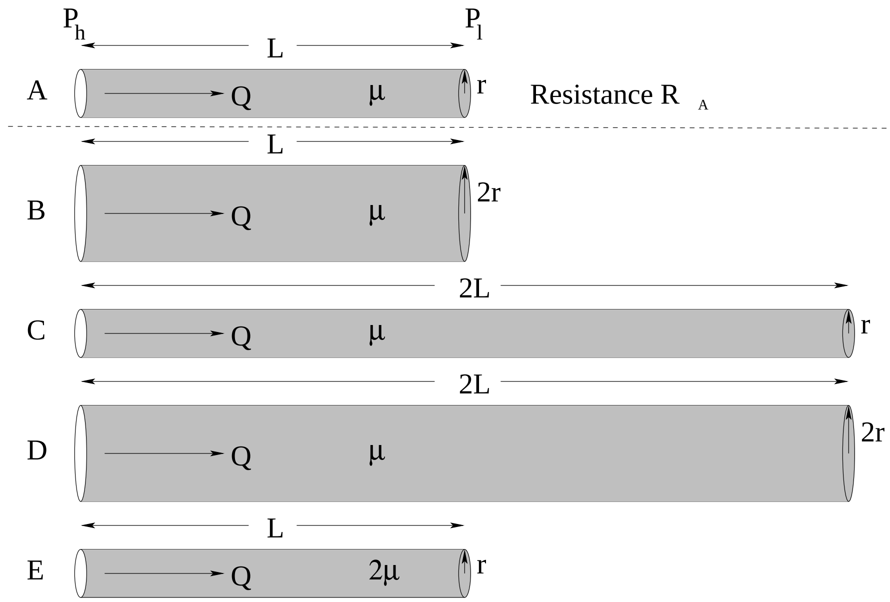
$\frac{R_B}{R_A} = \frac{1}{16}$

$\frac{R_C}{R_A} = 2$

$\frac{R_D}{R_A} = \frac{1}{8}$

$\frac{R_E}{R_A} = 2$

Problem 519. problems-1/fluids-sa-poiseiulles-law-3.tex



In the figure above, fluids of the given viscosities flow through circular pipes A-E with the given dimensions. In all cases the volumetric flow through the pipes is held ***constant*** at Q by varying the pressure difference $\Delta P_i = P_{\text{high}} - P_{\text{low}}$ across each $(i = A, B, C, D, E)$ pictured pipe segment.

The pressure difference that maintains flow Q fluid flow of circular pipe A is defined to be ΔP_A. What are the pressure differences across the other four pipes in terms of ΔP_A?

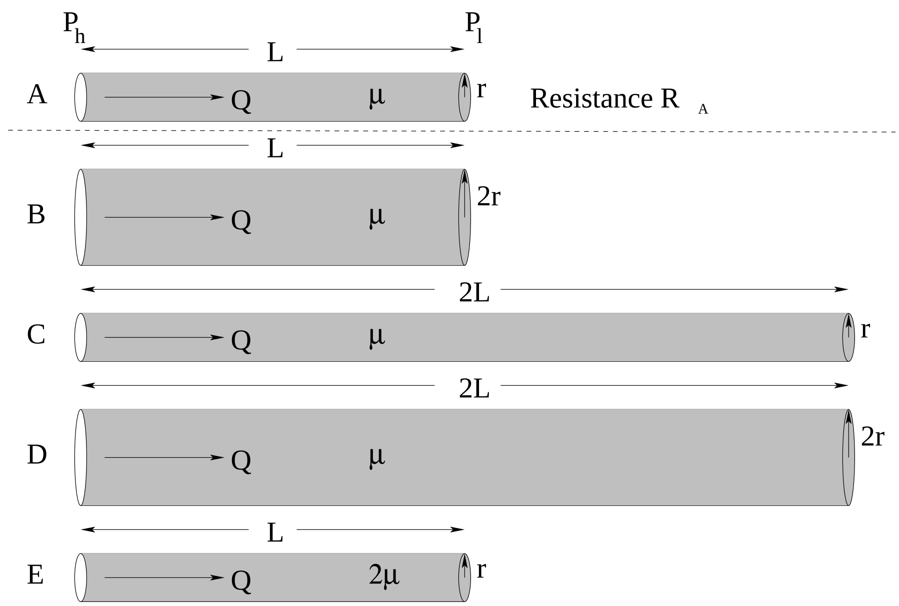
$\frac{\Delta P_B}{\Delta P_A} = \boxed{}$

$\frac{\Delta P_C}{\Delta P_A} = \boxed{}$

$\frac{\Delta P_D}{\Delta P_A} = \boxed{}$

$\frac{\Delta P_E}{\Delta P_A} = \boxed{}$

Problem 520. problems-1/fluids-sa-poiseiulles-law-3-soln.tex



In the figure above, fluids of the given viscosities flow through circular pipes A-E with the given dimensions. In all cases the volumetric flow through the pipes is held ***constant*** at Q by varying the pressure difference $\Delta P_i = P_{\text{high}} - P_{\text{low}}$ across each $(i = A, B, C, D, E)$ pictured pipe segment.

The pressure difference that maintains flow Q fluid flow of circular pipe A is defined to be ΔP_A. What are the pressure differences across the other four pipes in terms of ΔP_A?

The easy way to do this one (since Q is constant) is to use:

$$\Delta P = QR$$

That way all we have to consider is the scaling of the following expression:

$$R = \frac{8\mu L}{\pi r^4}$$

where most of the variables are the same from picture to picture and cancel. Thus:

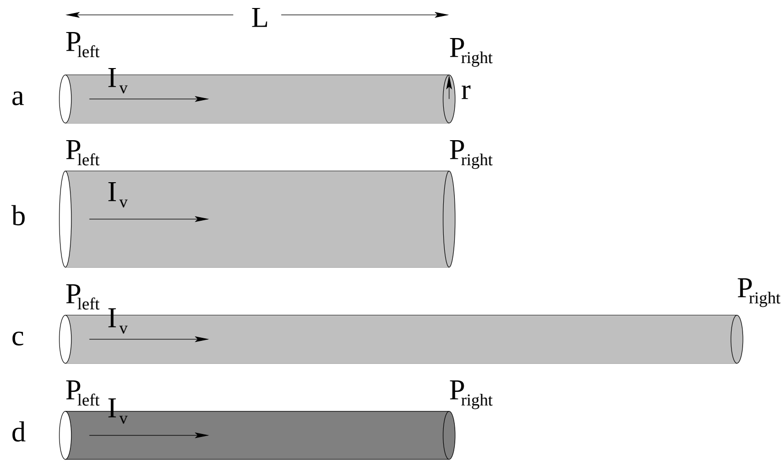
$$\frac{\Delta P_B}{\Delta P_A} = \frac{r^4}{(2r)^4} = \frac{1}{16}$$

$$\frac{\Delta P_C}{\Delta P_A} = \frac{2L}{L} = 2$$

$$\frac{\Delta P_D}{\Delta P_A} = \frac{2L}{L}\frac{r^4}{(2r)^4} = \frac{1}{8}$$

$$\frac{\Delta P_E}{\Delta P_A} = \frac{2\mu}{\mu} = 2$$

Problem 521. problems-1/fluids-sa-poiseiulles-law.tex



Use *Poiseuille's Law* to answer the following questions:

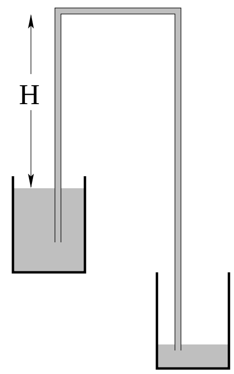
a) Is $\Delta P_a = P_{\rm left} - P_{\rm right}$ greater than, less than, or equal to zero in figure a) above, where blood flows at a rate I_v horizontally through a blood vessel with constant radius r and some length L against the **resistance** of that vessel?

b) If the radius r **increases** (while flow I_v and length L remain the same as in a), does the pressure difference ΔP_b increase, decrease, or remain the same compared to ΔP_a?

c) If the length **increases** (while flow I_v and radius r remains the same as in a), does the pressure difference ΔP_c increase, decrease, or remain the same compared to ΔP_a?

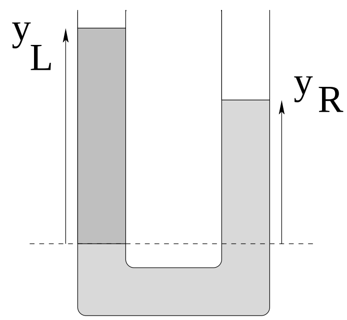
d) If the viscosity μ of the blood **increases** (where flow I_v, radius r, and length L are all unchanged compared to a) do you expect the pressure difference ΔP_d difference across a blood vessel to increase, decrease, or remain the same compared to ΔP_a?

Problem 522. problems-1/fluids-sa-siphon.tex



A **siphon** is a device for lifting water out of one (higher) reservoir and delivering it another (lower) reservoir as shown above. Estimate the probable maximum height H one can lift the water above the upper reservoir's water level before the tube descends into the lower reservoir. Explain your reasoning – how, and where, will the siphon fail?

Problem 523. problems-1/fluids-sa-utube-fluid-height.tex

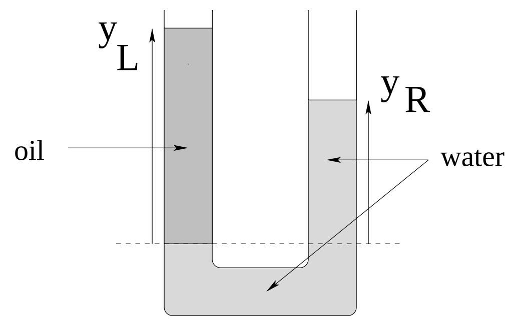


A vertical U-tube open to the air at the top is filled with oil (density ρ_o) on one side and water (density ρ_w) on the other, where $\rho_o < \rho_w$.

a) Make your own diagram of the problem and clearly label the oil and the water.

b) Find (derive) an expression for $\Delta y = y_L - y_R$, the ***difference*** in the heights of the two columns in terms of y_L.

Problem 524. problems-1/fluids-sa-utube-fluid-height-soln.tex



The static pressure at the interface must be the same accumulated descending from the ***same*** pressure on the top surface on the left or from the top surface on the right. Hence:

$$P_0 + \rho_L g y_L = P_0 + \rho_R g y_R$$

or

$$\frac{\rho_L}{\rho_R} = \frac{y_R}{y_L}$$

By inspection of this formula, the lower density has to be the higher column, so oil is on the right. We can also rearrange the formula:

$$y_R = \frac{\rho_L}{\rho_R} y_L$$

so that:

$$\Delta y = y_L - y_R = (1 - \frac{\rho_L}{\rho_R}) y_L$$

Problem 525. problems-1/fluids-sa-walking-in-a-pool.tex

People with vascular disease or varicose veins (a disorder where the veins in one's lower extremeties become swollen and distended with fluid) are often told to walk in water 1-1.5 meters deep. Explain why.

10.1.3 Ranking Problems

Problem 526. problems-1/fluids-ra-archimedes-objects-2.tex

Four large identical beakers are filled with water and also contain objects in static equilibrium with the water and beaker (they are not attached to or supported by anything outside of the beaker. The objects are, ***listed in the order of strictly decreasing density***:

a) A solid gold coin that has a mass of 100 grams;

b) A cast aluminum frog that has a mass of 100 grams;

c) An ice cube that has a mass of 100 grams;

d) A wooden carved monkey that has a mass of 100 grams.

You remove each object from the water in its beaker and measure the ***drop in water depth*** Δd_i, $i = a, b, c, d$.

Rank the Δd_i you expect to observe in this experiment from ***smallest to largest***. As always, in the case that some of the Δd_i are equal to neighbors, indicate that explicitly.

Problem 527. problems-1/fluids-ra-archimedes-objects-2-soln.tex

Four large identical beakers are filled with water and also contain objects in static equilibrium with the water and beaker (they are not attached to or supported by anything outside of the beaker. The objects are, ***listed in the order of strictly decreasing density***:

a) A solid gold coin that has a mass of 100 grams;

b) A cast aluminum frog that has a mass of 100 grams;

c) An ice cube that has a mass of 100 grams;

d) A wooden carved monkey that has a mass of 100 grams.

You remove each object from the water in its beaker and measure the ***drop in water depth*** Δd_i, $i = a, b, c, d$.

Rank the Δd_i you expect to observe in this experiment from ***smallest to largest***. As always, in the case that some of the Δd_i are equal to neighbors, indicate that explicitly.

Solution: The way to do this is to (mentally) rank the ***displaced volume*** of the objects. Gold is the densest, sinks completely, and therefore displaces the *smallest* volume. Aluminum also sinks, is less dense than gold, and will displace the *second smallest* volume. ***Both*** ice ***and*** wood (carved into a monkey or not) are expected to float, although one can probably find a very few woods that don't float even when dry and carved into monkeys, so they both *displace their identical weights in water* which is also the *most* water.

The water level will drop the most for the largest displacements, so:

$$\boxed{\Delta d_a < \Delta d_b < \Delta d_c = \Delta d_d}$$

Problem 528. problems-1/fluids-ra-archimedes-objects.tex

A large beaker is filled to a marked line with water. You have the following objects (in order of ***decreasing density***):

a) A solid gold coin that has a mass of 100 grams.

b) A cast aluminum frog that has a mass of 100 grams.

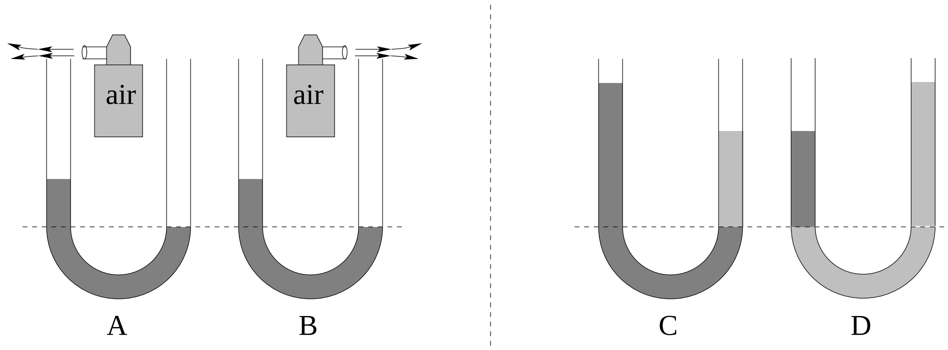
c) An ice cube that has a mass of 100 grams

d) A 100 gram chunk of shipping styrofoam.

You drop each item, one at a time, into the beaker in the water and record d_i, the change in water depth, and then remove it.

Rank the expected results for d_i for $i = a, b, c, d$. ***Indicate*** whether d_i is positive (so that the water in the beaker rises) or negative (falls). As always, in the case that some of the d_i are equal to neighbors, ***indicate that explicitly***.

Problem 529. problems-1/fluids-ra-four-utubes-venturi.tex

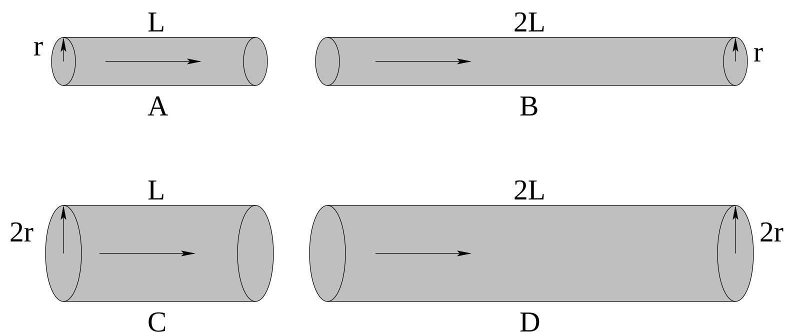


In the four u-tubes pictured above, only one of the two cases in each pair (A vs B and C vs D) make sense. In A vs B, a can of compressed air is blowing air ***across the top*** of one of the tube tops and the tube contains only a single fluid. In C vs D, the density of the immiscible fluids is indicated by the shading where the ***darker fluid has the greater density***.

Which two u-tubes DO make physical sense? (Circle one of each pair.)

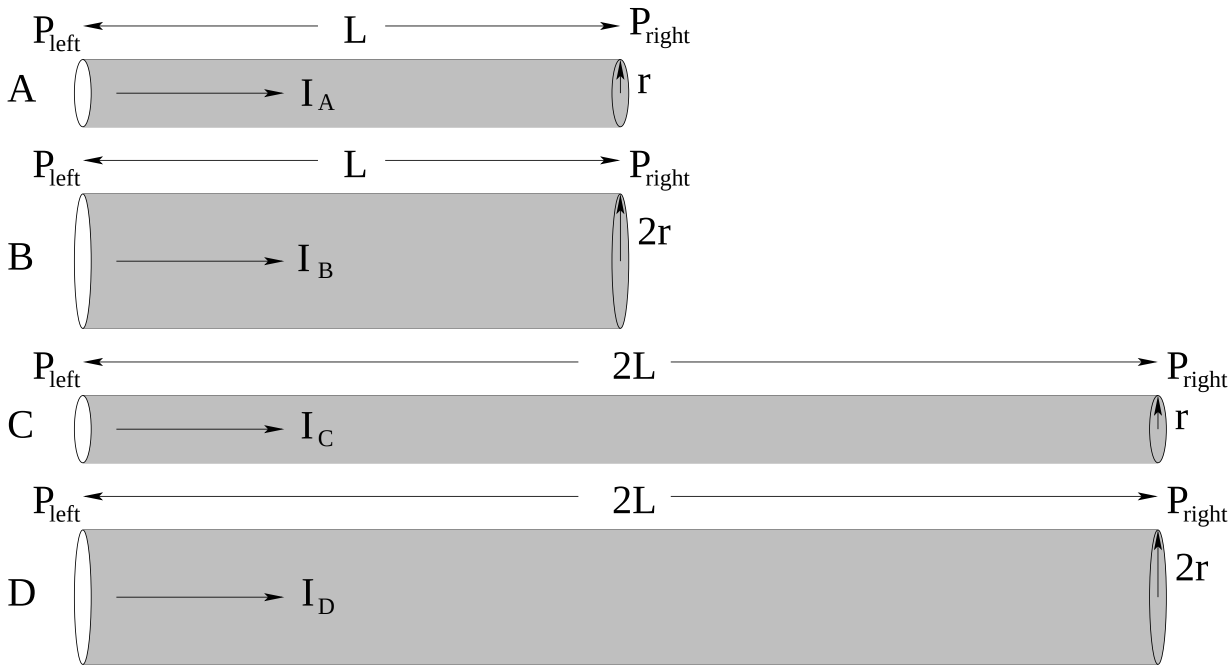
A **B** **C** **D**

Problem 530. problems-1/fluids-ra-poiseiulles-law-2.tex



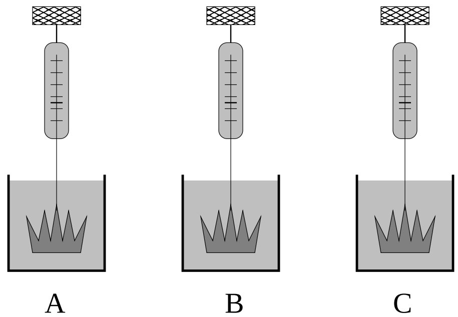
In the figure above, several circular pipes carry fluids with the same viscosity. Rank the pipes in the order of their **resistance** to laminar flow, from least to greatest. Equality is a possible answer. Think carefully about the dependence on r in Poiseuille's Law! This is why obstructions in arteries increase the resistance so dramatically!

Problem 531. problems-1/fluids-ra-poiseiulles-law.tex



Rank the volume flow I from the lowest to highest in the boxes below by filling A, B, C, and D into the large boxes and putting “<” or “=” signs into the small boxes in between for the four circular pipes illustrated in the figure above, assuming that in all cases that the flow, from left to right, is maintained by the *same* $\Delta P = P_{\rm left} - P_{\rm right} > 0$ and that the same fluid (with the same viscosity μ) is flowing through the pipes.

Problem 532. problems-1/fluids-ra-three-crowns-density.tex



Three crowns are shown above. Crown A is made of solid lead (specific gravity 11.3) covered with a thin veneer of gold leaf. Crown B is made of platinum (specific gravity 21.5), also covered with a thin veneer of gold leaf. Crown C is made of pure gold (specific gravity 19.3). The scale suspending the three crowns in water ***all read a weight of 5 newtons***. Rank the crowns in order of their ***true weight*** as measured in air from lowest to highest.

Problem 533. problems-1/fluids-ra-three-crowns-density-soln.tex

Although it isn't impossible to reason one's way through to the answer purely verbally, it is a *lot* easier to start with an equation that describes this static equilibrium. For any given crown:

$$F_{\text{tot}} = T + F_b - Mg = 0$$

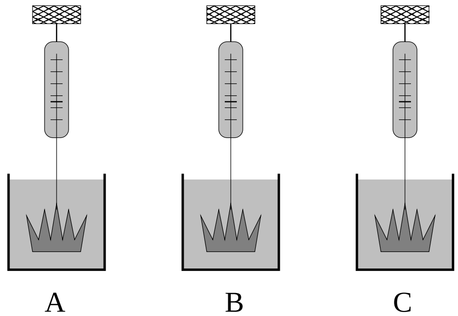
where the "true weight" of the crown is $Mg = \rho V g$ for its given density $\rho = \text{s.g.} * \rho_{\text{water}}$. T is what the scale reads and is common to all of the crowns. Hence we can rearrange this to solve for the true weight:

$$F_{\text{true}} = Mg = T + F_b = T + \rho_w V g$$

Now it is easy! The crown with the largest true weight is the one with the largest buoyant force, which in turn is the one with the largest submerged volume! True weight will ***increase*** as density ***decreases***. Hence we just need to sort out the weights by specific gravity, backwards:

$$M_B g < M_C g < M_A g$$

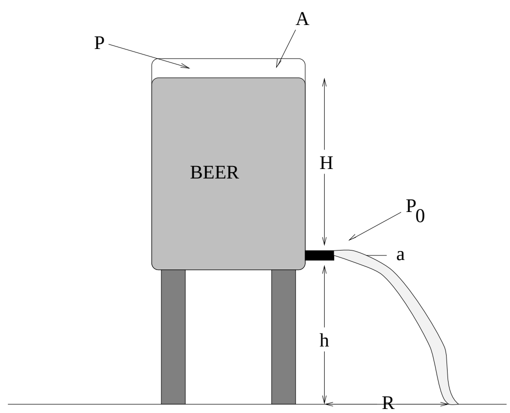
Problem 534. problems-1/fluids-ra-three-crowns.tex



Three crowns are shown above. Crown A is made of solid lead (specific gravity 11.3) covered with a thin veneer of gold leaf. Crown B is made of platinum (specific gravity 21.5), also covered with a thin veneer of gold leaf. Crown C is made of pure gold (specific gravity 19.3). All three crowns weigh **exactly 500 grams in air**. Rank the crowns in the order of effective weight while immersed in the water (what the three scales will read) lowest to highest.

10.1.4 Regular Problems

Problem 535. problems-1/fluids-pr-bernoulli-beer-keg-horizontal.tex



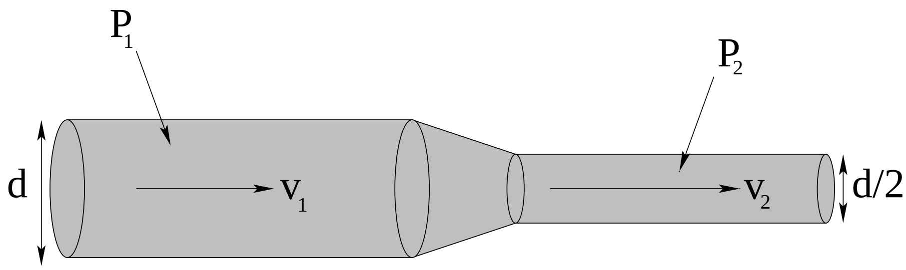
In the figure above, a CO_2 cartridge is used to maintain a pressure P on top of the beer in a beer keg, which is full up to a height H above the tap at the bottom (which is obviously open to normal air pressure) a height h above the ground. The keg has a cross-sectional area A at the top. Somebody has pulled the tube and valve off of the tap (which has a cross sectional area of a) at the bottom and it is spurting out onto the ground.

a) Find the speed with which the beer emerges from the tap. You may use the approximation $A \gg a$, but please do so only at the end of your algebra, not at the beginning. Assume laminar flow and no resistance.

b) Find the value of R at which you should place a pitcher (initially) to catch the beer.

c) Evaluate the answers to a) and b) for $A = 0.25$ m^2, $P = 2$ atmospheres, $a = 0.25$ cm^2, $H = 50$ cm, $h = 1$ meter and $\rho_{\rm beer} = 1000$ kg/m^3 (the same as water).

Problem 536. problems-1/fluids-pr-bernoulli-constricted-pipe.tex



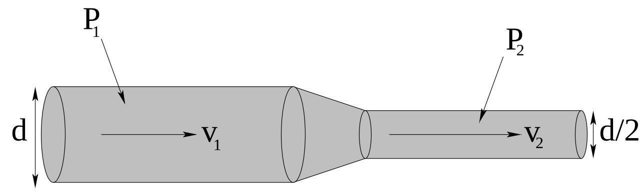
Water flows at a pressure P_1 and a speed v_1 in a circular storm culvert pipe of diameter d. The pipe narrows smoothly to a second pipe section where the diameter is only $d/2$.

a) Find v_2, the speed in the second pipe.

b) Find P_2, the pressure in the second pipe.

c) Write an algebraic expression in terms of the givens for the current (flow) I, the volume of water per second that passes through the pipe(s).

Problem 537. problems-1/fluids-pr-bernoulli-constricted-pipe-soln.tex



Water flows at a pressure P_1 and a speed v_1 in a circular storm culvert pipe of diameter d. The pipe narrows smoothly to a second pipe section where the diameter is only $d/2$.

a) The pipe area scales like $r^2 = d^2/4$, so $A_2 = A_1/4$. $A_1 v_1 = A_2 v_2$ is constant (conservation of flow). Hence:

$$\boxed{v_2 = 4v_1}$$

b) Bernoulli's Equation at constant height is:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Hence:

$$P_2 = P_1 + \frac{1}{2}\rho\left(v_1^2 - v_2^2\right) = P_1 + \frac{1}{2}\rho\left(v_1^2 - 16v_1^2\right)$$

or:

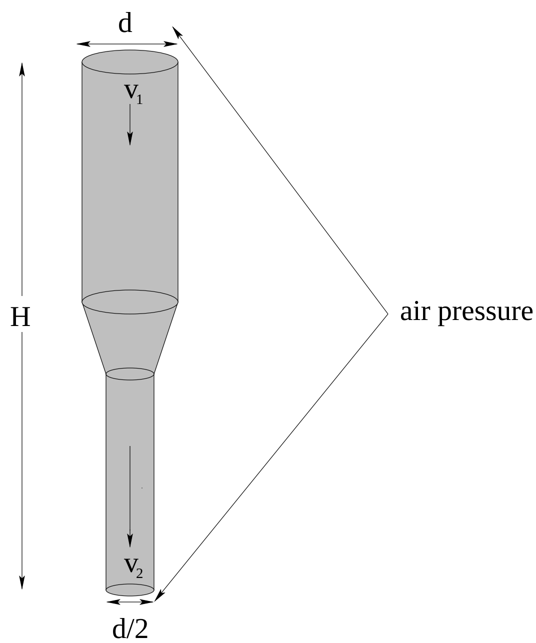
$$\boxed{P_2 = P_1 - \frac{15}{2}\rho v_1^2}$$

This answer is "interesting" because $P_2 \geq 0$, meaning that for any given P_1 there is an upper bound on v_1.

c)

$$\boxed{I = A_1 v_1 = A_2 v_2 = \frac{\pi d^2 v_1}{4}}$$

Problem 538. problems-1/fluids-pr-bernoulli-constricted-pipe-vertical.tex



A drain pipe in a house starts out at a **diameter** of d and narrows smoothly to a second pipe section where the **diameter** is only $d/2$. It is filled with water to a height H above the exit point of the lower pipe where it empties into a storm sewer. Both ends of the pipe are ***open to the air***.

a) Find v_1 and v_2, the speed of the flowing fluid in both pipe sections.

b) Write an algebraic expression in terms of the givens for the current (flow) Q, the volume of water per second that passes through the pipe(s). Give the expression in terms of d and v_1 and/or v_2 so that your answer does not depend on your answer to a).

c) How long Δt will it take for the water level in the top pipe to drop a distance $\Delta x \ll H$?

Problem 539. problems-1/fluids-pr-bernoulli-constricted-pipe-vertical-soln.tex

a) Find v_1 and v_2, the speed of the flowing fluid in both pipe sections.

Note that $A_1 = \pi d^2/4$ and $A_2 = \pi d^2/16 = A_1/4$. Then conservation of flow in a confined, incompressible fluid is:

$$I = \frac{dV}{dt} = A_1 v_1 = A_2 v_2 \qquad \text{or} \quad v_1 = \frac{A_2}{A_1} v_2 = \frac{v_2}{4}$$

Next, Bernoulli's formula/equation for this picture, ignoring viscosity and with both ends of the pipe open to the air at one atmosphere, is:

$$P_0 + \frac{1}{2}\rho v_1^2 + \rho g H = P_0 + \frac{1}{2}\rho v_2^2 + \rho g(0)$$

or

$$\begin{aligned} \frac{1}{2}\rho v_2^2(1 - 1/16) &= \rho g H \qquad \text{or} \\ v_2 &= \sqrt{\frac{32}{15} g H} \qquad \text{and} \\ v_1 &= \frac{v_2}{4} = \sqrt{\frac{2}{15} g H} \end{aligned}$$

b) Write an algebraic expression in terms of the givens for the current (flow) Q, the volume of water per second that passes through the pipe(s). Give the expression in terms of d and v_1 and/or v_2 so that your answer does not depend on your answer to a).

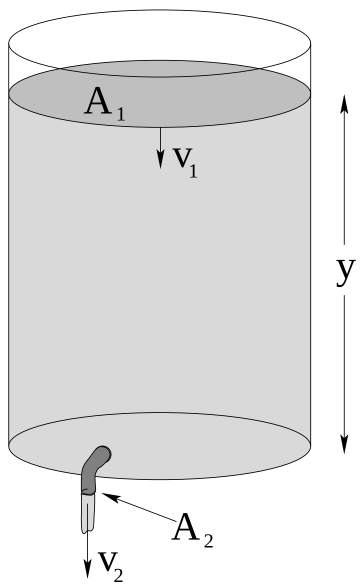
$$I = Av = \frac{\pi d^2}{4}\sqrt{\frac{2gH}{15}} = \pi d^2 \sqrt{\frac{gH}{120}}$$

c) How long Δt will it take for the water level in the top pipe to drop a distance $\Delta x \ll H$?

Assume v_1 is approximately constant over a short drop. Then:

$$\Delta x = v_1 \Delta t \qquad \text{or} \quad \Delta t = \frac{\Delta x}{v_1} = \sqrt{\frac{15\Delta x}{2gH}}$$

Problem 540. problems-1/fluids-pr-bernoulli-emptying-iced-tea-time.tex



A big cooler full of iced tea with cross-sectional area $A_1 = 1000$ cm^2 is ***open to the air on top***. The tap on the bottom has a hole with a cross-sectional area $A_2 = 1$ cm^2, and is opened at time $t = 0$ when the surface of the iced tea is (initially) at a height $y_0 = 50$ cm above the tap. Note that the density of the iced tea is the same as that of water, ρ_w.

Algebraically:

a) Find the rate at which the height of the iced tea drops – dy/dt – when the tap is opened, in terms of the givens and y, the current depth of the iced tea in the cooler

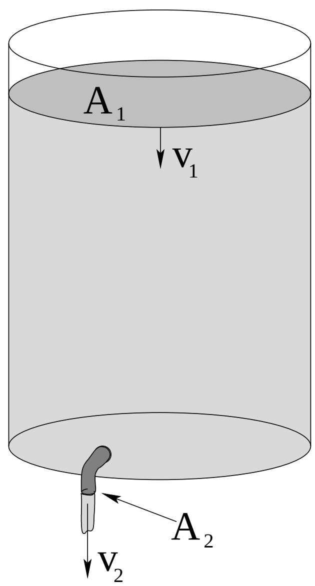
b) At what time t_f does the iced tea run out (that is, $y(t_f) = 0$)?

Numerically:

c) Evaluate t_f in seconds.

You may assume that $A_1 \gg A_2$ and use any approximations that may suggest.

Problem 541. problems-1/fluids-pr-bernoulli-emptying-iced-tea-time-soln.tex



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Algebraically:

a) Find the rate at which the height of the iced tea drops – dy/dt – when the tap is opened, in terms of the givens and y, the current depth of the iced tea in the cooler

b) At what time t_f does the iced tea run out (that is, $y(t_f) = 0$)?

Numerically:

c) Evaluate t_f in seconds.

You may assume that $A_1 \gg A_2$ and use any approximations that may suggest.

Solution: We start with the Bernoulli formula at arbitrary depth y plus conservation of flow (in the limit that $v_1 \ll v_2$ since $A_1 \gg A_2$):

$$\cancel{P_0} + \cancel{\frac{1}{2}\rho_w v_1^2} + \rho_w g y = \cancel{P_0} + \frac{1}{2}\rho_w v_2^2 + \cancel{\rho_w g(0)}$$
$$\rho_w g y = \frac{1}{2}\rho_w v_2^2$$

Hence the speeds at the top (1) and bottom (2) are:

$$v_2 = \sqrt{2gy} \text{ (recall Torricelli's rule)} \qquad \Rightarrow \qquad v_1 = -\frac{dy}{dt} = \frac{A_2}{A_1}v_2 = \frac{A_2}{A_1}\sqrt{2gy}$$

Hence:

$$\boxed{\frac{dy}{dt} = -\frac{A_2}{A_1}\sqrt{2gy}}$$

To answer the second part, we have to integrate to find $y(t_f) = 0$. We separate variables and integrate:

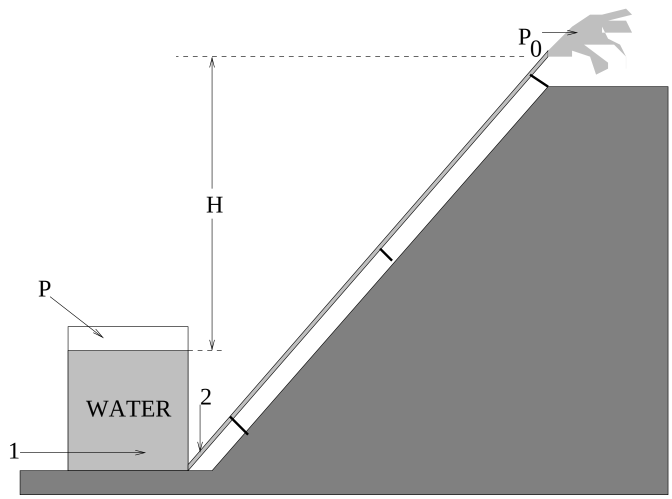
$$\int_H^0 \frac{dy}{y^{1/2}} = -\frac{\sqrt{2g}A_2}{A_1}\int_0^{t_f} dt \quad \Rightarrow \quad -2H^{1/2} = -\frac{\sqrt{2g}A_2}{A_1}t_f \quad \Rightarrow \quad \boxed{t_f = \sqrt{\frac{2}{g}}\frac{A_1}{A_2}H^{1/2}}$$

Finally, we can substitute in numbers:

$$\boxed{t_f = \frac{\sqrt{2}}{\sqrt{10}} \times 1000 \times \sqrt{\frac{1}{2}} \approx \frac{1000}{3.16} \approx 316 \text{ seconds}}$$

(or, around five and a quarter minutes). Note that this answer doesn't depend on the density! Beer, iced tea, any liquid where we can neglect the viscosity – they'll all take the same amount of time to empty. Note that I used $\sqrt{10} = 3.16$, although I could equally well have used the approximation $\sqrt{10} = \pi$ (good to less than 1% error) without much change in the answer. There would be a much larger error due to neglected viscosity and the inaccuracy over that last centimeter when the water doesn't cover the hole.

Problem 542. problems-1/fluids-pr-bernoulli-irrigation-pipe.tex



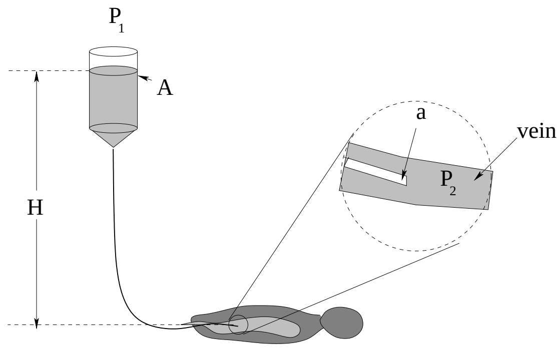
In the figure above, a pump maintains a pressure of P in the air at the top of a tank of water with a cross sectional area A. An irrigation pipe at the bottom leads up a slope to a farmer's field. The vertical distance between the top surface of water in the tank and the opening of the pipe is H. The cross-sectional area of the pipe is a. The top pipe is open to air pressure $P_0 = 1$ atm. Recall that the density of water is $\rho = 10^3$ kg/m^3.

a) What is the velocity of the water coming from the pipe? (Find this ***algebraically*** from the appropriate law(s).)

b) Is the pressure at the bottom of the tank greater inside the main vessel (point 1 on figure above) or inside the pipe (point 2)? ***Briefly*** explain.

c) ***After*** finding the answer to a) algebraically and answering b), evaluate v numerically using: $P = 2.5$ atm, $A = 10$ m^2, $H = 10$ m, and $a = 4$ cm^2. You shouldn't need a calculator for this.

Problem 543. problems-1/fluids-pr-bernoulli-IV.tex



Although mechanized and precise in modern first-world medicine, IV fluid delivery in the rest of the world is an imprecise gravity-driven system. A bag or bottle filled with a saline solution, plasma, blood, or medicine is hung above a patient's bed and a tube delivers that fluid directly into a patient's vein. A physician practicing medicine in many clinics or hospitals around the world may well need to be able to estimate things like the time of delivery of a bolus of fluid by a gravity-driven IV line for a given needle size.

Make such an estimate below, assuming that the bag of cross-sectional area A holds a fluid of density ρ, is effectively open to air pressure in the room P_1, and is suspended a height H above the level of the patient as shown. Use a as the cross-sectional area of the needle. Ignore viscosity and the fluid flow resistance of the tubing. ***Express all your answers algebraically in terms of*** A***,*** a***,*** ρ***,*** P_1 ***and*** P_2 ***for full credit***.

a) What is the minimum height H_{min} such that flow is from the bag ***to*** the patient instead of from the patient ***back*** towards the bag? (We don't want the patient to inadvertently donate blood!)

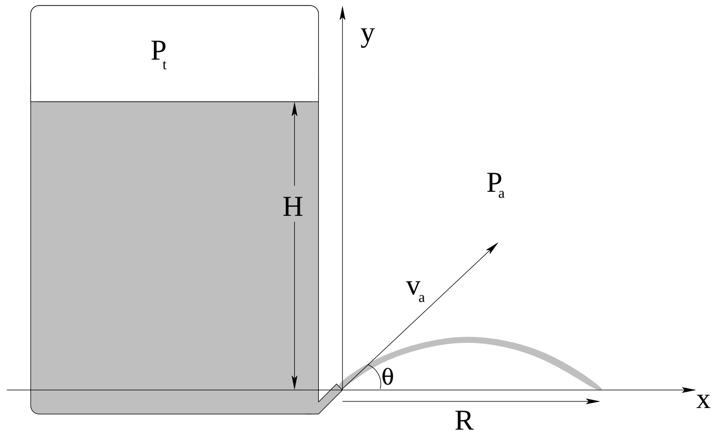
b) Suppose you raise the bag height to $H = 2H_{\text{min}}$. With what velocity does the fluid flow into the patient?

c) If the bag holds a fluid volume V ***estimate*** how long does it will take to deliver all of the fluid in the bag into the patient at this new height. Assume that H does not change (much) while the bag empties.

d) If one included viscosity and the drop in fluid height as the bag empties, would it increase or decrease the time from this rough estimate?

After finding the algebraic answers, you may estimate the numerical values of these quantities without a calculator for one point of EXTRA credit per answer for a maximum of three extra points. Assume that the fluid is water, $V = 500$ cubic centimeters, $P_1 = 1$ atm, $P_2 = 1.1$ atm, $A = 2 \times 10^{-3}$ m^2, and $a = \sqrt{5} \times 10^{-7}$ m^2. (Note that leaving radicals like $\sqrt{5}$ in your answers is OK).

Problem 544. problems-1/fluids-pr-bernoulli-range-arc.tex

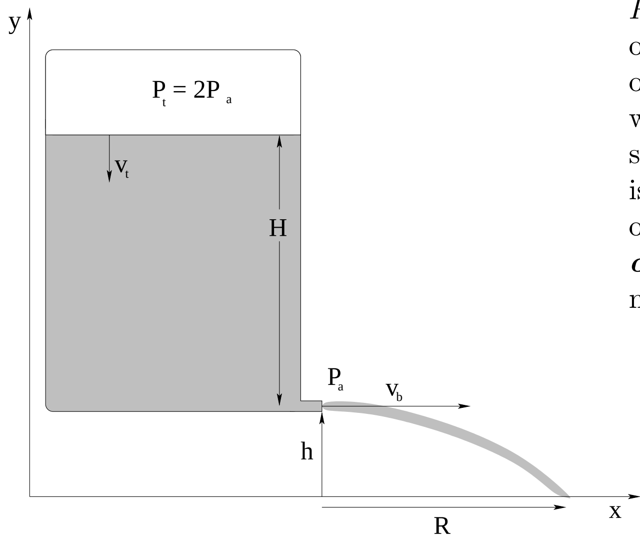


A sealed tank of water (density ρ) is shown above. Inside it is pressurized to a pressure $P_t = 3P_a$ (where P_a is the pressure outside of the tank, one atmosphere). The water escapes through a ***small pipe at the bottom*** where the stream is angled up at an angle θ with respect to the ground as shown. The cross-sectional area of the tank A is much larger than the cross-sectional area a of the small pipe at the bottom, $A \gg a$. (Picture is not necessarily to scale.)

a) What is the (approximate) speed v_a with which the water exits the small pipe? Express your answer (for this part only) in terms of ρ, g, P_t, P_a and possibly A and a.

b) What is the ***horizontal range*** of the stream of water, R, measured from the tip of the spout as shown. Express your answer (for this part only) in terms of v_a.

Problem 545. problems-1/fluids-pr-bernoulli-range-horizontal.tex



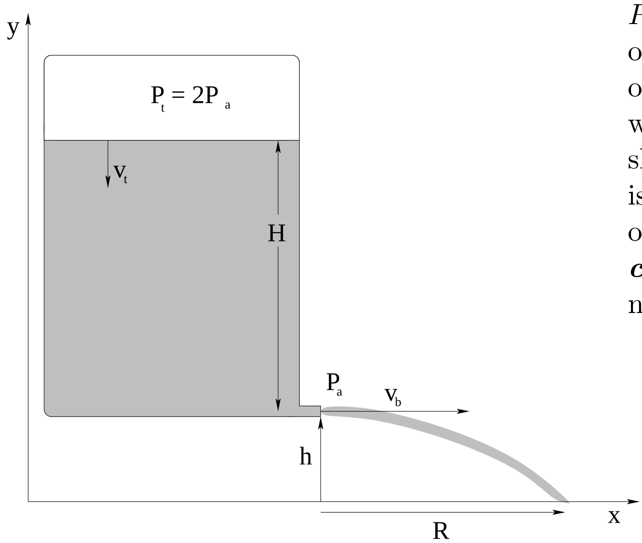
A sealed tank of water (density ρ_w) is shown above. Inside the tank, the air at the t(op) is pressurized above the water to a pressure $P_t = 2P_a$ (where P_a is the air pressure outside of the tank, one atmosphere). The water spurts out through a ***small pipe at the b(ottom)*** with speed v_b, initially parallel to the ground as shown. The cross-sectional area of the tank A is ***much larger*** than the cross-sectional area a of the small pipe at the bottom. ***Neglect viscosity and flow resistance***. Picture is not necessarily to scale.

a) Find the (approximate) speed v_b with which the water exits the small (bottom) pipe. You may assume $A \gg a$.

b) Find the ***horizontal range*** R of the stream of water measured from the tip of the spout as shown. Express your answer in terms of v_b, so that it needn't depend on getting a) correct.

$v_b =$ $\boxed{}$ $R =$ $\boxed{}$

Problem 546. problems-1/fluids-pr-bernoulli-range-horizontal-soln.tex



A sealed tank of water (density ρ_w) is shown above. Inside the tank, the air at the t(op) is pressurized above the water to a pressure $P_t = 2P_a$ (where P_a is the air pressure outside of the tank, one atmosphere). The water spurts out through a ***small pipe at the b(ottom)*** with speed v_b, initially parallel to the ground as shown. The cross-sectional area of the tank A is ***much larger*** than the cross-sectional area a of the small pipe at the bottom. ***Neglect viscosity and flow resistance***. Picture is not necessarily to scale.

a) Find the (approximate) speed v_b with which the water exits the small (bottom) pipe. You may assume $A \gg a$.

b) Find the ***horizontal range*** R of the stream of water measured from the tip of the spout as shown. Express your answer in terms of v_b, so that it needn't depend on getting a) correct.

$$v_b = \boxed{\sqrt{2\left(\frac{P_a}{\rho_w} + gH\right)}} \qquad R = \boxed{v_b\sqrt{\frac{2h}{g}}}$$

Solution:

a) This is just Bernoulli plus conservation of flow:

$$(P_t = 2P_a) + \rho_w gH + \frac{1}{2}\rho_w v_t^2 = P_a + \rho_w g(0) + \frac{1}{2}\rho_w v_b^2$$

and:

$$Av_t = av_b \implies v_t = \frac{a}{A}v_b \approx 0$$

or (rearranging):

$$P_a + \rho_w gH = \frac{1}{2}\rho_w v_b^2 \implies \boxed{v_b = \sqrt{2\left(\frac{P_a}{\rho_w} + gH\right)}}$$

b) This is just a chapter 1 kinematics problem. The time required to fall a distance h with no initial y-velocity is:

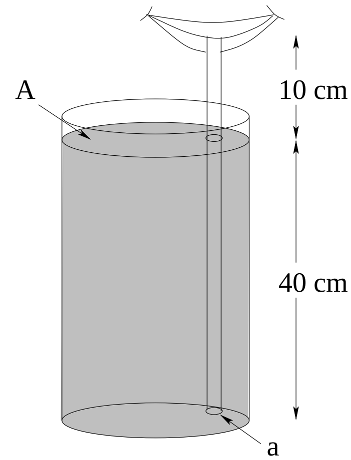
$$h = \frac{1}{2}gt_g^2 \implies t_g = \sqrt{\frac{2h}{g}}$$

The x-distance travelled in this time is the range:

$$\boxed{R = v_b t_g = v_b \sqrt{\frac{2h}{g}} = \sqrt{4h\left(\frac{P_a}{\rho_w g} + H\right)}}$$

(where you only need to find t_g and write the first part to get credit, but it is always nice to get the whole answer).

Problem 547. problems-1/fluids-pr-bernoulli-sipping-through-straw.tex

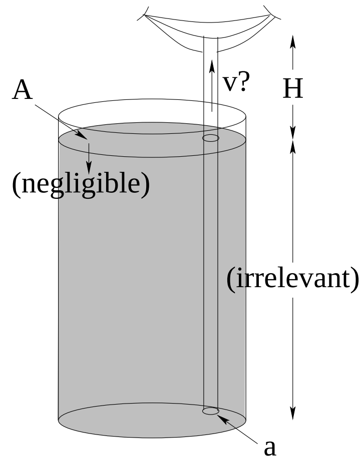


When you drink ***water*** through a straw, you create a pressure P_m in your mouth that is less than atmospheric pressure. Suppose $P_m = 9 \times 10^4$ Pa, and that the end of the straw in your mouth is 10cm above the surface of your 40cm high drink as shown above. You may assume that the cross-sectional area of the straw a is much less than the cross-sectional area A of the water at the top of your glass.

a) At what speed will the water in the straw be moving into your mouth? (Use $P_0 = 10^5$ Pa for the pressure of the air, $\rho = 1000$ kg/m^3, $g = 10$ m/s^2 and compute a number ***after showing how you obtained an algebraic expression for the answer.***)

b) Find an algebraic expression for how long it will take to sip a small volume ΔV of your water through the straw. Assume that the water height in the container makes a negligible change during this sip, and express your answer in terms of v_{straw} only (no need to substitute) to make it independent of your answer to a).

Problem 548. problems-1/fluids-pr-bernoulli-sipping-through-straw-soln.tex



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a) At what speed will the water in the straw be moving into your mouth? (Use $P_0 = 10^5$ Pa for the pressure of the air, $\rho = 1000$ kg/m^3, $g = 10$ m/s^2 and compute a number ***after showing how you obtained an algebraic expression for the answer.***)

b) Find an algebraic expression for how long it will take to sip a small volume ΔV of your water through the straw. Assume that the water height in the container makes a negligible change during this sip, and express your answer in terms of v_{straw} only (no need to substitute) to make it independent of your answer to a).

Solution: ***First***, ignore the numbers! Use Bernoulli, conservation of flow leading to the Torricelli approximation neglecting v at the top of the water in the glass, and choose the zero in height at the top of the water in the glass so only H indicated in the figure above is relevant! Then:

$$Av_{\text{top}} = av_{\text{straw}} \quad \Rightarrow \quad v_{\text{top}} = \frac{a}{A} v_{\text{straw}} \approx 0$$

and thus:

$$P_a + \frac{1}{2}\rho \cancel{v_{\text{top}}^2} + \rho g(0) = P_m + \frac{1}{2}\rho v_{\text{straw}}^2 + \rho g H$$

Now we just solve for v_{straw}:

$$v_{\text{straw}} = \sqrt{2\frac{(P_a - P_m) - \rho g H}{\rho}}$$

and (only now) plug in numbers (any of the three answers at the end OK):

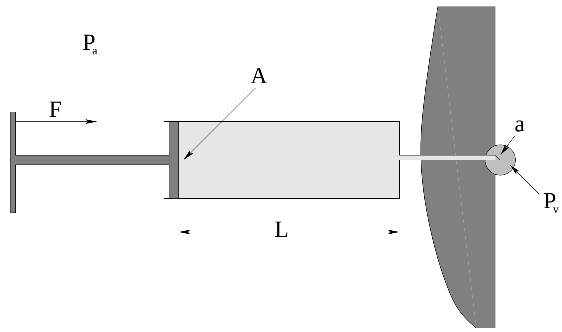
$$\boxed{v_{\text{straw}} = \sqrt{2 \times \frac{10^4 - 10^3}{1000}} = \sqrt{18} = 3\sqrt{2} \approx 4.2 \text{ m/sec}}$$

For b) recall that flow is just:

$$I = \frac{\Delta V}{\Delta t} = av_{\text{straw}} \quad \Rightarrow \quad \boxed{\Delta t = \frac{\Delta V}{av_{\text{straw}}}}$$

You do not need to substitute in either numbers or the algebraic form from a) to get full credit as the problem explicitly indicates.

Problem 549. problems-1/fluids-pr-bernoulli-syringe.tex



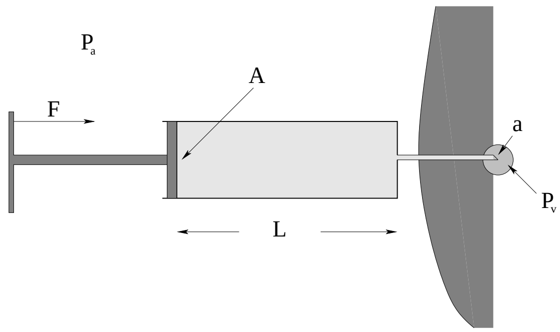
You are in a room at normal air pressure P_a and are given a hypodermic syringe full of medicine that we will treat as a zero-viscosity fluid with the density of water. The syringe tube has length L and cross-sectional area A and hence contains a volume AL of fluid. The cross-sectional area of the needle aperture is $a \ll A$. Holding the syringe ***horizontally*** as shown, you press on the (frictionless) plunger to inject the medicine into a patient's vein where the (given) blood pressure is $P_v > P_a$.

a) What force $F_{\rm min}$ (magnitude) do you have to exert on the plunger to hold the fluid in ***static equilibrium*** once the needle is in the patient?

b) Suppose you push with a force $F > F_{\rm min}$ on the plunger. Find an expression for the speed v_v with which the fluid flows *through the needle* into the vein. Don't forget the pressure of the air in the room!

c) Find an expression for the time required to empty the syringe ***in terms of*** $\boldsymbol{v_v}$ (so you do not have to use the results for b) or get b) correct to get full credit for c)).

Problem 550. problems-1/fluids-pr-bernoulli-syringe-soln.tex



a) In static equilibrium, the fluid does not flow and hence the pressure in the syringe tube must equal P_v. There are several ways, then, to consider the syringe plunger to get:

$$F_{\rm min} + P_a A = P_v A$$

or

$$F_{\rm min} = (P_v - P_a)A$$

Don't forget that air pressure on the outside helps push in on the plunger piston too!

b) Now we need to use all of Bernoulli's formula, using the pressure just inside the piston on the left:

$$(\frac{F}{A} + P_a) + \frac{1}{2}\rho v_s^2 = P_v + \frac{1}{2}\rho v_v^2$$

If we assume that $A \gg a$, we can neglect the v_s term (kinetic energy per unit volume in the syringe) and get:

$$v_v = \sqrt{2\frac{F - (P_v - P_a)A}{A\rho}}$$

or

$$v_v = \sqrt{2\frac{F - F_{\rm min}}{A\rho}} = \sqrt{2\frac{\Delta F}{A\rho}}$$

where ΔF is the *extra* force applied to the pressure relative to the force required to hold the syringe in equilibrium. Note that there is no $\rho g y$ term because whatever y is, it is the same on both sides.

If you don't use the inequality to simplify, it is still easy to solve for v_v. You just have to move the v_s term to the right and (using $v_s A = v_v a$) write:

$$(\frac{F}{A} + P_a) - P_v = \frac{1}{2}\rho v_v^2 \left(1 - \frac{a^2}{A^2}\right)$$

or:

$$v_v = \sqrt{2\frac{F - (P_v - P_a)A}{A\rho\left(1 - \frac{a^2}{A^2}\right)}}$$

c) The "flow" (volume per unit time) that moves through the needle is:

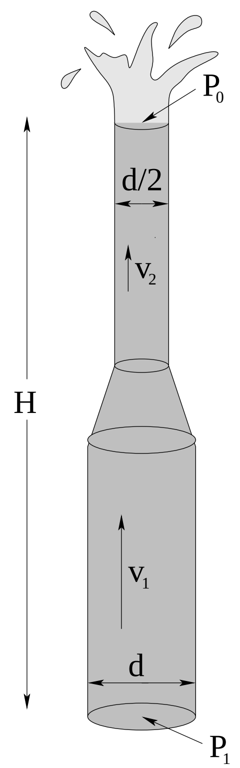
$$\frac{\Delta V}{\Delta t} = v_v a = v_s A$$

Hence:

$$\Delta t = \frac{AL}{v_v a} = \frac{AL}{v_s A} = \frac{L}{v_s}$$

The first form is "the answer", but the third form is *obviously correct* as this is the time required to move the plunger a distance L at constant speed v_v.

Problem 551. problems-1/fluids-pr-bernoulli-vertical-pipe-fountain.tex

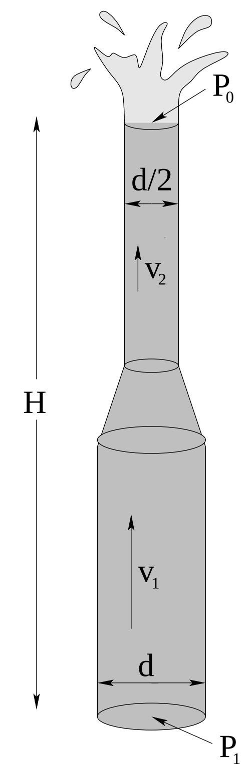


A small fountain used in a zen rock garden is pictured to the left. A pump (not shown) maintains a given pressure P_1 at the base of a pipe of diameter d that lifts the water (density ρ) and narrows to a diameter of $d/2$ at the top to speed it up. The water exits into air at pressure $P_0 = 1$ atm. The overall pipe has height H between the pump and the exit. Find an ***algebraic expressions in terms of the givens*** for:

a) v_1 and v_2, the speed of the flowing fluid in both the lower and the upper pipe sections.

b) The current (flow) Q, the ***volume of water per second*** that passes through the pipe(s). ***Give this expression in terms of v_1 and/or v_2 as needed*** so that your answer can be correct even if you get part a) wrong.

Problem 552. problems-1/fluids-pr-bernoulli-vertical-pipe-fountain-soln.tex



A small fountain used in a zen rock garden is pictured to the left. A pump (not shown) maintains a given pressure P_1 at the base of a pipe of diameter d that lifts the water (density ρ) and narrows to a diameter of $d/2$ at the top to speed it up. The water exits into air at pressure $P_0 = 1$ atm. The overall pipe has height H between the pump and the exit. Find an ***algebraic expressions in terms of the givens*** for:

a) v_1 and v_2, the speed of the flowing fluid in both the lower and the upper pipe sections.

b) The current (flow) Q, the ***volume of water per second*** that passes through the pipe(s). ***Give this expression in terms of v_1 and/or v_2 as needed*** so that your answer can be correct even if you get part a) wrong.

Solution: This is a Bernoulli Equation problem, so we write it out at the bottom and top of the pipe as follows:

$$P_1 + \rho g(0) + \frac{1}{2}\rho v_1^2 = P_0 + \rho g H + \frac{1}{2}\rho v_2^2$$

This is only one equation and we have two unknowns, so we also need to use ***conservation of flow***:

$$Q = A_1 v_1 = \boxed{\frac{\pi d^2}{4} v_1 = \frac{\pi d^2}{16} v_2} = A_2 v_2$$

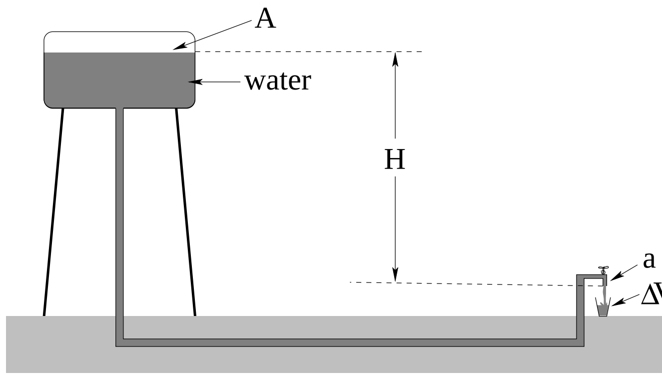
boxed because it is the answer to part b), or:

$$v_2 = 4v_1$$

Substituting this into the first equation, solving for v_1, and backsubstituting to get v_2 we get::

$$(P_1 - P_0) - \rho g H = 8\rho v_1^2 - \frac{1}{2}\rho v_1^2 = \frac{15}{2}\rho v_1^2 \quad \Rightarrow \quad \boxed{v_1 = +\sqrt{\frac{2\{(P_1 - P_0) - \rho g H\}}{15\rho}}, v_2 = +4\sqrt{\frac{2\{(P_1 - P_0) - \rho g H\}}{15\rho}}}$$

Problem 553. problems-1/fluids-pr-city-water-supply-tank.tex



The figure above represents the water distribution system of a typical city or town. An elevated tank is filled with water from a purified source. Sealed pipes descend from the tank and extend through the ground to your house, where your closed water tap holds in the pressure. When you open the tap, water flows from the tank, through the pipe, and out into your glass.

Suppose that the top of the tank has a cross-sectional area $A \gg a$, where a is the cross-sectional area of your spigot. A pump (not shown) maintains the water height in the tank so that it remains a height H above your spigot as shown whether the tap is open or closed. The tank is filled with water of density ρ, and both the top of the tank and the spigot are open to air at the same pressure (one atmosphere). Assume laminar flow and zero viscosity.

a) When your tap is closed, what is the pressure of the water just inside the tap?

b) When the tap is opened, with what speed does water flow out of the tap?

c) How long will it take to fill the cup of volume ΔV shown with water?

Problem 554. problems-1/fluids-pr-city-water-supply-tank-soln.tex

a) When your tap is closed, what is the pressure of the water just inside the tap?

From statics/Pascal ***or*** Bernoulli (which works fine for statics too!):

$$P_b = P_a + \rho g H$$

b) When the tap is opened, with what speed does water flow out of the tap?

From Bernoulli:

$$P_a + \rho g H + \frac{1}{2}\rho v_t^2 = P_a + \frac{1}{2}\rho v_b^2$$

Note that $v_t \ll v_b$, so we can ignore it, cancel the pressures, and get:

$$v_b = \sqrt{2gH}$$

which is *Torrecelli's Rule.* We can actually use this to get an excellent approximation to $v_t = v_b(a/A)$

c) How long will it take to fill the cup of volume ΔV shown with water?

The rate of flow is:

$$\frac{dV}{dt} = Av_t = av_b$$

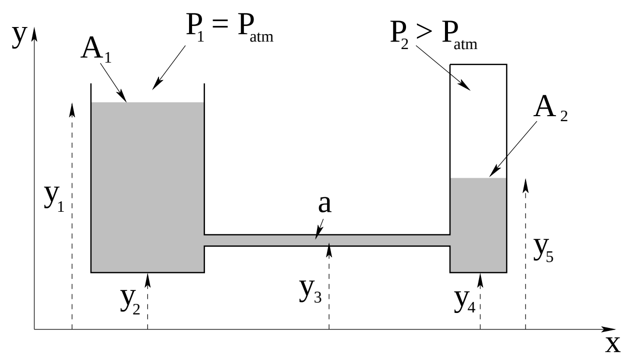
and doesn't change appreciably while filling a small cup (as H doesn't vary by much). This means that:

$$\Delta V = av_b\Delta t = \sqrt{2gH}a\Delta t$$

or:

$$\Delta t = \frac{\Delta V}{\sqrt{2gH}a} = \frac{\Delta V\sqrt{2gH}}{2gHa}$$

Problem 555. problems-1/fluids-pr-flow-between-containers.tex



In the figure above, water (density ρ) is being pushed through a pipe of cross-sectional area a from a small (sealed) tank to a larger one open to the air at normal pressure $P_1 = P_{\rm atm}$ by the pressure difference when $P_2 > P_{\rm atm}$. The cross-sectional areas of the two containers are given in terms of the cross-sectional area of the pipe by $A_1 = 400a$ and $A_2 = 100a$.

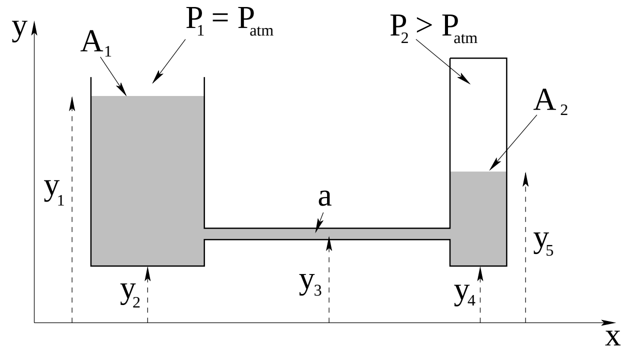
The givens are: a, ρ, P_1, and the various heights y_i labeled in the figure that ***may or may not be of interest*** . Neglect viscosity and drag/resistance in the containers or pipe.

a) Find the smallest pressure $P_{2,\rm min}$ that will cause water to flow *from* the smaller container *to* the larger one instead of the other way around.

b) Suppose the pressure P_2 is ***larger*** than this minimum pressure (so water *does* flow from the smaller to the larger container). What is the speed v_p of the fluid in the **pipe** as this happens? Hint: the velocity in *both containers* is negligible compared to the velocity in the *pipe!* What is the (nearly static) force that drives water one way or the other through the pipe?

c) What are the speeds v_1 with which the water ***rises*** in the first container and v_2 ***falls*** in the second?

Problem 556. problems-1/fluids-pr-flow-between-containers-soln.tex



First, this is a hard problem. Full credit will be given for any solution attempt that gets a) correct and indicates that you know both Bernoulli's formula and the equation for conservation of flow. A five point bonus of extra credit will be given if you have the insight that you need to neglect ***both*** tank velocities compared to the velocity in the pipe!

Indeed, this problem can only consistently be solved by applying Bernoulli's formula *across the pipe*. This is because there are *three* velocities, and $v_1, v_2 \ll v_p$, so that *only* the kinetic energy term in the pipe (also called the "dynamical pressure") is not negligible compared to that in *either* tank. Neglecting the kinetic energy terms in both tanks is equivalent to using the the *static pressure difference* at the bottom of the two tanks at the entrance and exit of the pipe as the source of the "work" that drives the water through the pipe. This makes sense! If you actually get this point, you will get a bonus of five points on the problem and the exam! If you don't, you will encounter serious difficulties (e.g. imaginary numbers) trying to apply the Bernoulli formula to tanks 1 and 2 while ignoring the pipe.

We start, then, by evaluating the (approximately!) static pressure in the bottoms of both tanks at the height of the pipe. On the left end of the pipe (tank 1) is:

$$P_{b1} = P_1 + \rho g(y_1 - y_3)$$

on the right (tank 2) it is:

$$P_{b2} = P_2 + \rho g(y_5 - y_3)$$

For part a), Fluid will not flow if this static pressure matches across the pipe! This result is exact, and everybody should be able to get it.

$$P_{b1} = P_1 + \rho g(y_1 - y_3) = P_2 + \rho g(y_5 - y_3) = P_{b2}$$

or (with $P_1 = P_{\text{atm}}$):

$$P_{2,\text{min}} = P_{\text{atm}} + \rho g(y_1 - y_5)$$

and:

$$\boxed{P_2 > P_{\text{atm}} + \rho g(y_1 - y_5)}$$

will make the fluid flow uphill into the larger container. This makes complete sense. It *also* suggests, if you think about it, that the work (per unit volume) that speeds the water up effectively from "rest" is:

$$(P_2 - P_{\rm atm}) - \rho g(y_1 - y_5) = \frac{1}{2}\rho v_p^2$$

To answer both b) and c) we need to use Bernoulli where the pipe pressure equals the static pressure at the bottom of tank 1 (the tank the water is flowing in*to*), and where we equate the formula to the static pressure on the bottom of tank 2:

$$P_{b1} + \rho g y_3 + \frac{1}{2}\rho v_p^2 = P_{b2} + \rho g y_3$$

Note that even if there is a small contribution from the motion of the fluids in tanks 1 and 2, it is a negligible correction to P_{b1} and P_{b2} respectively compared to the $\frac{1}{2}\rho v_p^2$ term! Then (cancelling the $\rho g y_3$ bits):

$$P_1 + \rho g y_1 + \frac{1}{2}\rho v_p^2 = P_2 + \rho g y_5$$

and indeed:

$$(P_2 - P_{\rm atm}) - \rho g(y_1 - y_5) = \frac{1}{2}\rho v_p^2$$

as we guessed above. Solving for v_p:

$$v_p = \sqrt{\frac{2\left((P_2 - P_{\rm atm}) - \rho g(y_1 - y_5)\right)}{\rho}}$$

As you can see, as long as $P_2 > P_{\rm atm} + \rho g(y_1 - y_5)$, the pressure at the bottom of tank 2 will be higher than the pressure at the bottom of tank 1, and the pressure difference will drive water from tank 2 to tank 1 at speed v_p in the pipe. If we go the other way, the role of starting and ending height reverse (changing sign) and we'll only get a real answer if $P_2 < P_{2,min}$.

To get the speeds at the top of the tanks is now simple. From flow conservation:

$$A_1 v_1 = a v_p = A_2 v_2$$

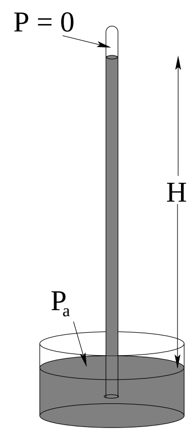
or

$$\boxed{v_1 = \frac{a}{A_1} v_p = 0.0025\sqrt{\frac{2\left((P_2 - P_{\rm atm}) - \rho g(y_1 - y_5)\right)}{\rho}}}$$

and:

$$\boxed{v_2 = \frac{a}{A_2} v_p = 4v_1 = 0.01\sqrt{\frac{2\left((P_2 - P_{\rm atm}) - \rho g(y_1 - y_5)\right)}{\rho}}}$$

Problem 557. problems-1/fluids-pr-compare-barometers.tex



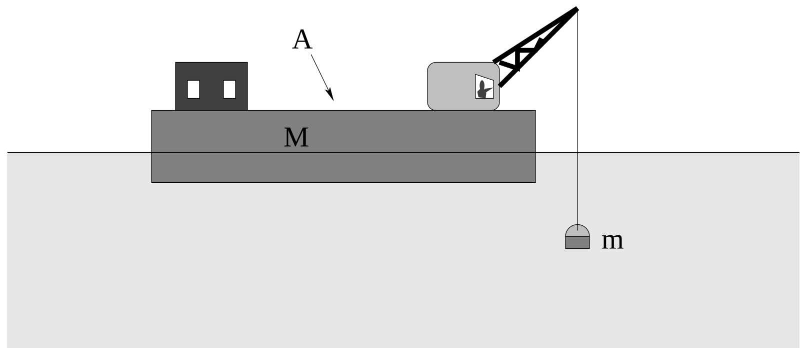
The idea of a barometer is a simple one. A tube filled with a suitable liquid is inverted into a reservoir. The tube empties (maintaining a seal so air bubbles cannot get into the tube) until the static pressure in the liquid is in balance with the *vacuum* that forms at the top of the tube and the ambient pressure of the surrounding air on the fluid surface of the reservoir at the bottom.

a) Suppose the fluid is water, with $\rho_w = 1000$ kg/m^3. Approximately how high will the water column be? Note that water is not an ideal fluid to make a barometer with because of the height of the column necessary and because of its annoying tendency to boil at room temperature into a vacuum.

b) Suppose the fluid is mercury, with a specific gravity of 13.6. How high will the mercury column be? Mercury, as you can see, *is* nearly ideal for fluids-pr-compare-barometers except for the minor problem with its extreme toxicity and high vapor pressure.

Fortunately, there are many other ways of making good fluids-pr-compare-barometers.

Problem 558. problems-1/fluids-pr-crane-2.tex

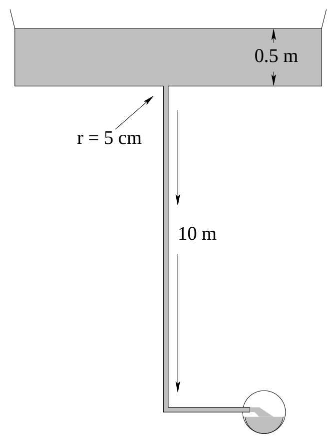


A barge with a crane mounted on it has a cross sectional area A, a total mass M, and straight sides. It is very slowly winching up a one of Blackbeard's treasure chests (of total mass m) from the ocean floor near Beaufort.

a) As the chest comes out of the water, does the boat sink or rise? Justify your answer with an equation or two and/ or a before and after figure.

b) Just before the crane turns to put the chest on the deck, Blackbeard's Ghost appears and cuts the cable of the crane so that the chest plunges back into the briny deep. Find an expression for the distance d the boat rises up in the water (after it stops bobbing) when this happens. Use the symbol ρ_s for the density of sea water.

Problem 559. problems-1/fluids-pr-dangerous-drain.tex



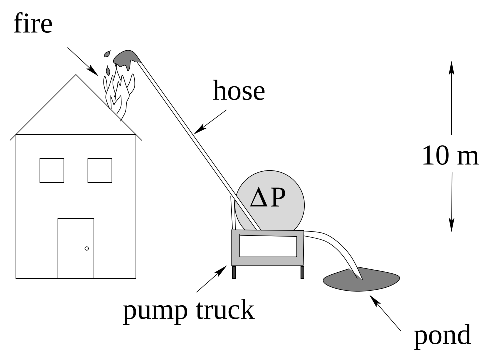
It is dangerous to build a drain for a pool or tub consisting of a single narrow pipe that drops down a long ways before encountering air at atmospheric pressure. This was demonstrated tragically in 1993 in an accident that occurred (no fooling!) within two miles from where you are sitting. A baby pool was built with just such a drain and one day a little girl sat down on the drain and was severely injured. In 2008 another young girl in Minneapolis was killed!

In this problem you will analyze why.

Suppose the mouth of a drain is a circle five centimeters in radius, and the pool has been draining long enough that its drain pipe is filled with water (and no bubbles) to a depth of ten meters below the top of the drain, where it exits in a sewer line open to atmospheric pressure. The pool is 50 cm deep. If a thin steel plate is dropped to suddenly cover the drain with a watertight seal, what is the force one would have to exert to remove it straight up?

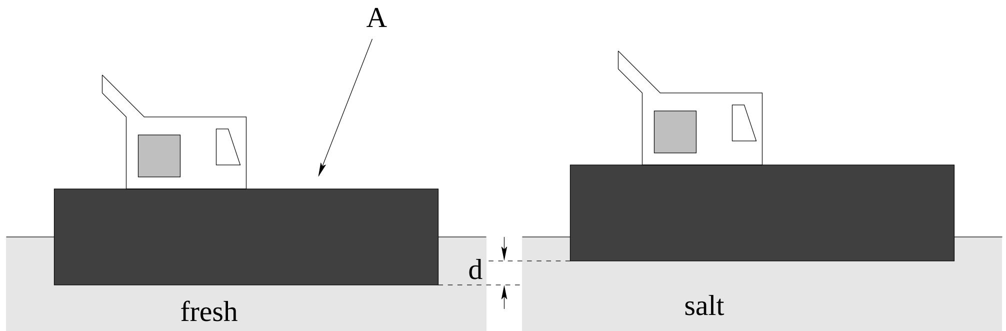
Note carefully this force relative to the likely strength of mere flesh and bone (or even thin steel plates!) Ignorance of physics can be actively dangerous.

Problem 560. problems-1/fluids-pr-firefighters-pump.tex



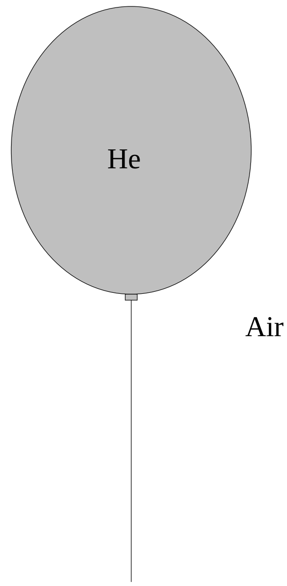
Firefighters arrive at a fire in the country and have to use water from the farm pond to try to battle the blaze. Their pump firetruck takes in water from the pond at one atmosphere (P_0) and increases the pressure at the **bottom of the hose** to an adjustable pressure $P_0 + \Delta P$ that can be set at any value of ΔP from 0 to 2 atmospheres of pressure. What is the **minimum** value ΔP_{min} one can set the pump to that will lift the water as high as the second floor (ten meters up above the ground, two meters above the fire)? ***Show all work and justify your answer with a physical principle or two!***

Problem 561. problems-1/fluids-pr-floating-freighter.tex



A rectangular ocean barge with horizontal area A (viewed from the top) floats in fresh water (ρ_w). It floats downriver and enters the ocean ($\rho_s = 1.1\rho_w$). As it does so, the ship bobs up an additional distance d from its earlier (freshwater) waterline. Find the total mass of the ship in terms of A, ρ_w, ρ_s and d. Hint – since you don't know either the height of the ship or its displacement in fresh water as given, concentrate on the *difference* in the forces (and the displacement) as it sails from fresh to salt.

Problem 562. problems-1/fluids-pr-helium-balloon.tex

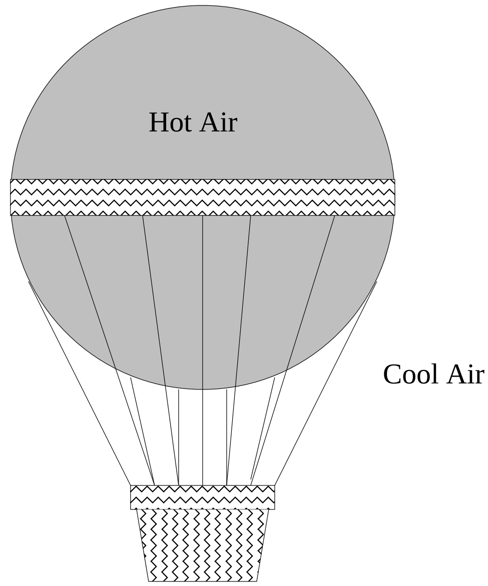


In the figure above, a helium balloon ($\rho_{\mathrm{He}} = 0.18$ kg/m^3) is suspended in air ($\rho_a = 1.28$ kg/m^3) by a string.

a) Assuming that the volume of the helium balloon is approximately 4000 cubic centimeters (4×10^{-3} m^3), find the total 'lift' of the balloon (the tension in the string). Neglect the mass of the balloon itself and the string.

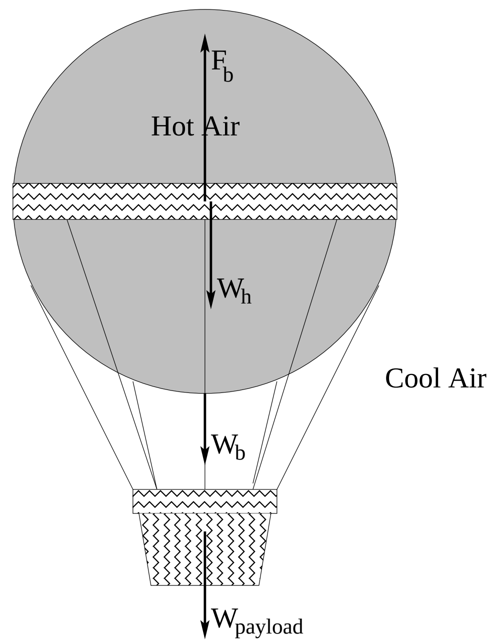
b) In the movies, humans are shown grabbing a few dozen helium balloons and being pulled up into the sky. Assuming that a reasonable human payload (including the mass of all of the balloon rubber and strings) is 100 kg, approximately how many balloons would *really* be required to lift a person?

Problem 563. problems-1/fluids-pr-hot-air-balloon.tex



A hot air balloon is drawn in the figure above. Estimate its total 'lift', assuming that the density of cool air is approximately constant at $\rho_a = 1.28$ kg/m^3, the density of hot air in the baloon is $\rho_h = 0.64$ kg/m^3, and that the balloon proper has a (filled) volume of 1000 m^3 (corresponding to a spherical balloon roughly 13 meters in diameter). If the balloon, basket, and rigging have a mass of 340 kg, what is the maximum payload it can carry?

Problem 564. problems-1/fluids-pr-hot-air-balloon-soln.tex



Compute separately the buoyant force F_b, the weight of the hot air in the balloon, and the weight of the given mass of the balloon, basket, and rigging. The difference is the leftover lift of the balloon that can lift a payload.

$$F_b = \rho_a \Delta V_b g \approx 1.28 \times 10^4 \quad \mathrm{N}$$

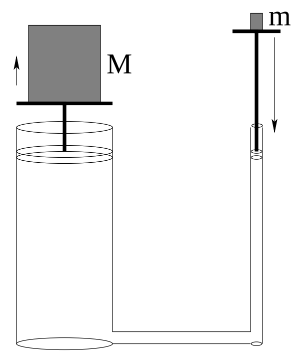
$$W_h = \rho_h \Delta V_b g \approx 0.64 \times 10^4 \quad \mathrm{N}$$

$$W_b = 0.34 \times 10^4 \quad \mathrm{N}\ \text{(given)}$$

$$W_{\text{payload}} <= F_b - W_h - W_b = 0.30 \times 10^4 = 3000 \quad \mathrm{N}$$

or the balloon can lift a maximum of around 300 kg of mass, call it three adult males. Note that we used $g \approx 10$ so this answer is only good to 2-3%.

Problem 565. problems-1/fluids-pr-hydraulic-lift.tex

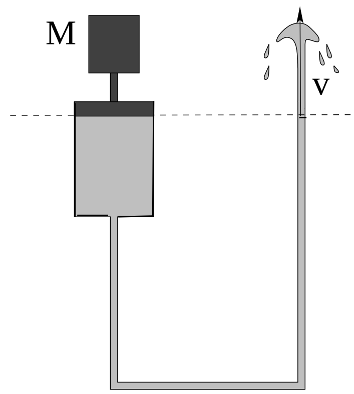


The figure above illustrates the principle of hydraulic lift. A pair of coupled cylinders are filled with an incompressible, very light fluid (assume that the mass of the fluid is zero compared to everything else).

a) If the mass M on the left is 1000 kilograms, the cross-sectional area of the left piston is 100 cm^2, and the cross sectional area of the right piston is 1 cm^2, what mass m should one place on the right for the two objects to be in balance?

b) Suppose one pushes the right piston down a distance of one meter. How much does the mass M rise?

Problem 566. problems-1/fluids-pr-piston-pump-1.tex

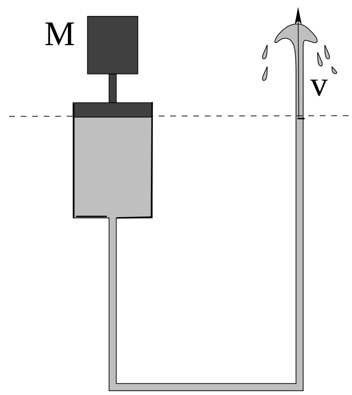


A piston and weight has a total mass M and is pressing on water confined in a cylinder of cross sectional area A. The water is then pushed into a pipe with a cross sectional area of a that is open to the air at the same height as the piston. Neglect viscosity.

a) What does M have to be to make the water spurt from the pipe with a speed v? You should use the approximation $a \ll A$ to develop your algebraic answer.

b) Find the numerical value for M that will produce a speed $v = 5$ m/sec for the following data: $A = 100$ cm^2, $a = 1$ cm^2. The density of water is $\rho_w = 10^3$ kg/meter^3.

Problem 567. problems-1/fluids-pr-piston-pump-1-soln.tex



A piston and weight has a total mass M and is pressing on water confined in a cylinder of cross sectional area A. The water is then pushed into a pipe with a cross sectional area of a that is open to the air at the same height as the piston. Neglect viscosity.

a) What does M have to be to make the water spurt from the pipe with a speed v? You should use the approximation $a \ll A$ to develop your algebraic answer.

b) Find the numerical value for M that will produce a speed $v = 5$ m/sec for the following data: $A = 100$ cm^2, $a = 1$ cm^2. The density of water is $\rho_w = 10^3$ kg/meter^3.

Solution: Use Bernoulli's Formula plus conservation of flow, $Av_p = av$ (where v_p is the speed of the *slowly* descending piston). The force pushing the piston *down* is $P_a A + Mg$. It is hardly moving, so the *total* force on the piston is approximately zero, or the pressure just under the piston is:

$$P_p = \frac{P_a A + Mg}{A} = P_a + \frac{Mg}{A}$$

The two points are at the same height (set equal to 0 for convenience). Thus the Bernoulli formula is:

$$\cancel{P_a} + \frac{Mg}{A} + \frac{1}{2}\rho \cancel{v_p^2} = \cancel{P_a} + \frac{1}{2}\rho v^2$$

where we've cancelled atmospheric pressure on both sides and neglected the kinetic energy term for the slowly moving fluid just under the piston. This lets us solve for the answer requested in a):

$$\boxed{v = \sqrt{\frac{2Mg}{\rho A}}}$$

If we care, we also know that:

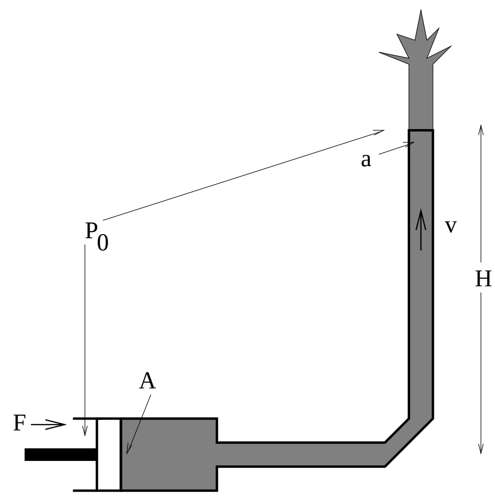
$$v_p = \frac{a}{A} v = \sqrt{\frac{2Mga^2}{\rho A^3}}$$

and sure, $v_p \ll v$ as long as $a \ll A$. For b) we now do a second piece of algebra plus some 'simple' arithmetic:

$$M = \frac{1}{2}\frac{\rho A v^2}{g} = \frac{1}{2}\frac{1000 \times 0.01 \times 25}{10} = \boxed{12.50 \text{ kg}}$$

Note we used SI units throughout and did not use $a = 10^{-4}$ m^2 except to note that $1 \ll 100$.

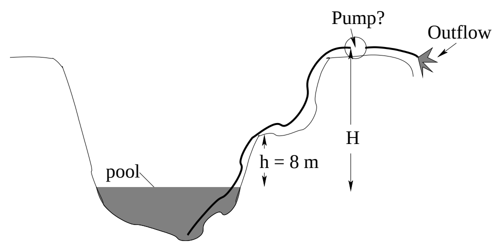
Problem 568. problems-1/fluids-pr-piston-pump-2.tex



A piston is pressed with a force $\vec{\mathbf{F}}$ on a hydraulic cylinder containing water ($\rho = 10^3$ kg/m^3). The cross sectional area of the cylinder is $A = 400$ cm^2. The water therein is forced into a pipe with a cross sectional area of $a = 2$ cm^2 that rises vertically a height $H = 40$ meters. Both the end of the pipe (at the top) and the back of the piston (at the bottom) are open to atmospheric pressure.

What does F have to be to make the water spurt from the pipe with a speed of 10 meters/sec at the top? Solve this problem beginning from (stated) physical principles, showing all work.

Problem 569. problems-1/fluids-pr-pump-water-up-cliff.tex



This problem will help you learn required concepts such as:

- Static Pressure
- Barometers

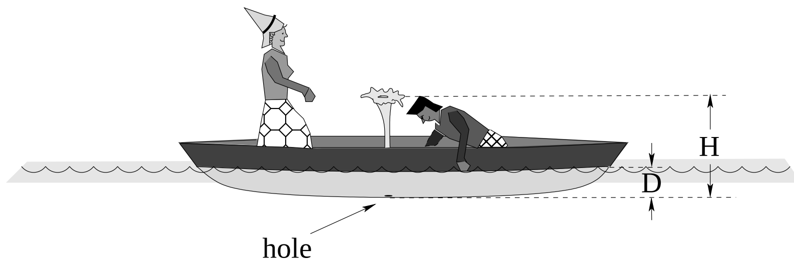
so please review them before you begin.

A pump is a machine that can maintain a pressure differential between its two sides. A particular pump that can maintain a pressure differential of as much as 10 atmospheres of pressure between the low pressure side and the high pressure side is being used on a construction site.

a) Your construction boss has just called you into her office to either explain why they aren't getting any water out of the pump on top of the $H = 25$ meter high cliff shown above. Examine the schematic above and show (algebraically) why it cannot possibly deliver water that high. Your explanation should include an invocation of the appropriate physical law(s) and an explicit calculation of the highest distance the a pump *could* lift water in this arrangement. Why is the notion that the pump "sucks water up" misleading? What really moves the water up?

b) If you answered a), you get to keep your job. If you answer b), you might even get a raise (or at least, get full credit on this problem)! Tell your boss where this single pump should be located to move water up to the top and show (draw a picture of) how it should be hooked up.

Problem 570. problems-1/fluids-pr-romeo-and-juliet.tex



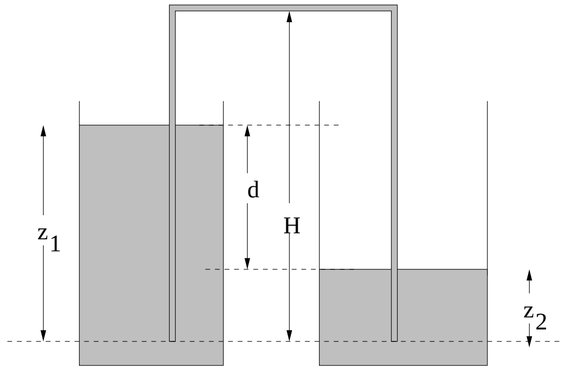
Romeo and Juliet are out in their boat again when Juliet's Salvatore Ferragamo heels poke a circular hole of radius r in the bottom of the boat. The boat has a ***draft*** of D (this is the distance the boat's bottom lies underwater as shown).

a) Romeo tries to cover the hole with his hand. What is the minimum force he must apply to keep it covered?

b) Juliet convinces Romeo that a little water fountain would be romantic, so he moves his hand. How fast does the water move through the hole?

c) To what height H does Juliet's fountain spout up from the bottom of the boat? (The height drawn is to illustrate the quantity H only and may not be at all correct.)

Problem 571. problems-1/fluids-pr-siphon-two-tank.tex



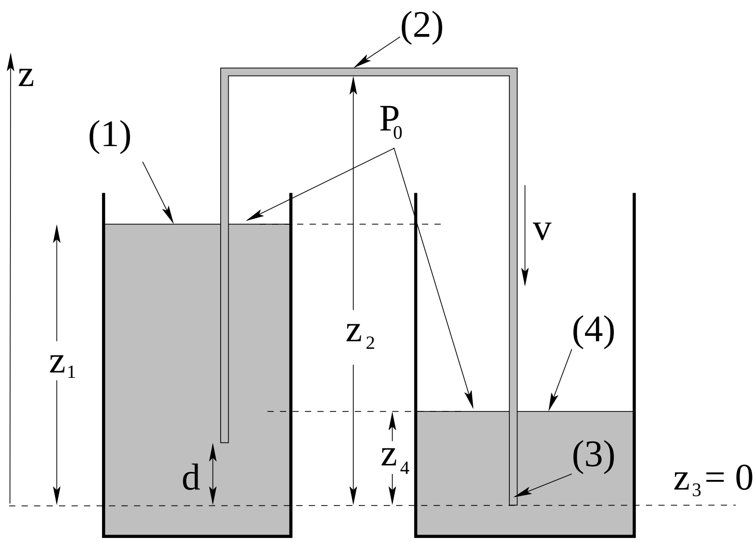
In the figure about two tanks are partly filled with fluid to the heights indicated. A siphon tube of a certain height H at its heighest point is started between them. The fluid is assumed to have a density ρ (which could well be "water", for example) and viscosity and fluid resistance are to be ignored. You may also assume that the surface area at the top of either tank is large compared to the cross-sectional area of the tube.

a) Find the velocity v of the fluid in the siphon tube at the instant shown.

b) Find the pressure P_H at the heighest point of the siphon tube.

c) Find the maximum height obstacle H_{max} that a siphon tube can go over (relative to the geometry shown) and still function.

Problem 572. problems-1/fluids-pr-siphon-two-tank-soln.tex



In the figure about two tanks are partly filled with fluid to the heights indicated. A siphon tube of a certain height H at its heighest point is started between them. The fluid is assumed to have a density ρ (which could well be "water", for example) and viscosity and fluid resistance are to be ignored. You may also assume that the surface area at the top of either tank is large compared to the cross-sectional area of the tube.

a) Find the velocity v of the fluid in the siphon tube at the instant shown.

b) Find the pressure P_H at the heighest point of the siphon tube.

c) Find the maximum height obstacle $H_{\rm max}$ that a siphon tube can go over (relative to the geometry shown) and still function.

Solution

It is useful to consider points 1 through 4 in the figure above. If we write Bernoulli's formula for points 1, 2 and 3, all three formulas must be equal for the fluid moving through the "pipe" represented by the two tanks and intermediary tube. Note that $P_1 = P_4 = P_0$, atmospheric pressure at the top of both tanks. Thus

$$\begin{aligned} P_0 + \rho g z_1 + \frac{1}{2}\rho v_1^2 \quad (1) &= P_0 + \rho g z_2 + \frac{1}{2}\rho v^2 \quad (2) \\ &= P_3 + \rho g(0) + \frac{1}{2}\rho v^2 \quad (3) \end{aligned}$$

where v is the desired velocity in the tube. We will assume that $v_1 \ll v$ and $v_2 \ll v$ and throw them both out relative to v.

In our previous tank problems like this, P_3 is the pressure in the fluid at the point where the system exits the fluid. The reason the fluid flows in the tube at all is that the pressure at this height is *different* on the dashed line in the two vessels. *Within* the two tanks (not in the tube) the fluid is nearly static, so the pressure $P_3 = P_0 + \rho g z_4$. This is the key to solving the problem, because if you naively write Bernoulli's formula for points 1 and 4 and equate them, you get a contradiction. For all values of z less than the top of the fluid, the pressure in the second tank is less than the pressure in the first at the same height.

We can now do some algebra between points 1 and 3:

$$\begin{aligned} P_0 + \rho g z_1 + \frac{1}{2}\rho v_1^2 \quad (1) &= P_0 + \rho g z_4 + \frac{1}{2}\rho v^2 \\ \frac{1}{2}\rho v^2 &= \rho g(z_1 - z_4) \\ v &= \sqrt{2g(z_1 - z_2)} \end{aligned}$$

We get an answer that looks "like" Torricelli's Law even though the tube exit per se is no longer the relevant height and even though the fluid at the tank *tops* is moving slowly compared to this in *both* tanks! The force that pushes the fluid from the first to the second tank is evidently the pressure in the *first* tank at the depth of the surface of the *second* tank, $\Delta z = z_1 - z_4$.

Now let's equate the Bernoulli formulas for points 1 and 2 and solve for P_2:

$$\begin{aligned} P_0 + \rho g z_1 + \frac{1}{2}\rho v_1^2 &= P_2 + \rho g z_2 + \frac{1}{2}\rho v^2 \\ P_2 &= P_0 + \rho g(z_1 - z_2) - \frac{1}{2}\rho v^2 \\ &= P_0 + \rho g(z_1 - z_2) - \rho g(z_1 - z_4) \\ &= P_0 - \rho g(z_2 - z_4) = P_0 - \rho g h_{\rm max} \end{aligned}$$

where we have substituted our answer for v in. This answer leads us to some "issues". If we start at the outflow pressure (inside the tube) and go uphill, the pressure must decrease. The pressure at point 2 must be lower than P_0 by $\rho g h_{\rm max}$, the decrease in static pressure with height, because the speed of the fluid in the tube does not change.

If, however, we keep lowering the second tank (increasing Δz between the surfaces of the two tanks) and thereby making v larger, we also make P_2 smaller until eventually it becomes zero! If we lower it any further, the pressure in the tube cannot be negative, as ***before*** it ever reaches 0 (a vacuum) nearly any real fluid will "come apart" at the molecular level in a process called cavitation. But what if we consider just the first line of 1 = 2:

$$P_0 + \rho g z_1 = P_2 + \rho g z_2 + \frac{1}{2}\rho v^2$$

and suppose we set $P_2 = 0$, the point where cavitation occurs. This equation then becomes:

$$P_0 + \rho g z_1 = \rho g z_2 + \frac{1}{2}\rho v^2$$

and we can solve for v this way as well:

$$v = \sqrt{2\left(P_0 - \rho g(z_2 - z_1)\right)/\rho}$$

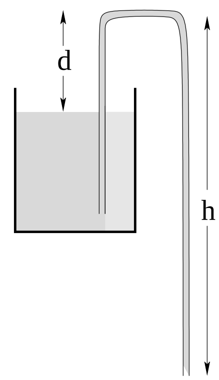
This equation *seems* to imply that as we increase z_2 holding z_1 constant, we decrease the maximum speed of flow in the tube until it is zero *and* the pressure is zero, but be careful. This is increasing z_2 ***holding*** $\boldsymbol{P_2 = 0}$, which is to say, maintaining $z_2 - z_4 = P_0/(\rho g)$!

We must therefore carefully think about the conditions for cavitation. Will the fluid cavitate at zero pressure *while in uniform motion in the tube* with $v \neq 0$? Or does the fluid *both* have to be *stationary* in the tube to cavitate? If we reach zero pressure at the top with $z_2 - z_1$ too small to make v zero, then if we increase $h_{\rm max}$ we lower the point on the right hand side of the tube where zero pressure exists and we have a serious problem. There is no longer any downward directed pressure gradient above that point because the pressure cannot go below zero. Gravity is pulling down fluid elements in the tube. If there were no pressure gradient to oppose gravity, those elements would *speed up*. But they cannot speed up and maintain a uniform flow (just as a uniformly falling stream of water splits up into droplets).

So when does the siphon "break" and e.g. water stop flowing? I think that the answer is best understood by considering that uniformly falling stream. If $z_2 - z_4 > 10$ meters, there is a stretch at the top where the velocity entering the vicinity of P_2 from a solution to Bernoulli's equation on the left hand side only ($1 = 2$ above) is ***smaller than*** v for the collective tube ($1 = 3$ above). There is a false continuity implied by $1 = 3$ through pressures that are implicitly less than zero, but this is impossible. The fluid in the tube cavitates *continually*, basically breaking up into drops that accelerate as they fall freely under gravity from right where the top of the tube bends down on the right until they match the flow velocity implied on the lower part of the tube continuously matched to the pressure at the bottom of the right hand side. Fluid flows, but it is no longer the case that it is flowing uniformly or that the tube itself remains continuously full.

As $z_2 - z_1$ is *separately* increased to 10 meters, v (now dominated by $z_2 - z_1$ at zero pressure at point 2 and P_0 at z_1) decreases to zero and the fluid on both sides of the tube stops flowing forming two "water barometers" on either side. Not so obvious!

Problem 573. problems-1/fluids-pr-siphon.tex

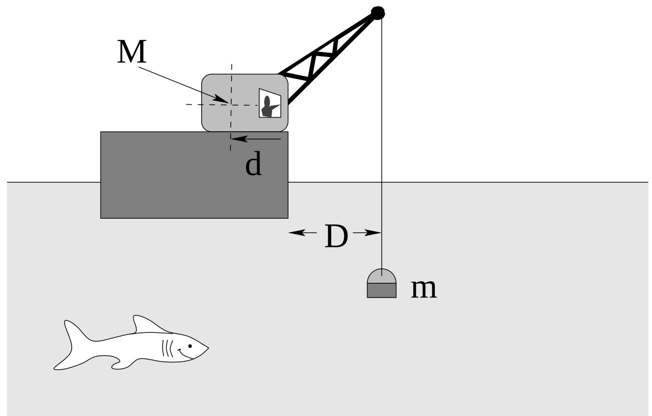


Water is being drained from a large container by means of a siphon as shown. The highest point in the siphon is distance d above the level of water in the container, and the total height of the long arm of the siphon is h. The distance h can be varied. The mass density of water is ρ_w, and air pressure is P_0. Express all answers in terms of d, h, ρ_w, P_0, and g.

a) What is the maximum possible value of h for which the siphon will work? (***Hint:*** The pressure cannot be negative anywhere in the siphon, in particular, in the long arm of the siphon.)

b) For that maximum value of h, what is the speed of the water coming out of the siphon?

Problem 574. problems-1/fluids-pr-static-crane.tex

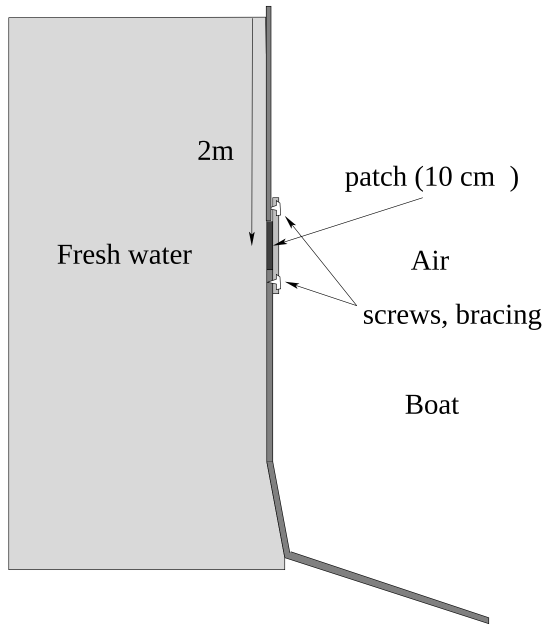


The crane above has a nearly massless boom. It is being used to salvage some of Blackbeard's treasure – a chest of mass m filled with very dense gold.

a) Find the maximum weight that the crane can lift, assuming that all of the weight of the crane itself acts downward at its center of mass to counterbalance it at the position shown, a horizontal distance d to the left of the bottom right corner of the crane. The crane's boom is fixed so that its moment arm (shown) is always D. Your answer should be expressed in M, g and the given lengths d and D.

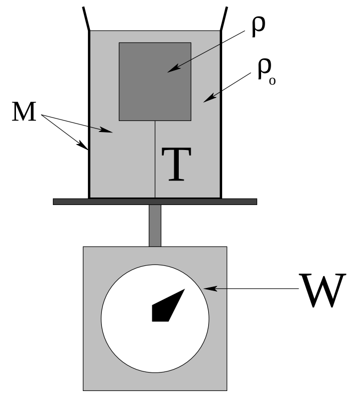
b) Suppose that Blackbeard's treasure is so massive that the crane is *almost* tipping over as it very slowly lifts it up **through the water**. What will happen when the crane tries to lift the mass out of the water, and **why**? "Why" should involve certain forces and a good before and after picture.

Problem 575. problems-1/fluids-pr-static-hole-in-a-boat.tex



Your yacht has a hole in it! Oh, no! The hole is 2 meters below the waterline, and has a cross-sectional area of 10 cm^2 (that's ten square centimeters, not ten centimeter's squared!). You patch it, and need to brace the patch with screws that can each hold at most a force of 5 Newtons. How many screws (at *least*) should you use to be sure of being able to withstand the force of the ocean pressing in against your patch?

Problem 576. problems-1/fluids-pr-tension-suspends-immersed-mass.tex



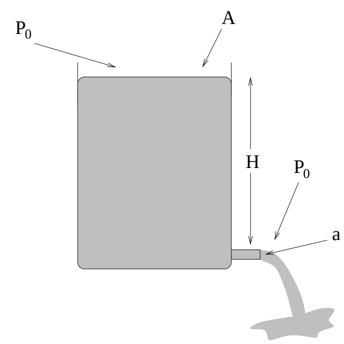
A floating block of density ρ and volume V is suspended, fully immersed, by a thin thread attached to the bottom in a jar of oil (density $\rho_o > \rho$) that is resting on a scale as shown. The total mass of the oil and jar (alone) is M.

a) What is the buoyant force exerted by the oil on the block?

b) What is the tension T in the thread?

c) What does the scale read?

Problem 577. problems-1/fluids-pr-time-to-empty-open-vat.tex



This problem will help you learn required concepts such as:

- Bernoulli's Equation
- Torricelli's Law

so please review them before you begin.

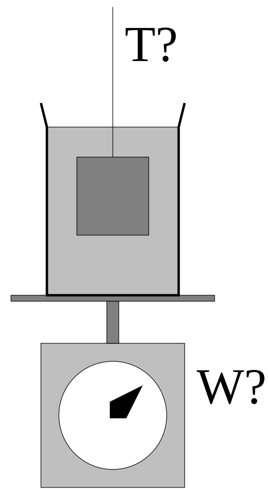
In the figure above, a large drum of water is open at the top and filled up to a height H above a tap at the bottom (which is also open to normal air pressure). The drum has a cross-sectional area A at the top and the tap has a cross sectional area of a at the bottom.

a) Find the speed with which the water emerges from the tap. Assume laminar flow without resistance. Compare your answer to the speed a mass has after falling a height H in a uniform gravitational field (after using $A \gg a$ to simplify your final answer, Torricelli's Law).

b) How long does it take for all of the water to flow out of the tap? (Hint: Start by *guessing* a reasonable answer using dimensional analysis and insight gained from a). That is, think about how you expect the time to vary with each quantity and form a simple expression with the relevant parameters that has the right units. Next, find an expression for the velocity of the top. Integrate to find the time it takes for the top to reach the bottom.) Compare your answer(s) to each other and the time it takes a mass to fall a height H in a uniform gravitational field. Does the correct answer make dimensional and physical sense?

c) Evaluate the answers to a) and b) for $A = 0.50$ m^2, $a = 0.5$ cm^2, $H = 100$ cm.

Problem 578. problems-1/fluids-pr-weight-of-immersed-mass.tex



A block of density ρ and volume V is suspended by a thin thread and is immersed completely in a jar of oil (density $\rho_o < \rho$) that is resting on a scale as shown. The total mass of the oil and jar (alone) is M.

a) What is the buoyant force exerted by the oil on the block?

b) What is the tension T in the thread?

c) What does the scale read?

Problem 579. problems-1/fluids-pr-weight-of-immersed-mass-soln.tex

a) What is the buoyant force exerted by the oil on the block?

$$F_b = \rho_o V g$$

b) What is the tension T in the thread?

$$T + \rho_o V g - mg = 0$$

or (since $mg = \rho V g$):

$$T = (\rho - \rho_o) V g$$

c) What does the scale read?

$$T + W = mg + Mg$$

so combining the previous two equations or alternatively directly from Newton's Third Law:

$$W = Mg + \rho_o V g$$

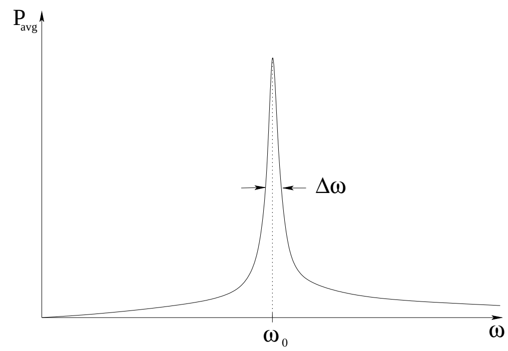
Chapter 11

Oscillations

11.1 Oscillations

11.1.1 Multiple Choice Problems

Problem 580. problems-1/oscillation-mc-change-resonance-3.tex



In the figure above, the curve shows the (average) power $P_{\text{avg}}(\omega)$ delivered to a damped, driven oscillator with equation of motion:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos(\omega t)$$

Recall that the "width" of the curve $\Delta\omega$ is the ***full width at half maximum power***. Suppose the damping constant b is ***doubled*** while k of the spring, m, and the driving force magnitude F_0 are ***kept unchanged***. What happens to the curve?

a) The curve becomes narrower (smaller $\Delta\omega$) at the same frequency;

b) The curve becomes narrower at a higher frequency;

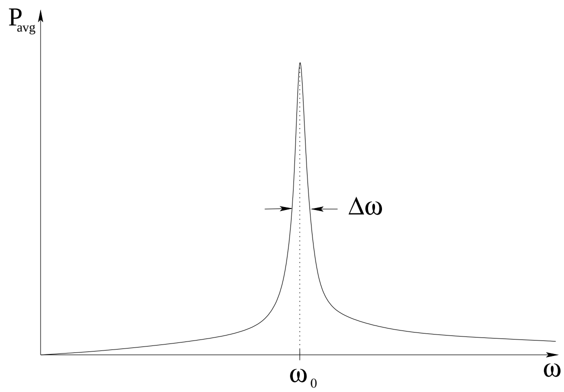
c) The curve becomes broader (larger $\Delta\omega$) at the same frequency

d) The curve becomes broader at a different frequency;

e) The curve does not change;

f) There is not enough information to determine the changes of the curve.

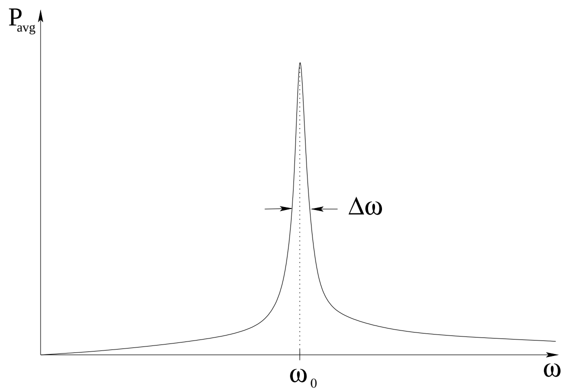
Problem 581. problems-1/oscillation-mc-increase-k-m-resonance-curve.tex



The curve in the figure shows the (average) power $P_{\text{avg}}(\omega)$ delivered to a damped, driven oscillator. Recall that the "width" of the curve $\Delta\omega$ is the ***full width at half maximum power*** as shown. If both k of the spring and the mass m are ***doubled*** while the damping constant b and driving force magnitude F_0 are ***kept unchanged***, what happens to the curve?

- ☐ The curve does not change;
- ☐ The curve becomes broader (larger $\Delta\omega$) at the same frequency
- ☐ The curve becomes broader at a different frequency;
- ☐ The curve becomes narrower (smaller $\Delta\omega$) at the same frequency;
- ☐ The curve becomes narrower at a higher frequency;
- ☐ There is not enough information to determine the changes of the curve.

Problem 582. problems-1/oscillation-mc-increase-k-m-resonance-curve-soln.tex



The curve in the figure shows the (average) power $P_{\text{avg}}(\omega)$ delivered to a damped, driven oscillator. Recall that the "width" of the curve $\Delta\omega$ is the ***full width at half maximum power*** as shown. If both k of the spring and the mass m are ***doubled*** while the damping constant b and driving force magnitude F_0 are ***kept unchanged***, what happens to the curve?

- ☐ The curve does not change;
- ☐ The curve becomes broader (larger $\Delta\omega$) at the same frequency
- ☐ The curve becomes broader at a different frequency;
- ☒ The curve becomes narrower (smaller $\Delta\omega$) at the same frequency;
- ☐ The curve becomes narrower at a higher frequency;
- ☐ There is not enough information to determine the changes of the curve.

Solution:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2k}{2m}}$$

does not change. But:

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{m\omega_0}{b} = \frac{\sqrt{km}}{b}$$

and

$$Q_{\text{new}} = \frac{2m\omega_0}{b} = \frac{\sqrt{4km}}{b} = 2Q = \frac{\omega_0}{\Delta\omega_{\text{new}}} \Rightarrow \Delta\omega_{\text{new}} = \frac{\Delta\omega}{2}$$

independent of F_0. Since Q doubles when k *and* m double together, ***$\boldsymbol{\Delta\omega}$ goes down by a factor of $\frac{1}{2}$*** so the curve becomes narrower (smaller $\Delta\omega$) at the same frequency;

Problem 583. problems-1/oscillation-mc-stride-resonance.tex

You have to take a long hike on level ground, and are in a hurry to finish it. On the other hand, you don't want to waste energy and arrive more tired than you have to be.

Your *stride* is the length of your steps. Your *pace* is the frequency of your steps, basically the number of steps you take per minute. Your *average speed* is the product of your pace and your stride: the distance travelled per minute is the number of steps you take per minute times the distance you cover per step.

Your best strategy to cover the distance faster but with minimum additional energy consumed is to:

a) Increase your stride but keep your pace about the same.

b) Increase your pace, but keep your stride about the same.

c) Increase your pace and your stride.

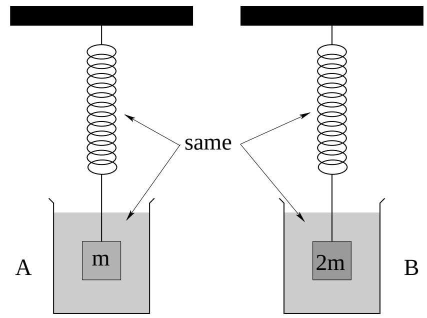
d) Increase your stride but decrease your pace.

e) Increase your pace but decrease your stride.

(in all cases so that your average speed increases).

Note well that this is a physics problem, so be sure to *justify your answer* with a physical argument. You might want to think about *why* one answer will probably accomplish your goal within the constraints and the others will not.

Problem 584. problems-1/oscillation-mc-two-damped-oscillators.tex



Two ***identical springs*** support two masses of the ***same size and shape*** in the ***same damping fluid***. However, $m_B = 2m_A$.

Both systems are pulled to an initial displacement from equilibrium of X_0 and released, and the exponential decay times τ_A and τ_B required for the initial amplitude of oscillation of each mass to decay to $X_0 e^{-1}$ is measured. We expect that:

a) $\tau_A = 2\tau_B$

b) $2\tau_A = \tau_B$

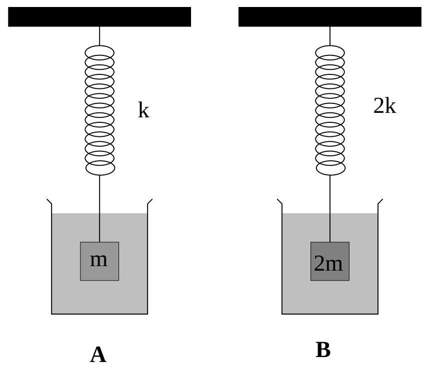
c) $\tau_A = \tau_B$

d) $4\tau_A = \tau_B$

e) We cannot predict the ***relative*** decay times without more information.

11.1.2 Short Answer Problems

Problem 585. problems-1/oscillation-sa-damped-oscillation.tex



Two springs with different spring constants (k and $2k$, respectively) support two blocks of the ***same size and shape***, but different masses, $m_B = 2m_A$. The blocks are fully submerged in the ***same damping fluid***, therefore, they have the same coefficient of damping.

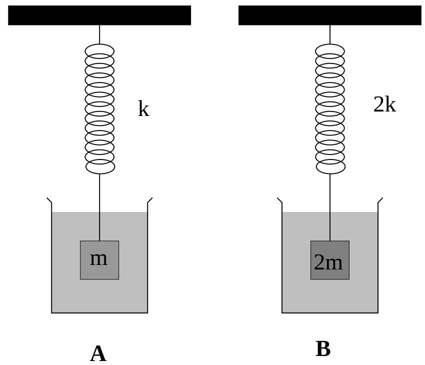
Both systems are pulled to an initial displacement from equilibrium and released to undergo damped oscillations with the blocks remaining fully submerged in the fluid at all times. The natural frequencies of two systems are ω_A and ω_B, respectively. The measured exponential decay times of the oscillation amplitude are τ_A and τ_B, respectively.

a) Write down the ratio for natural frequencies: $\dfrac{\omega_B}{\omega_A} = \boxed{\quad}$

b) Write down the ratio for damping times: $\dfrac{\tau_B}{\tau_A} = \boxed{\quad}$

c) Which oscillator will damp out its initial energy faster/sooner (A or B): $\boxed{\quad}$.

Problem 586. problems-1/oscillation-sa-damped-oscillation-soln.tex



Two springs with different spring constants (k and $2k$, respectively) support two blocks of the ***same size and shape***, but different masses, $m_B = 2m_A$. The blocks are fully submerged in the ***same damping fluid***, therefore, they have the same coefficient of damping.

Both systems are pulled to an initial displacement from equilibrium and released to undergo damped oscillations with the blocks remaining fully submerged in the fluid at all times. The natural frequencies of two systems are ω_A and ω_B, respectively. The measured exponential decay times of the oscillation amplitude are τ_A and τ_B, respectively.

a) Write down the ratio for natural frequencies: $\dfrac{\omega_B}{\omega_A} = \boxed{1}$

b) Write down the ratio for damping times: $\dfrac{\tau_B}{\tau_A} = \boxed{2}$

c) Which oscillator will damp out its initial energy faster/sooner (A or B): $\boxed{\text{A}}$.

Solution: "Natural frequencies" here refers to the *undamped* frequencies $\omega = \sqrt{k/m}$, so:

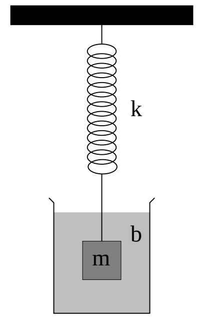
$$\frac{\omega_B}{\omega_A} = \frac{\sqrt{2k/2m}}{\sqrt{k/m}} = 1$$

The damping time $\tau = 2m/b$ for each of them, so:

$$\frac{\tau_B}{\tau_A} = \frac{4m/b}{2m/b} = 2$$

We can use Q or τ either one to answer this. Let's use $Q = m\omega/b$, since ω and b are the same. Then $Q_B/Q_A = 2$, and the one with higher Q loses energy **more slowly**, so A decays faster. That's also obvious from the comparison of $\tau_B = 2\tau_A$, larger (amplitude) τ decays more slowly.

Problem 587. problems-1/oscillation-sa-damping-variation.tex



The damped oscillator above is set in motion at time $t = 0$. Fill in the following table with x's in the provided boxes. τ is the exponential damping time of the amplitude, and ω_0 is the natural frequency.:

If b increases: τ increases decreases remains unchanged. ω_0 increases decreases remains unchanged

☐ ☐ ☐ ☐ ☐ ☐

If m increases: τ increases decreases remains unchanged. ω_0 increases decreases remains unchanged

☐ ☐ ☐ ☐ ☐ ☐

If k increases: τ increases decreases remains unchanged. ω_0 increases decreases remains unchanged

☐ ☐ ☐ ☐ ☐ ☐

Problem 588. problems-1/oscillation-sa-damping-variation-soln.tex

The damped oscillator above is set in motion at time $t = 0$. Fill in the following table with x's in the provided boxes. τ is the exponential damping time of the amplitude, and ω_0 is the natural frequency.:

In case you don't understand the question, recall that a general exponential process is:

$$f(t) = f_0 e^{-t/\tau}$$

where τ is the time required for f to decay to $1/e$ of its value at any time, e.g. $f(t+\tau) = f(t)/e$.)

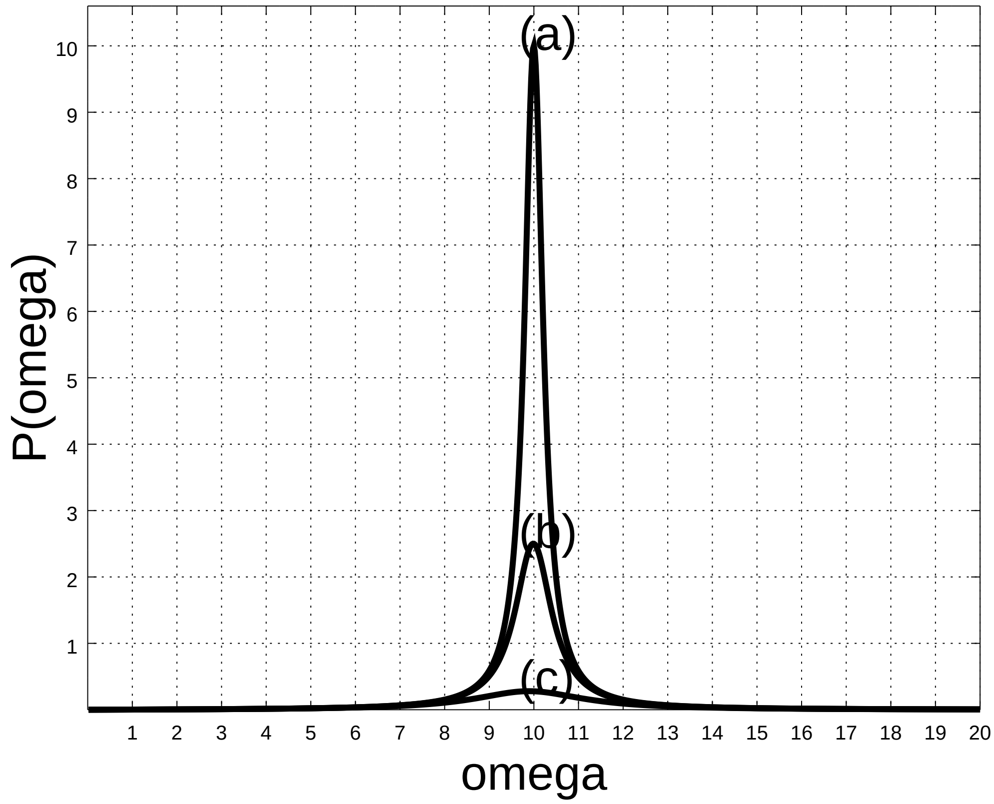
Note well that:

$$x(t)Ae^{-\frac{bt}{2m}}\cos(\omega' t + \delta)$$

so that the exponential damping time is $\tau = 2m/b$ and scales linearly with m and inversely with b. On the other hand, the natural angular frequency $\omega_0 = \sqrt{k/m}$ is independent of b.

So increasing b decreases τ (which damps it *faster*, as expected) but doesn't change ω_0. Increasing m increases τ and decreases ω_0. Increasing k doesn't change τ but increases ω_0.

Problem 589. problems-1/oscillation-sa-estimate-Q-resonance-curve.tex



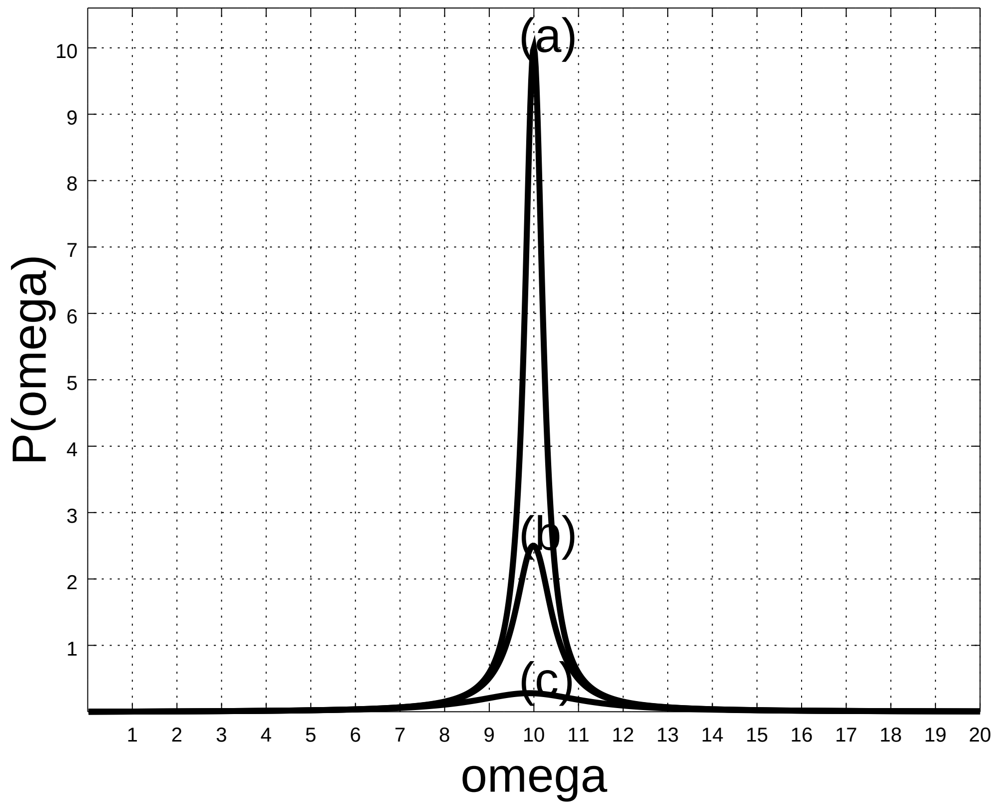
In the figure above, three resonance curves showing the amplitude of steady-state driven oscillation $A(\omega)$ as functions of ω. In all three cases the resonance frequency ω_0 is the same. Put down an *estimate* of the Q-value of each oscillator by looking at the graph. It may help for you to put down the definition of Q most relevant to the process of estimation on the page.

a)

b)

c)

Problem 590. problems-1/oscillation-sa-estimate-Q-resonance-curve-soln.tex



a) $Q_a \approx 15 - 16$ (larger than 10, smaller than 20).

b) $Q_b \approx 10$.

c) $Q_c \approx 3 - 4$ (larger than 2, smaller than 5).

Problem 591. problems-1/oscillation-sa-match-the-solution.tex

You are presented with three identical simple harmonic oscillators, **A,B,C**, which oscillate with a known harmonic frequency ω. They differ only in their initial conditions. ***At time t = 0***, the attached masses have an initial position and velocity (x_i, v_i) given by:

A $(x_A = x_0, \quad v_A = 0)$

B $(x_B = 0, \quad v_B = v_0)$

C $(x_C = x_0, \quad v_C = x_0 * \omega)$

where x_0 and v_0 are ***positive numbers not equal to zero*** in the appropriate units.

Match each set of initial conditions to the corresponding solution from the list of possible solution ***forms*** below. Note that you do not have to identify the specific amplitude A that corresponds to the initial conditions; you are basically only identifying the correct phase.

Put A, B or C into the correct box, where "No solution present" is a possible answer for one or more of them:

☐ $x(t) = A\cos(\omega t)$

☐ $x(t) = A\cos(\omega t - \pi/4)$

☐ $x(t) = A\sin(\omega t)$

☐ $x(t) = A\cos(\omega t + \pi/4)$

☐ No solution with the correct phase present.

Problem 592. problems-1/oscillation-sa-match-the-solution-soln.tex

You are presented with three identical simple harmonic oscillators, **A,B,C**, which oscillate with a known harmonic frequency ω. They differ only in their initial conditions. ***At time*** $\boldsymbol{t = 0}$, the attached masses have an initial position and velocity (x_i, v_i) given by:

A $(x_A = x_0, \quad v_A = 0)$

B $(x_B = 0, \quad v_B = v_0)$

C $(x_C = x_0, \quad v_C = x_0 * \omega)$

where x_0 and v_0 are ***positive numbers not equal to zero*** in the appropriate units.

Match each set of initial conditions to the corresponding solution from the list of possible solution ***forms*** below. Note that you do not have to identify the specific amplitude A that corresponds to the initial conditions; you are basically only identifying the correct phase.

Put A, B or C into the correct box, where "No solution present" is a possible answer for one or more of them:

First is simply $x(t) = (A = x_0)\cos(\omega t)$.

Second is equally simply $x(t) = (A = \frac{x_0}{\omega})\sin(\omega t)$.

The third is tricky. We need (at $t = 0$) $\cos(\phi) = -\sin(\phi) = \frac{\sqrt{2}}{2}$ or $\phi = -\pi/4 = -45°$, so: $x(t) = (A = \sqrt{2}x_0)\cos(\omega t - \pi/4)$, $v(t) = -\sqrt{2}x_0\omega\sin(\omega t - \pi/4)$.

$\boxed{A}$ $x(t) = A\cos(\omega t)$

$\boxed{C}$ $x(t) = A\cos(\omega t - \pi/4)$

$\boxed{B}$ $x(t) = A\sin(\omega t)$

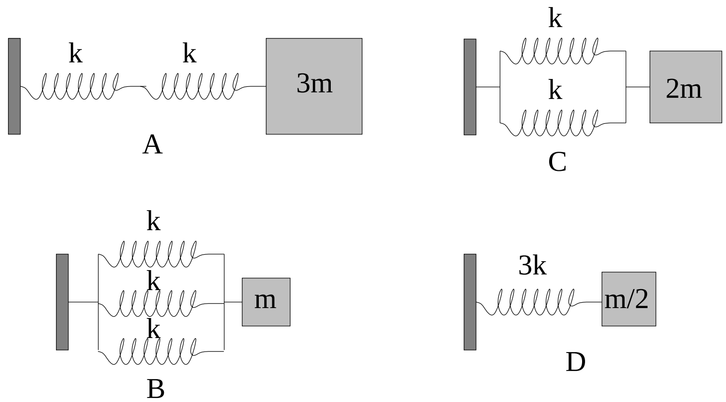
$\boxed{}$ $x(t) = A\cos(\omega t + \pi/4)$

$\boxed{}$ No solution with the correct phase present.

Problem 593. problems-1/oscillation-sa-roman-soldiers-bridge-resonance.tex

Roman soldiers (like soldiers the world over even today) marched in step at a constant frequency – except when crossing wooden bridges, when they broke their march and walked over with random pacing. Why? What might have happened (and originally did sometimes happen) if they marched across with a collective periodic step?

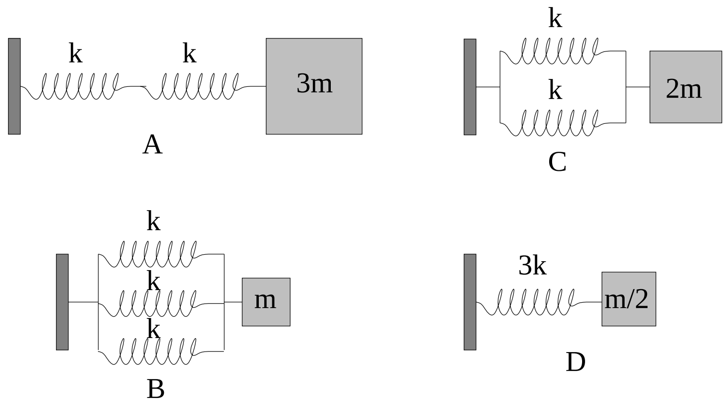
Problem 594. problems-1/oscillation-sa-series-parallel-frequency.tex



Find the ratio of the angular frequencies of each spring-mass combination above to $\omega_0 = \sqrt{k/m}$.

$$\frac{\omega_A}{\omega_0} = \boxed{} \quad \frac{\omega_B}{\omega_0} = \boxed{} \quad \frac{\omega_C}{\omega_0} = \boxed{} \quad \frac{\omega_D}{\omega_0} = \boxed{}$$

Problem 595. problems-1/oscillation-sa-series-parallel-frequency-soln.tex



Find the ratio of the angular frequencies of each spring-mass combination above to $\omega_0 = \sqrt{k/m}$.

$$\frac{\omega_A}{\omega_0} = \boxed{\sqrt{6}/6} \quad \frac{\omega_B}{\omega_0} = \boxed{\sqrt{3}/3} \quad \frac{\omega_C}{\omega_0} = \boxed{1} \quad \frac{\omega_D}{\omega_0} = \boxed{\sqrt{6}}$$

Solution: The idea is simple. For N springs in series, $k_{\text{eff}} = k/N$. For N springs in parallel, $k_{\text{eff}} = Nk$. Hence:

$$\omega_A = \sqrt{\frac{k/2}{3m}} = \frac{\sqrt{6}}{6}\omega_0 \qquad \omega_B = \sqrt{\frac{k/3}{m}} = \frac{\sqrt{3}}{3}\omega_0$$

$$\omega_C = \sqrt{\frac{2k}{2m}} = \omega_0 \qquad \omega_D = \sqrt{\frac{3k}{m/2}} = \sqrt{6}\,\omega_0$$

Problem 596. problems-1/oscillation-sa-sho-true-facts.tex

The one-dimensional motion of a mass m is described by $x(t) = A\sin(\omega t)$. Identify the true and false statements among the following by placing a T in the provided box for true statements and an F in the provided box for false statements:

a) $\square$ If A and ω are constant (i.e. – independent of time t) the motion is simple harmonic motion.

b) $\square$ The mass m starts at $t = 0$ with zero velocity.

c) $\square$ If the motion of mass m is simple harmonic oscillation, the potential energy of the mass can be written $U(x) = \frac{1}{2}m\omega^2 x^2$.

d) $\square$ If the motion of the mass m is simple harmonic oscillation, the total force acting on the mass can be written $F_x = -m\omega^2 x$.

Problem 597. problems-1/oscillation-sa-sho-true-facts-soln.tex

The one-dimensional motion of a mass m is described by $x(t) = A\sin(\omega t)$. Identify the true and false statements among the following by placing a T in the provided box for true statements and an F in the provided box for false statements:

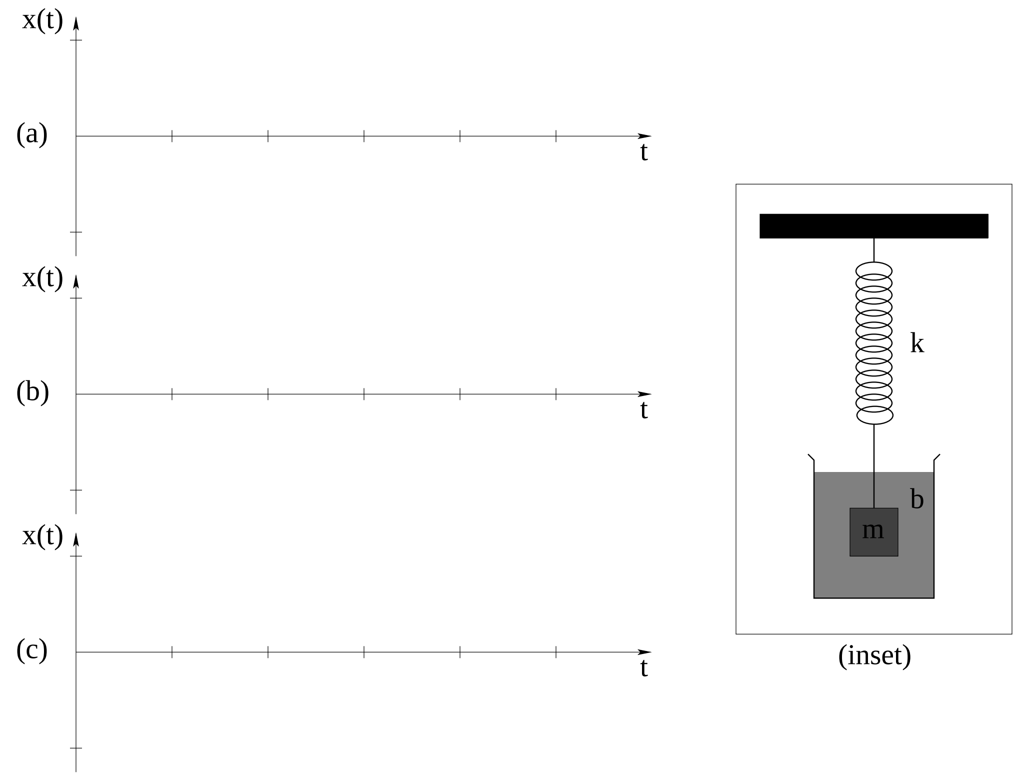
a) If A and ω are constant (i.e. – independent of time t) the motion is simple harmonic motion. **True**

b) The mass m starts at $t = 0$ with zero velocity. **False**

c) If the motion of mass m is simple harmonic oscillation, the potential energy of the mass can be written $U(x) = \frac{1}{2}m\omega^2 x^2$. **True**

d) If the motion of the mass m is simple harmonic oscillation, the total force acting on the mass can be written $F_x = -m\omega^2 x$. **True**

Problem 598. problems-1/oscillation-sa-sketch-damped-oscillation.tex



A mass m is attached to a spring with spring constant k and immersed in a damping fluid with linear damping coefficient b as shown in the inset figure above. Equilibrium is at $x = 0$ meters. At time $t = 0$ seconds the mass is pulled to $x(0) = 1$ meters and released from rest. The ***period*** of the oscillator in the *absence* of damping is $T = 2$ seconds. On the provided axes with integer tick-marks above, sketch the following:

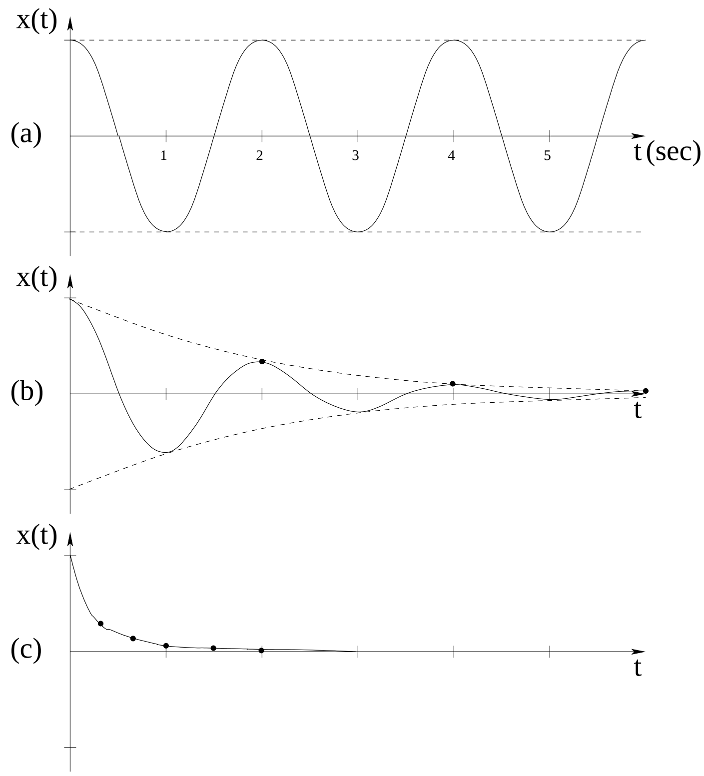
a) $x(t)$ in the absence of damping.

b) $x(t)$ if $b/2m = 1/3$ (underdamped, assume that $\omega' \approx \omega_0$).

c) $x(t)$ in the case where $b/2m = \pi$ (critically damped).

The second two curves only need to be *qualitatively* correct (you don't have to plot them exactly), but they should also not be crazily out of scale. You may use $e = 2.72 \approx 3$ to make drawing the curves easier without needing a calculator.

Problem 599. problems-1/oscillation-sa-sketch-damped-oscillation-soln.tex



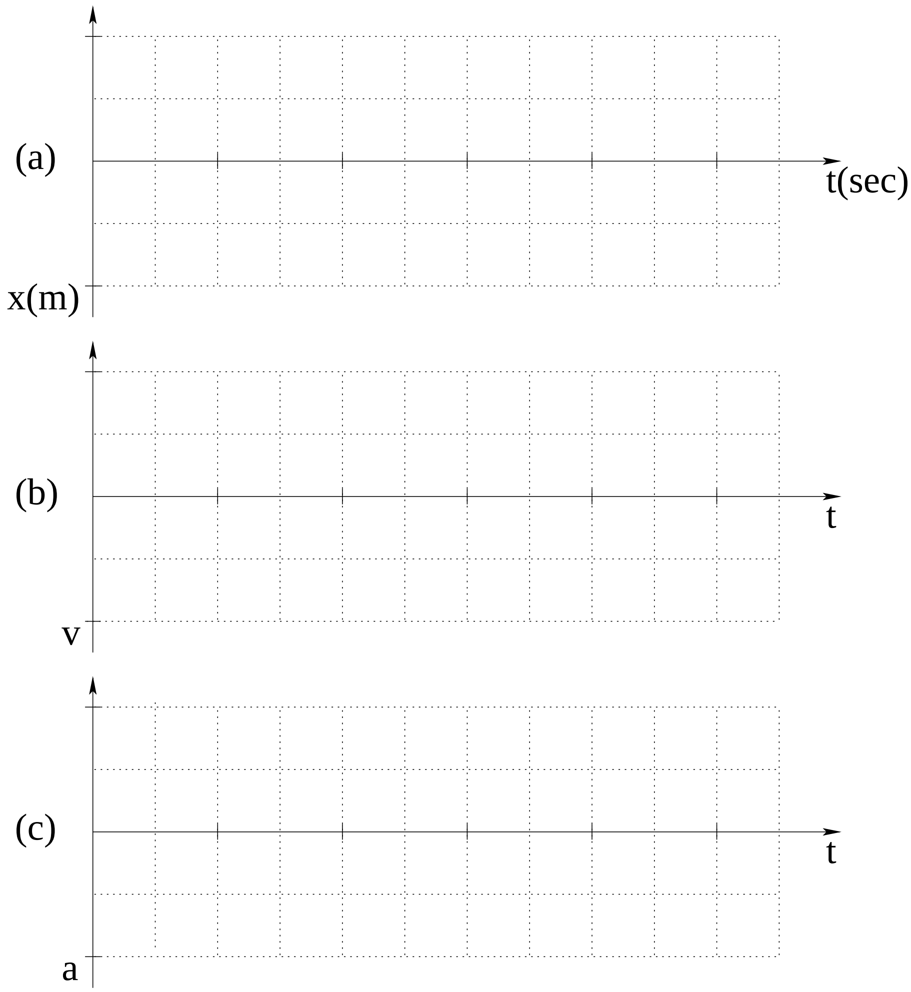
Note the dots to help draw the exponential(s).

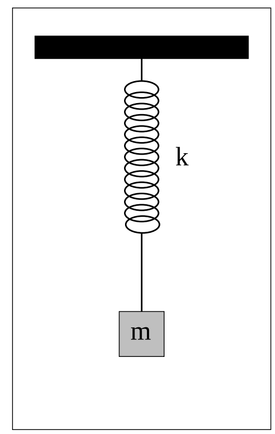
a) $x(t)$ in the absence of damping.

b) $x(t)$ if $b/2m = 1/3$ (underdamped, assume that $\omega' \approx \omega_0$).

c) $x(t)$ in the case where $b/2m = \pi$ (critically damped).

Problem 600. problems-1/oscillation-sa-sketch-oscillation-plus-damping.tex



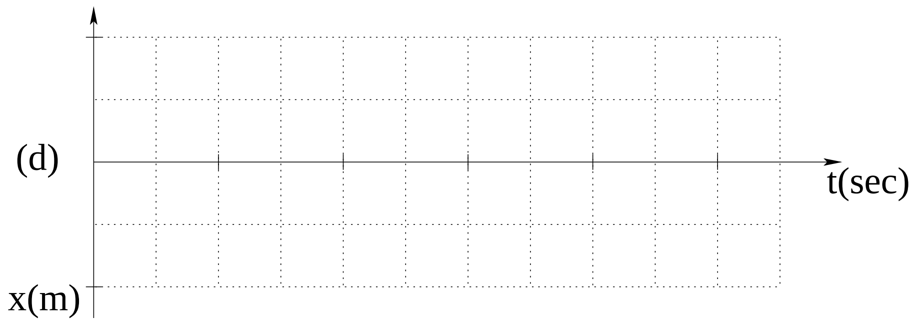


(inset)

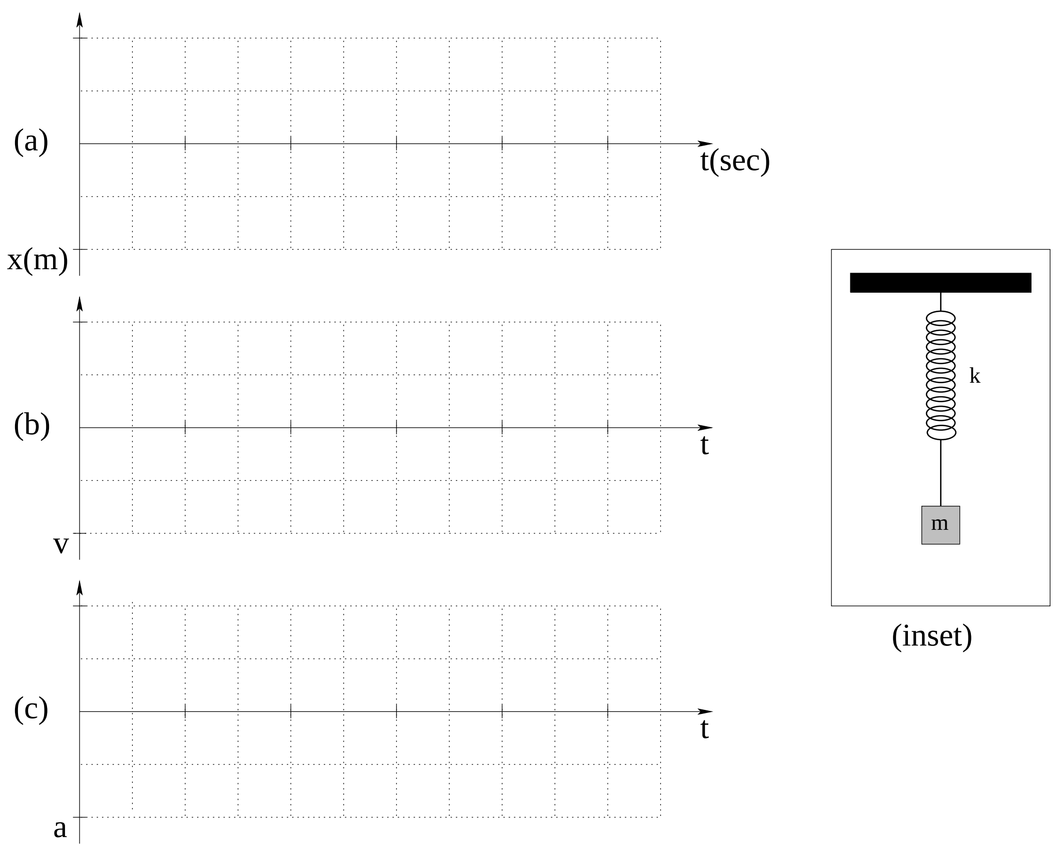
A mass m is attached to a spring with spring constant k as shown in the inset figure above.

On the provided axes above, sketch $x(t)$, $v(t)$ and $a(t)$, given that at time $t = 0$ the mass is pulled to $x(0) = X_0 = 1$ meter (relative to equilibrium) and released from rest, assuming no damping. The ***period*** is $T = 1$ second, and you should use the tic-marks on the t axis as seconds. Your graphs should have the correct sign, phase, period, and you should label the peak positive value in terms of the givens on the ordinate axes.

Suppose that the block is then placed in a damping fluid. On the axes labelled d) below, sketch the position as a function of time for an oscillator with period $T = 1$ seconds and damping time $\tau = 4$ seconds, again assuming that $x(0) = X_0 = 1$ meter.



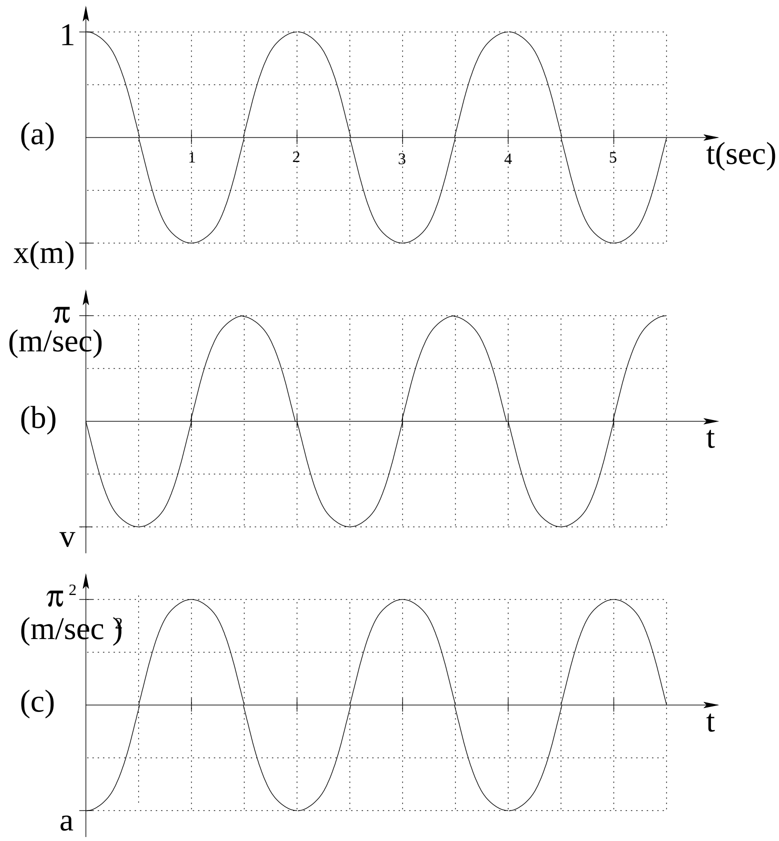
Problem 601. problems-1/oscillation-sa-sketch-oscillation.tex



A mass m is attached to a spring with spring constant k as shown in the inset figure above. There ***is no damping***.

On the provided axes above, sketch $x(t)$, $v(t)$ and $a(t)$, given that at time $t = 0$ the mass is pulled to $x(0) = X_0 = 1$ (relative to equilibrium) and released from rest. The ***period*** is $T = 2$ seconds, and you should use the tic-marks on the t axis as seconds. Your graphs should have the correct sign, phase, period, and you should label the peak positive value in terms of the givens on the ordinate axes.

Problem 602. problems-1/oscillation-sa-sketch-oscillation-soln.tex



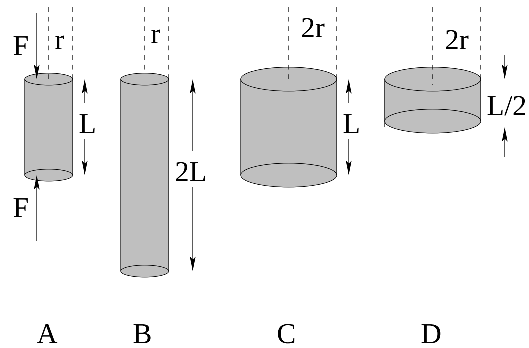
$$x(t) = A\cos(\omega t)$$
$$v(t) = -A\omega\sin(\omega t)$$
$$a(t) = -A\omega^2\cos(\omega t)$$

and

$$\omega = \frac{2\pi}{T} = \pi \text{rad/sec}$$

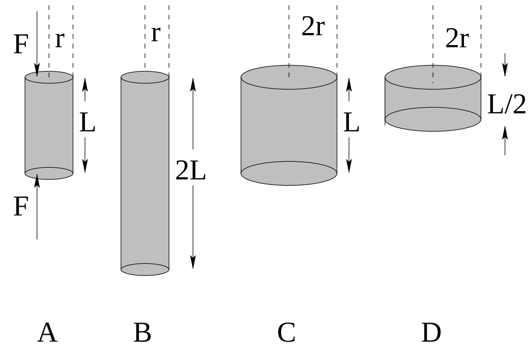
11.1.3 Ranking Problems

Problem 603. problems-1/oscillation-ra-compressed-rods-youngs-modulus-scaling.tex



Rank the magnitude of the compression ΔL of the rods (made of the same material) above when a force with magnitude F is exerted between the ends as shown in case A. Equality is a possibility. Your answer should look something like $C = D > A > B$.

Problem 604. problems-1/oscillation-ra-compressed-rods-youngs-modulus-scaling-soln.tex



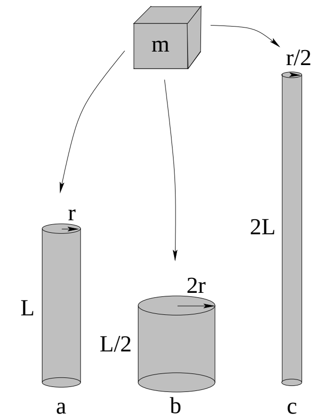
Fast and Easy: The compression ΔL scales like:

$$\Delta L \propto Y \frac{FL}{A}$$

where F is the same and (since they are all the same material) Y (Young's Modulus) is the same. Hence

D ¡ C ¡ A ¡ B

Problem 605. problems-1/oscillation-ra-compression-three-rods-1.tex



In the figure above three rods made out of copper are shown with the dimensions given. In (a), a mass m is placed on top of the rod (which rests on a rigid table) and the rod is observed to be compressed and shrinks by a length ΔL. By what length ΔL_i do you expect rods (b) and (c) to be compressed by if the *same* mass m is placed on top of them? (Express your answer as a pure number times ΔL_a.)

$\Delta L_b = \boxed{}$ $\Delta L_c = \boxed{}$

Problem 606. problems-1/oscillation-ra-compression-three-rods-1-soln.tex

The rule is "stress equals (minus) Young's modulus times strain" so:

$$\text{stress} = \frac{F}{A} = -Y\frac{\Delta L}{L} = -Y \times \text{stress}$$

or (rearranging, noting same materials have same Y, for case a) where $A = \pi r^2$):

$$|\Delta L| = \frac{FL}{YA}$$

In b) $A_b = 4A$, $L_b = L/2$, so $\Delta L_b = \Delta L/8$. In c) $A_c = A/4$, $L_c = 2L$, so $\Delta L_c = 8\Delta L$.

Problem 607. problems-1/oscillation-ra-mass-spring-double-displacement.tex

Two identical masses are attached to two identical springs. The first mass is pulled to a distance x_0 from equilibrium. The second one is pulled to a distance $2x_0$ from equilibrium. At time $t = 0$ they are released. The first mass reaches its equilibrium point at time t_1, the second one at time t_2.

What is the ratio t_2/t_1?

$$\frac{t_2}{t_1} = \boxed{}$$

Problem 608. problems-1/oscillation-ra-mass-spring-double-displacement-soln.tex

Two identical masses are attached to two identical springs. The first mass is pulled to a distance x_0 from equilibrium. The second one is pulled to a distance $2x_0$ from equilibrium. At time $t = 0$ they are released. The first mass reaches its equilibrium point at time t_1, the second one at time t_2.

What is the ratio t_2/t_1?

$$\frac{t_2}{t_1} = \boxed{\quad 1 \quad}$$

Solution: This is a very simple problem. The answer is obviously 1, because ***the period of an ideal oscillator does not depend on the amplitude of oscillation!***

The algebra to "prove" this answer – which is not required, and should not really be needed to find it – is that for both masses, the equation of motion is:

$$k_1 = k_2 = k \qquad m_1 = m_2 = m$$

so:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \quad \Rightarrow \quad \omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k}}$$

for both masses. The time required to return to equilibrium for either mass is:

$$t_{1,2} = T/4 = \frac{\pi}{2}\sqrt{\frac{m}{k}}$$

(a quarter of the period) independent of the magnitude of the initial position so:

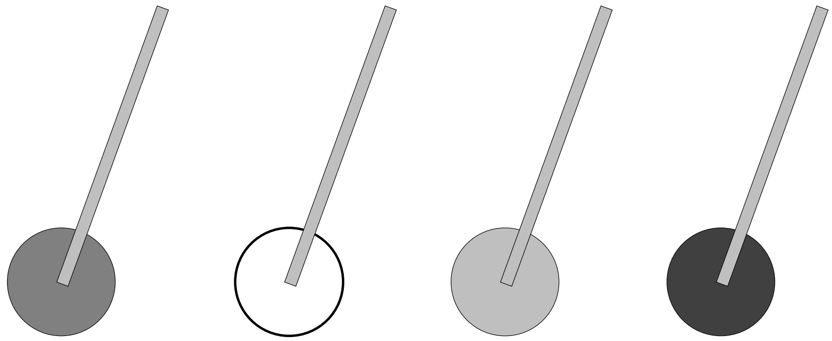
$$\boxed{\frac{t_2}{t_1} = 1}$$

Problem 609. problems-1/oscillation-ra-mass-swing-double-displacement.tex

Two kids are sitting on swings of equal length. One of them has about twice the mass of the other (but they are about the same height). The lighter one is pulled back to an initial (small) angle θ_0. The heavier one is pulled back to a (still small!) angle $2\theta_0$. At $t = 0$ they are both released. It takes the lighter one a time t_l to reach the lowest point of his trajectory, and the heavier one a time t_h.

What is the ratio t_h/t_l?

Problem 610. problems-1/oscillation-ra-physical-pendula-periods.tex



In the figures above, four physical pendulums are drawn. All consist of a light (massless) rod of length L to the center of mass of different shaped masses connected to the end. All of the shapes have the same mass M and the same primary length scale R. Rank the **periods** of the physical pendulums from lowest (highest frequency!) to the highest (lowest frequency!). Equality is a possibility.

The moments of inertia of the round objects (about their centers of mass) are:

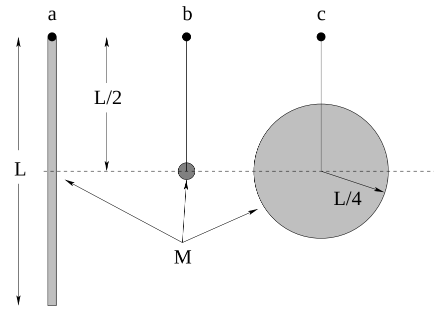
A) $I = \frac{1}{2}MR^2$ (disk)

B) $I = MR^2$ (hoop)

C) $I = \frac{2}{3}MR^2$ (hollow ball)

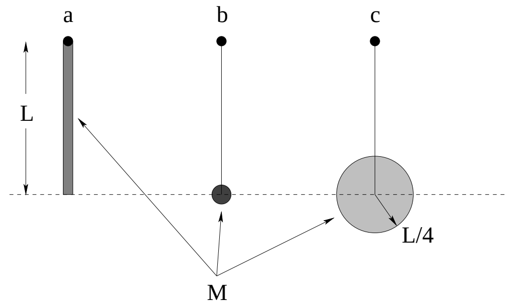
D) $I = \frac{2}{5}MR^2$ (solid ball)

Problem 611. problems-1/oscillation-ra-physical-pendulums-1.tex



In the figure above, three pendulums are suspended from frictionless pivots. The first is a rod of mass M and length L. The second is a "point" mass M with negligible radius. The third is a disk of mass M and radius $L/2$. In all three cases, the center of mass of the pendulum is a distance $L/2$ from the pivot and the mass is constrained to rotate around the pivot (physical pendulum). Rank the angular frequencies (where equality is allowed) so that an answer might be (but probably isn't) $\omega_a > \omega_b = \omega_c$.

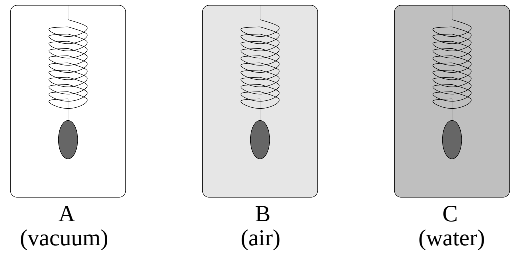
Problem 612. problems-1/oscillation-ra-physical-pendulums-2.tex



In the figure above, three pendulums are suspended from frictionless pivots. The first is a thick rod of mass M and length L. The second is a "point" mass M with negligible radius on a thin (massless) rod of length L. The third is a disk of mass M and radius $L/4$ on the end of a thin (massless) rod of so that its center of mass is a distance L away from the pivot. In all three cases, the mass is constrained to rotate around the pivot as a ***physical pendulum***.

Rank the ***angular frequencies in increasing order*** (where equality is allowed) so that an answer might be (but probably isn't) $\omega_a > \omega_b = \omega_c$.

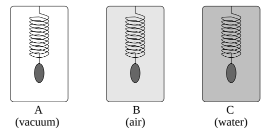
Problem 613. problems-1/oscillation-ra-rank-the-damped-frequency.tex



In the figure above identical masses are connected to identical springs and located in three different labelled containers. All three masses are pulled to the same distance from equilibrium and are released from rest. The container A contains a vacuum, container B is filled with ordinary room-temperature air at 1 atmosphere of pressure, and container C contains water.

Rank the ***frequencies*** of the oscillation of the three masses by their container letter, where (precise) equality is a possibility. That is, a possible answer might be $f_A = f_C < f_B$ (but probably isn't). (It is wise to explain your answer with a few words or an equation.)

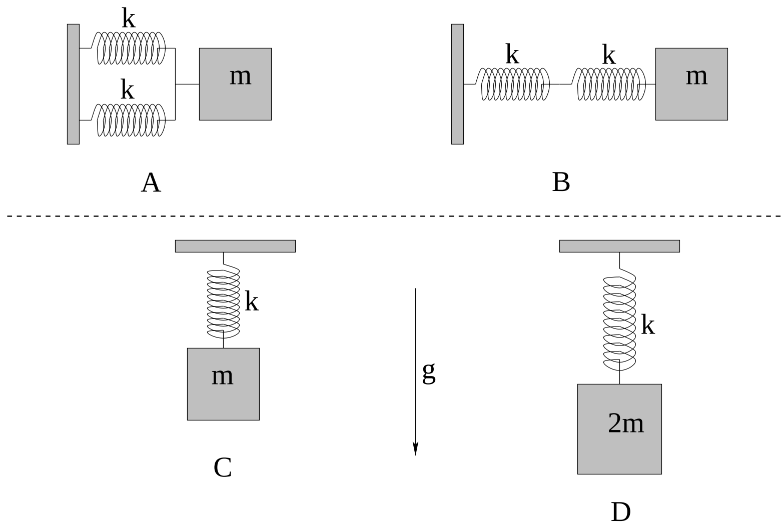
Problem 614. problems-1/oscillation-ra-rank-the-damped-periods.tex



In the figure above identical masses are connected to identical springs and located in three different labelled containers. All three masses are pulled to the same distance from equilibrium and are released from rest. The container A contains a vacuum, container B is filled with ordinary room-temperature air at 1 atmosphere of pressure, and container C contains water.

Rank the ***period*** of the oscillation of the three masses by their container number, where (precise) equality is a possibility. That is, a possible answer might be $T_a = T_c < T_b$ (but probably isn't). Explain your answer with a few words or an equation.

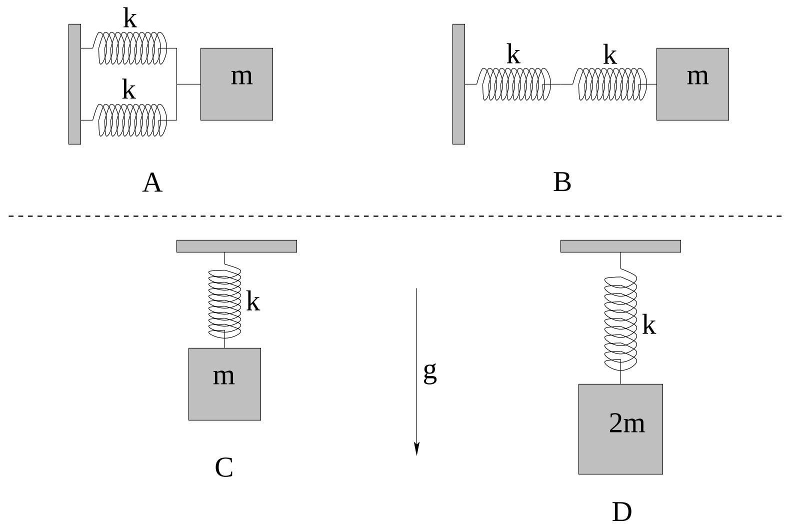
Problem 615. problems-1/oscillation-ra-rank-the-periods.tex



In the figure above, rank the ***periods*** of each ***pair*** of oscillators shown (where equality is allowed). That is, fill in the boxes in the two expressions below with a $<, >, =$ sign as appropriate.

$T_A \; \square \; T_B$ $\qquad\qquad$ $T_C \; \square \; T_D$

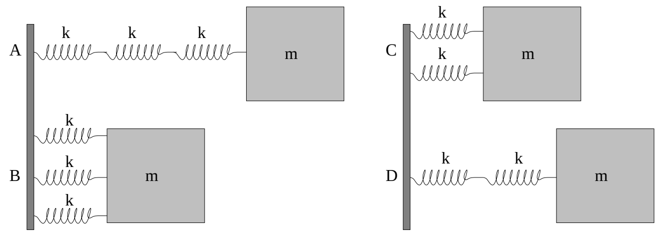
Problem 616. problems-1/oscillation-ra-rank-the-periods-soln.tex



The period is inversely proportional to the (angular) frequency $\sqrt{\frac{k}{m}}$, so:

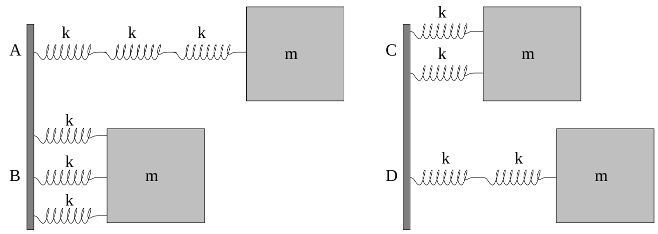
$$T_A < T_B \qquad\qquad T_C < T_D$$

Problem 617. problems-1/oscillation-ra-series-parallel-frequency-easy.tex



Rank the ***oscillation frequencies*** of the identical masses m connected to the springs in the figure above from ***lowest to highest*** with equality a possibility. The springs have spring constant k, and you should neglect damping. A possible answer is (as always) $D < A = B < C$ or the like.

Problem 618. problems-1/oscillation-ra-series-parallel-frequency-easy-soln.tex



Recall parallel and series addition rules:

$$k_{\text{tot}} = \sum_{i\text{parallel}} = k_i$$

$$\frac{1}{k_{\text{tot}}} = \sum_{i\text{series}} = \frac{1}{k_i}$$

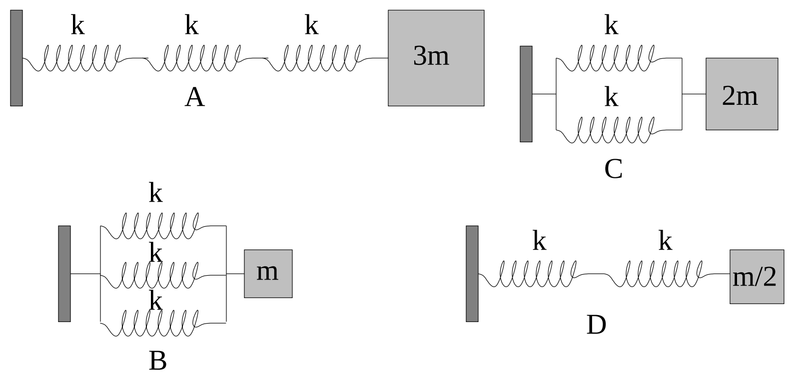
Start by evaluating the total effective spring constant for each configuration:

$$\begin{array}{llcl} \textbf{A} & k_A & = & k/3 \\ \textbf{B} & k_B & = & 3k \\ \textbf{C} & k_A & = & 2k \\ \textbf{D} & k_A & = & k/2 \end{array} \tag{11.1}$$

The angular frequency $\omega_i = \sqrt{\frac{k_i}{m}} = 2\pi f_i$, so the frequencies (linear or angular) ***scale*** with $\sqrt{k_i}$ as the masses are all equal. Hence:

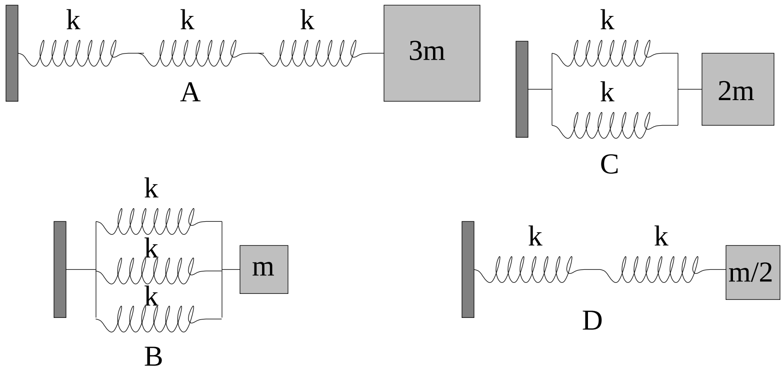
$$f_A < f_D < f_C < f_B$$

Problem 619. problems-1/oscillation-ra-series-parallel-frequency.tex



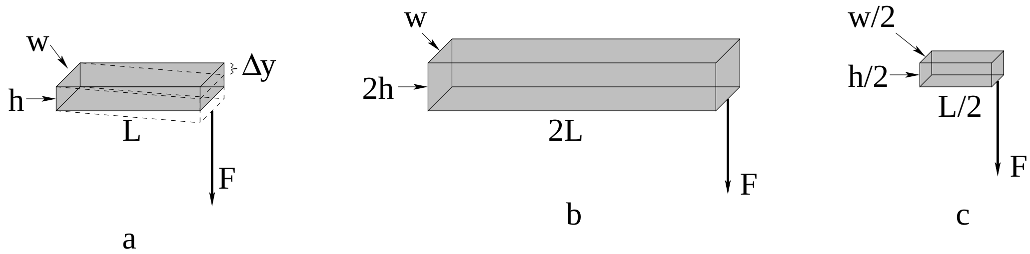
Rank the frequencies of the masses on the spring arrangements in the figure above, from **lowest to highest** with equality a possibility. Neglect damping. A possible answer is (as always) $D < A = B < C$ or the like.

Problem 620. problems-1/oscillation-ra-series-parallel-period.tex



Rank the ***period of oscillation*** of the masses on the spring arrangements in the figure above, from ***lowest to highest*** with equality a possibility. Neglect damping. A possible answer could be (as always) $D < A = B < C$ but probably isn't.

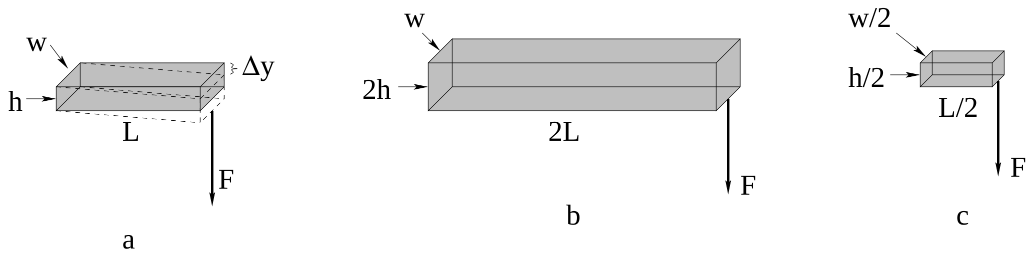
Problem 621. problems-1/oscillation-ra-shear-three-rods-1.tex



In the figure above, three light wooden boards and their relative dimensions are shown. The boards are each fixed in a vise (not shown) on the left hand side so that the left end of each board cannot move. A downward force $\vec{F}$ is applied at the right hand end of each board. The first board is bent by this force so that its right hand end is displaced downward by a distance Δy. By how much are the right hand ends of the other two boards displaced downward? (Express your answers as multiples of Δy.)

$\Delta y_b = \boxed{} \times \Delta y$ $\qquad$ $\Delta y_c = \boxed{} \times \Delta y$

Problem 622. problems-1/oscillation-ra-shear-three-rods-1-soln.tex



Solution: Use the formula for *shear* stress and *shear* strain:

$$\frac{F}{A} = M\frac{\Delta y}{L}$$

For the first one (a):

$$\Delta y = MF\frac{L}{A}$$

defines the scaling. M and F don't change, but for the second one (b) $L \to 2L$, $A \to 2A$, so $\boldsymbol{\Delta y_b}$ **is unchanged!**

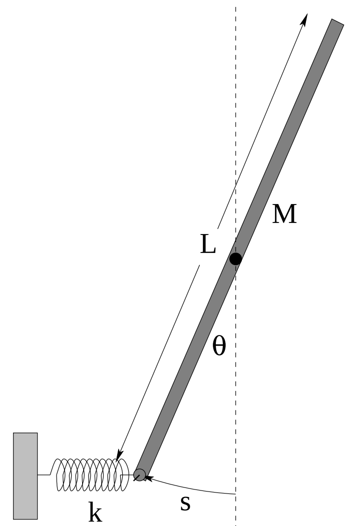
For the third one (c), $L \to L/2$, $A \to A/4$, so $\boldsymbol{\Delta y \to 2\Delta y}$.

That is:

$$\Delta y_b = \boxed{1} \times \Delta y \qquad\qquad \Delta y_c = \boxed{2} \times \Delta y$$

11.1.4 Regular Problems

Problem 623. problems-1/oscillation-pr-bar-and-spring-1.tex



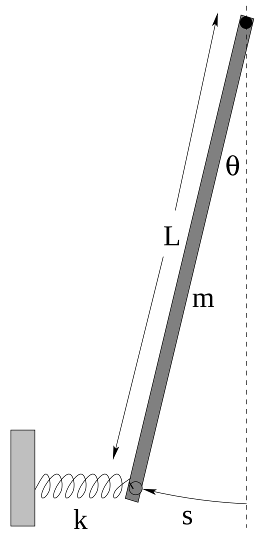
In the figure above a rigid rod of mass M and length L is pivoted in the center with a frictionless bearing. Its lower end is attached to a spring with spring constant k as shown that is unstretched (at equilibrium) when the rod is vertical and $\theta = 0$.

For *small displacements* $s \ll L$ (where one can use the small angle approximation), the spring will exert a restoring force $F_s = -ks \approx -k(L/2)\theta$ along the arc of motion of the end of the rod. It is pulled to an initial small displacement angle θ_0 and released at time $t = 0$.

a) What is the period of this oscillator for small oscillations?

b) What is the ***angular velocity*** Ω of the rod when it reaches its equilibrium position at $\theta = 0$? (Note well: Do not confuse ω_0, the angular frequency of oscillation, and $\Omega = \frac{d\theta}{dt}$, the angular velocity of the rod! Don't forget direction!)

Problem 624. problems-1/oscillation-pr-bar-and-spring-2.tex



In the figure above a rigid rod of mass m and length L is pivoted at the end with a frictionless bearing. Its lower end is attached to a spring with spring constant k as shown that is unstretched (at equilibrium) when the rod is vertical and $\theta = 0$.

For *small displacements* $s \ll L$ (where one can use the small angle approximation), the spring will exert a restoring force $F_s = -ks$ along the arc of motion of the end of the rod. It is pulled to an initial small displacement angle θ_0 and released at time $t = 0$, at which point it will begin to oscillate with angular frequency ω_0.

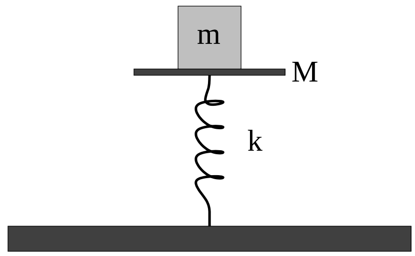
a) **Neglecting damping**, find the period T_0 of this oscillator for small oscillations and **sketch a qualitatively correct graph** of $\theta(t)$ for the rod. (Note well: *both* the spring *and* gravity contribute to the motion of the rod!)

b) What is the angular velocity of the rod $\omega = \frac{d\theta}{dt}$ when it reaches its equilibrium position at $\theta = 0$? Do not confuse the angular velocity of the rod with its angular frequency.

c) Suppose one compares the predicted motion $\theta(t)$ to the motion one would actually observe in the real world, where the system surely would be at least weakly damped. **Sketch a graph** that is **qualitatively** correct illustrating what $\theta(t)$ might **really** look like when weak damping is taken into account.

(Hint: The moment of inertia of a rod pivoted about one end is $\frac{1}{3}ML^2$.)

Problem 625. problems-1/oscillation-pr-block-on-vertically-oscillating-plate.tex



A block of mass m is sitting on a plate of mass M. It is supported by a vertical ideal massless spring with spring constant k. Gravity points down.

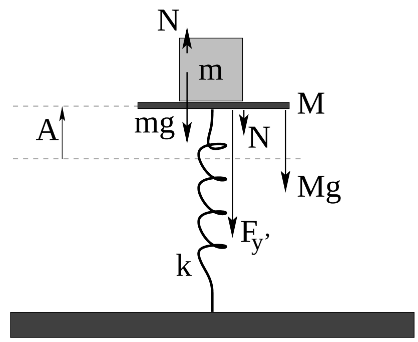
a) When the system is at rest, how much is the spring compressed from its completely uncompressed length?

b) The spring is pushed down an ***extra*** distance A and released. Assuming that the mass m ***remains on the plate***, what is its frequency of vertical oscillation?

c) What is the maximum value of A such that the small mass m will not leave the plate at any point in the motion?

Express all answers in terms of m, M, k, g.

Problem 626. problems-1/oscillation-pr-block-on-vertically-oscillating-plate-soln.tex



A block of mass m is sitting on a plate of mass M. It is supported by a vertical ideal massless spring with spring constant k. Gravity points down.

a) When the system is at rest, how much is the spring compressed from its completely uncompressed length?

b) The spring is pushed down an ***extra*** distance A and released. Assuming that the mass m ***remains on the plate***, what is its frequency of vertical oscillation?

c) What is the maximum value of A such that the small mass m will not leave the plate at any point in the motion?

Express all answers in terms of m, M, k, g.

Solution: a) is a static equilibrium question:

$$F_y = -ky_e - (m+M)g = 0 \quad \Rightarrow \quad \boxed{y_e = -\frac{(m+M)g}{k}}$$

where the negative sign isn't required in the answer but is useful for part b).

For b) we use N2:

$$F_y = -ky - (m+M)g = (m+M)\frac{d^2y}{dt^2}$$

and *change variables* to $y = y' + y_e$, cancelling the (constant) term with y_e to form the SHOE:

$$-ky' \cancel{- ky_e - (m+M)g} = (m+M)\frac{d^2y'}{dt^2} \quad \Rightarrow \quad -ky' = (m+M)\frac{d^2y'}{dt^2} \quad \Rightarrow \quad \frac{d^2y'}{dt^2} + \left(\frac{k}{m+M}\right)y' = 0$$

The circled part is ω^2 where $\omega = 2\pi f$ so:

$$\boxed{f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m+M}}}$$

It probably isn't worth taking a point off if students omit the $1/2\pi$ piece and give the angular frequency instead of the frequency, but technically the probably DOES as for the frequency, not the angular frequency.

The solution to part c) requires quite a lot of deep insight. First, since:

$$N - mg = ma_m$$

is the equation of motion for the block, and since $N \geq 0$ (in the positive direction ***only***) it is clear that the maximum acceleration of the upper mass will occur when $N \to 0$ and $a_m \to -g$. Under no circumstances can the upper block accelerate downward *faster* than g.

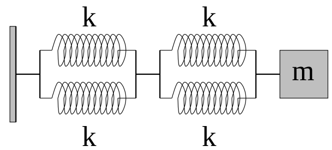
When will its downward acceleration (and that of the underlying plate) *equal* g? The answer – after a bit of meditation – is ***at the original unstretched/compressed equilibrium of the spring***. In words, when $y = 0$ the spring exerts no force at all on the plate. If the plate and block are released together at rest from $y = 0$, they will both freely fall with an acceleration of g downward and no normal force between them.

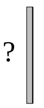
But $y = 0$ is the same as $y'_{\text{max}} = A_{\text{max}} = y_e$ from part a), or:

$$\boxed{A_{\text{max}} = \frac{(m + M)g}{k}}$$

so this must be our answer. There are several other ways to get the answer algebraically, but this one allows us to see *why* $y = 0$ must be the maximum height where the spring would not pull the plate "out from under" the (at best) freely falling block by giving it an acceleration *greater than* g downward.

Problem 627. problems-1/oscillation-pr-box-of-springs.tex





You are given a mass m, a box full of identical springs each with spring constant k, and a bunch of stiff wire you can bend and use to fasten the springs together to the wall and the mass in any combination of series and parallel you like.

I've drawn one such arrangement for you, one that will cause the mass m to oscillate harmonically on a smooth surface at angular frequency ω. Your job is to design an arrangement of springs that will make the mass oscillate at an angular frequency of $\sqrt{\frac{3k}{2m}}$, using only the (uncut) springs in the box.

a) Find the angular frequency of the four-spring oscillator I've drawn.

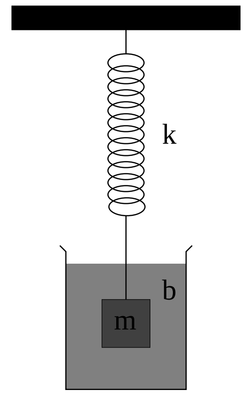
b) Draw a new arrangement on the bar underneath (or elsewhere on your paper) that will have an angular frequency of $\sqrt{\frac{3k}{2m}}$. Note well that there is more than one way to get the right answer, but some ways need (a lot) more springs than others. Try to get an answer with no more than six springs

c) Prove/show that your answer is correct.

Problem 628. problems-1/oscillation-pr-car-on-springs-resonance.tex

A car with a mass of $M = 1000$ kg rests on shock absorber springs with a collective spring constant of $k = 10^5$ N/m. It is driving down a road which has raised expansion joints every 5 meters that bounce the car. At what speed would you expect the ride to be roughest?

Problem 629. problems-1/oscillation-pr-damped-oscillation.tex



A mass m is attached to a spring with spring constant k and immersed in a medium with damping coefficient b. (Gravity, if present at all, is irrelevant as shown in class). The net force on the mass when displaced by x from equilibrium and moving with velocity v_x is thus:

$$F_x = ma_x = -kx - bv_x$$

(in one dimension).

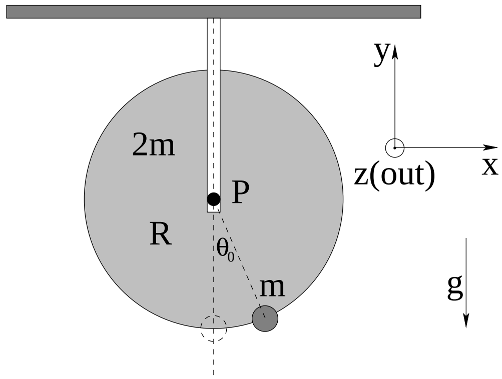
a) Convert this equation (Newton's second law for the mass/spring/damping fluid arrangement) into the equation of motion for the system, a "second order linear homogeneous differential equation" as done in class.

b) Optionally *solve* this equation, finding in particular the exponential damping rate of the solution (the real part of the exponential time constant) and the shifted frequency ω', assuming that the motion is underdamped. You can put down any form you like for the answer; the easiest is probably a sum of exponential forms. However, you may also simply **put down the solution** derived in class if you plan to *just* memorize this solution instead of learn to derive and understand it.

c) Using your answer for ω' from part b), write down the criteria for damped, underdamped, and critically damped oscillation.

d) Draw *three* qualitatively correct graphs of $x(t)$ if the oscillator is pulled to a position x_0 and released at rest at time $t = 0$, one for each damping. Note that you should be able to do this part even if you cannot derive the curves that you draw or ω'. Clearly label each curve.

Problem 630. problems-1/oscillation-pr-disk-with-rim-weight.tex



A uniform disk of radius R and mass $2m$ can freely rotate about a fixed frictionless horizontal axis passing through its fixed center P as shown. The disk has a point mass m fixed on its rim so that in equilibrium, this mass is at the lowest point ($\theta = 0$).

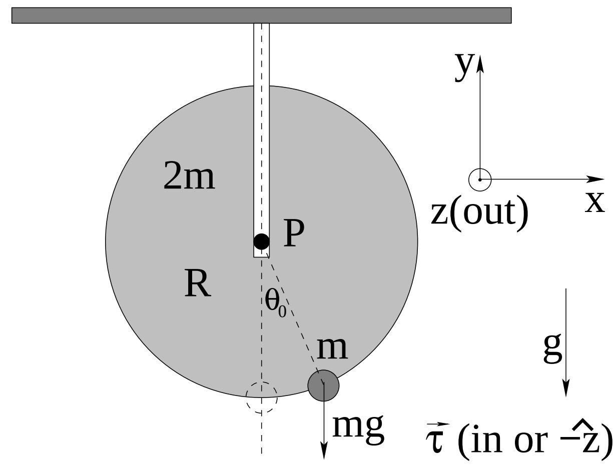
At time $t = 0$, the disk is gently rotated by the ***small, positive*** angle θ_0 ("out" or $+\hat{z}$) ***as shown*** and released ***from rest***. Answer the following questions:

a) Just after it is released, what is the net torque ***vector*** $\vec{\tau}$ about P acting on the disk (***magnitude and direction***) as a function of θ and the givens? Use the provided coordinate frame for direction.

b) After the disk is released, it oscillates. What is the angular ***frequency*** $\boldsymbol{\omega}$ of the oscillation?

c) What is $\vec{\boldsymbol{\Omega}}(t)$, the ***angular velocity*** of the point mass as a function of time? Again, give ***direction*** too!

Problem 631. problems-1/oscillation-pr-disk-with-rim-weight-soln.tex



A uniform disk of radius R and mass $2m$ can freely rotate about a fixed frictionless horizontal axis passing through its fixed center P as shown. It has a point mass m fixed on its rim, so that in equilibrium, the disk is oriented such that $\theta = 0$. At time $t = 0$, the disk is gently rotated by the ***small, positive*** angle θ_0 (out or $\hat{\boldsymbol{z}}$) as shown and released ***from rest***.

a) Just after it is released, what is the net torque ***vector*** $\vec{\boldsymbol{\tau}}$ about P acting on the disk (magnitude ***and direction*** as a function of θ and the givens, using provided coordinate frame for directions)?

b) After the disk is released, it oscillates. What is the angular ***frequency*** $\boldsymbol{\omega}$ of the oscillation?

c) Find $\theta(t)$, i.e., the angular position of the point mass as a function of time.

Solution: a) Using the right hand rule and expression for the magnitude of the cross-product (easiest, not only):

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}} = -mgR\sin(\theta_0) \text{ (out of the page or)} - \hat{\boldsymbol{z}}$$

It only asks for the magnitude though, so to be safe we should probably eliminate both the sign and the indication of direction and write:

$$\tau = mgR\sin(\theta_0)$$

For b), following the usual recipe for oscillator problems, we write Newton's Second Law for the (rotational) motion:

$$\tau = -mgR\sin(\theta) = I\alpha = \left(\frac{1}{2}(2m)R^2 + mR^2\right)\frac{d^2\theta}{dt^2} = 2mR^2\frac{d^2\theta}{dt^2}$$

We use the small angle approximation $\sin(\theta) \approx \theta$, rearrange this into a (standard form) equation of motion and circle/identify ω^2:

$$\frac{d^2\omega}{dt^2} + \frac{mgR}{2mR^2}\theta = \frac{d^2\omega}{dt^2} + \left(\omega^2 = \frac{g}{2R}\right)\theta = 0$$

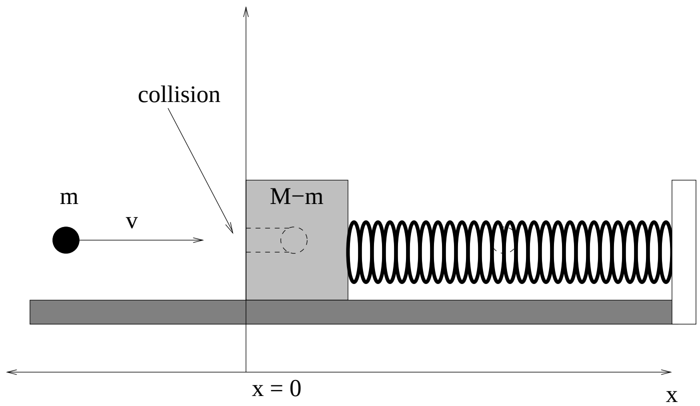
so

$$\omega = \sqrt{\frac{g}{2R}}$$

Finally, c) is now easy. It starts at its maximum angle, at rest, and oscillates like:

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{2R}}t\right)$$

Problem 632. problems-1/oscillation-pr-inelastic-collision-mass-on-spring.tex



A bullet of mass m, travelling at speed v, hits a block of mass $M - m$ with a pre-drilled hole resting connected at the equilibrium position to a connected spring with constant k and ***sticks in the hole***. The block is sitting on a frictionless table (i.e. – ignore damping). Assume that the collision occurs at $t = 0$. All answers below should be given in terms of m, M, k, v.

a) What is the maximum displacement X_0 of the block?

b) What is the angular frequency ω of oscillation of the combined bullet-block system?

c) Write down $x(t)$, the position of the block as a function of time.

Problem 633. problems-1/oscillation-pr-inelastic-collision-mass-on-spring-soln.tex

a) What is the maximum displacement X_0 of the block?

First, momentum conservation in the inelastic collision:

$$p = p_i = mv = Mv_f = p_f$$

Second, energy conservation ***after*** the collision:

$$E_i = K_f = \frac{p^2}{2M} = \frac{1}{2}kX_0^2 = U_f = E_f$$

b) What is the angular frequency ω of oscillation of the combined bullet-block system?

It's just the usual angular frequency for the combined bullet+block mass M:

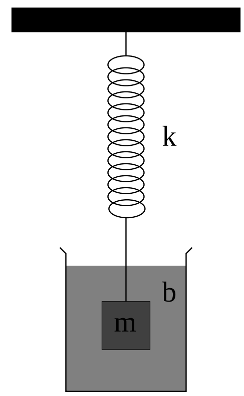
$$\omega = \sqrt{\frac{k}{M}}$$

c) Write down $x(t)$, the position of the block as a function of time

This is a simple harmonic oscillator. It starts ***at the origin*** at time $t = 0$, so:

$$x(t) = X_0 \sin(\omega t)$$

Problem 634. problems-1/oscillation-pr-mass-on-spring-damped.tex



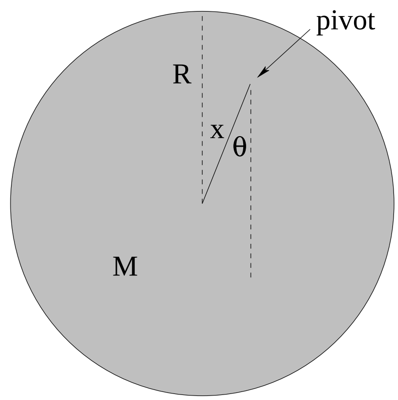
A mass m is attached to a spring with spring constant k and immersed in a medium with damping coefficient b. The net force on the mass when displaced by x from its equilibrium position is thus:

$$F_x = ma_x = -kx - bv_x$$

Convert this equation (Newton's second law for the mass/spring/damping fluid arrangement) into a second order linear homogeneous differential equation and solve it, finding the damping rate and the shifted frequency ω'. You may leave the final answer in exponential form or convert it to cosine as you wish.

Also Draw a qualitatively correct graph of $x(t)$ if the oscillator is pulled to a position x_0 and released at rest at time $t = 0$. Note that you should be able to do this part even if you cannot derive the curves that you draw or ω'.

Problem 635. problems-1/oscillation-pr-minimize-period-of-disk.tex



A uniform disk of mass M and radius R has a hole drilled in it a distance $0 \leq x < R$ from its center. It is then hung on a (frictionless) pivot, pulled to the side through a *small* angle θ_0, and released from rest to oscillate harmonically.

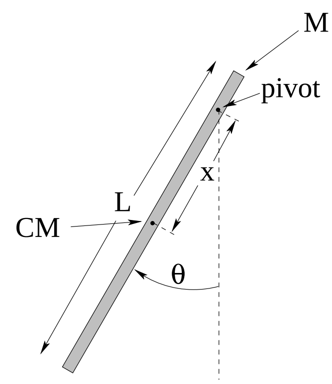
a) What is the moment of inertia of the disk about this pivot?

b) Write $\tau = I\alpha$ for this disk, make the small angle approximation, and turn it into the differential *equation of motion*.

c) Write an expression for T, the period of oscillation of the disk, as a function of d.

d) 5 point extra credit bonus question! What value of d ***minimizes*** this period? That is, if we wanted to make a disk oscillate with the shortest possible period, how far from the end would we drill a pivot hole?

Problem 636. problems-1/oscillation-pr-minimize-period-of-rod.tex



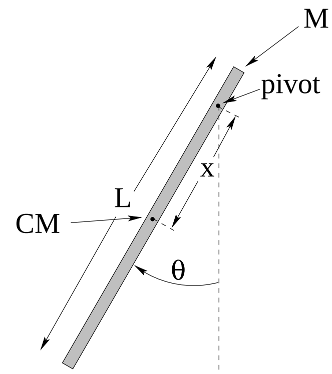
A rod of mass M and length L is pivoted a distance x from the center as shown above. Gravity acts on the rod, pulling it down (as usual) at its center of mass.

a) What is the moment of inertia of the rod about this pivot?

b) Write $\tau = I\alpha$ for this rod, make the small angle approximation, and turn it into the differential *equation of motion*. Use this to write an expression for T, the period of oscillation of the rod, as a function of x.

c) What value of x ***minimizes*** this period? That is, if we wanted to make a rod oscillate with the shortest possible period, how far from the end would we drill a pivot hole?

Problem 637. problems-1/oscillation-pr-minimize-period-of-rod-soln.tex



A rod of mass M and length L is pivoted a distance x from the center as shown above. Gravity acts on the rod, pulling it down (as usual) at its center of mass.

a) What is the moment of inertia of the rod about this pivot?

$$I = \frac{1}{12}ML^2 + Mx^2 \quad \text{(parallel axis theorem)}$$

b) Write $\tau = I\alpha$ for this rod, make the small angle approximation, and turn it into the differential *equation of motion*. Use this to write an expression for T, the period of oscillation of the rod, as a function of x.

$$\tau = -Mgx\sin\theta = \left(\frac{1}{12}ML^2 + Mx^2\right)\frac{d^2\theta}{dt^2} = I\alpha$$

or (linearizing and rearranging):

$$\frac{d^2\theta}{dt^2} + \left(\frac{Mgx}{\frac{1}{12}ML^2 + Mx^2}\right)\theta = 0$$

so that:

$$\omega = \frac{2\pi}{T} = \left(\frac{Mgx}{\frac{1}{12}ML^2 + Mx^2}\right)^{1/2}$$

and:

$$T = 2\pi\left(\frac{1}{12}\frac{L^2}{gx} + \frac{x}{g}\right)^{1/2}$$

c) What value of x ***minimizes*** this period? That is, if we wanted to make a rod oscillate with the shortest possible period, how far from the end would we drill a pivot hole?

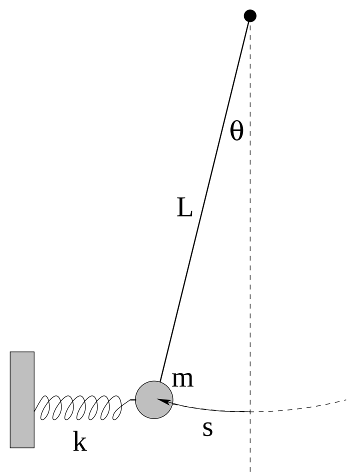
Easiest, we can ignore the square root and just set the derivative of the contents of the square root to zero and solve for x:

$$\frac{d}{dx}\left(\frac{1}{12}\frac{L^2}{gx}+\frac{x}{g}\right)=-\frac{1}{12}\frac{L^2}{gx^2}+\frac{1}{g}=0$$

or:

$$x=\sqrt{\frac{1}{12}}L$$

Problem 638. problems-1/oscillation-pr-pendulum-with-spring.tex

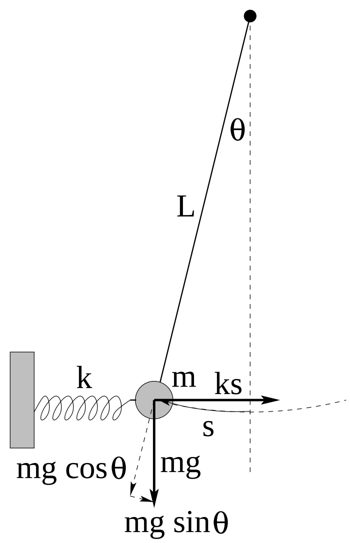


In the figure above a mass m on the end of a massless string of length L forms a pendulum. A light (massless) spring of spring constant k is attached to the mass so that for *small oscillations* $s \ll L$ (where one can use the small angle approximation), $F_s = -ks$ where s is the distance along the arc of motion from the equilibrium position in the center. When released, both gravity and the spring contribute to its motion, with the force exerted by the spring remaining **approximately tangent to the trajectory throughout**.

a) Find the period of this oscillator for small oscillations.

b) If it is started at an angle θ_0 and released, how fast is the mass m moving as it crosses equilibrium at $\theta = 0$?

Problem 639. problems-1/oscillation-pr-pendulum-with-spring-soln.tex



a) Find the period of this oscillator for small oscillations.

Start with Newton's Second Law, using the force diagram above and (for small angles) treating the spring force as if it acts along the curved tangent to the circle:

$$-ks - mg\sin(\theta) = -kL\theta - mg\sin(\theta) = m\frac{d^2s}{dt^2} = mL\frac{d^2\theta}{dt^2}$$

or

$$\frac{d^2\theta}{dt^2} + \frac{k}{m}\theta + \frac{g}{L}\theta = \frac{d^2\theta}{dt^2} + \left(\frac{k}{m} + \frac{g}{L}\right)\theta = 0$$

so

$$\omega^2 = \frac{4\pi^2}{T^2} = \frac{k}{m} + \frac{g}{L} = \frac{kL + mg}{mL}$$

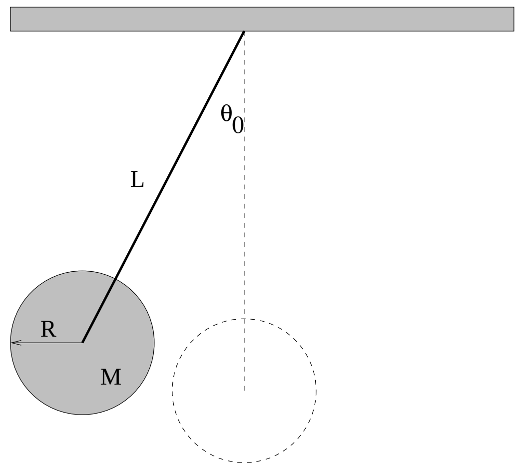
and

$$T = 2\pi\sqrt{\frac{mL}{kL + mg}}$$

b) If it is started at an angle θ_0 and released, how fast is the mass m moving as it crosses equilibrium at $\theta = 0$?

$$v_{\max} = \omega L\theta_0$$

Problem 640. problems-1/oscillation-pr-physical-pendulum-ball-on-stick.tex



A physical pendulum is constructed from a thin rod of negligible mass inserted into a uniform ball of mass M and radius R. The rod has length L from the pivot point to the center of the ball. At time $t = 0$ the ball is released from rest when the rod is at an initial **small** angle θ_0 with respect to its vertical equilibrium position.

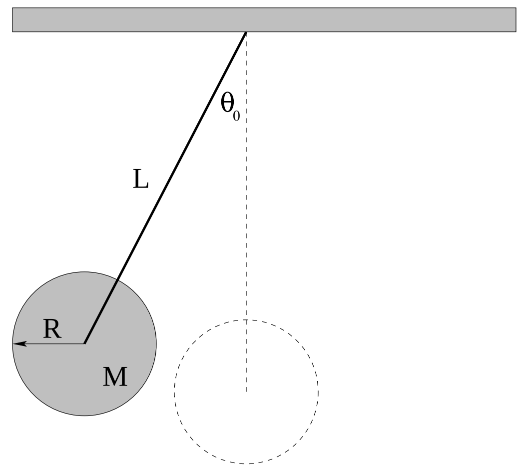
Answer all the questions below in terms of M, R, L, g, θ_0. You may make the small angle approximation where appropriate.

a) Determine the equation of motion for the system, solving for $\alpha = \frac{d^2\theta}{dt^2}$.

b) Determine the angular frequency of oscillation ω and write down $\theta(t)$ for the ball.

c) Find the maximum speed v of the ball. Is this larger or smaller than it would have been if the ball had been a point mass M at the end of the rod? Why?

Problem 641. problems-1/oscillation-pr-physical-pendulum-disk-on-stick-1.tex



A physical pendulum is constructed from a thin rod of negligible mass rigidly inserted into a uniform disk of mass M and radius R. The rod has length L from the pivot point at the top of the rod to the center of the disk. At time $t = 0$ the disk is released from rest when the rod is at an initial ***small*** angle θ_0 with respect to its vertical equilibrium position.

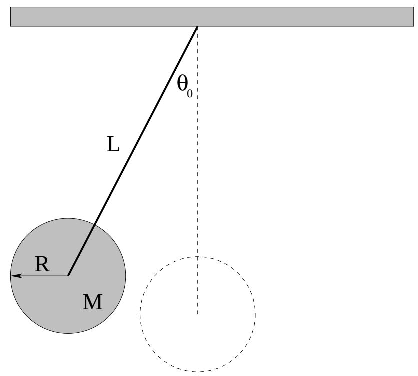
Answer all the questions below in terms of M, R, L, g, θ_0. You may make the **small angle approximation** where appropriate.

a) Find the ***vector torque*** $\vec{\tau}$ about the pivot point *at the instant the ball is released*, assuming $\theta_0 > 0$ (positive) as drawn.

b) Determine the ***period*** $\boldsymbol{T}$ of the resulting oscillation.

c) Find the ***maximum speed*** $\boldsymbol{v}$ of the center of mass of the disk as it oscillates.

Problem 642. problems-1/oscillation-pr-physical-pendulum-disk-on-stick.tex



A physical pendulum is constructed from a thin rod of negligible mass inserted into a uniform disk of mass M and radius R. The rod has length L from the pivot point to the center of the disk. At time $t = 0$ the disk is released from rest when the rod is at an initial ***small*** angle θ_0 with respect to its vertical equilibrium position. You may make the small angle approximation where appropriate.

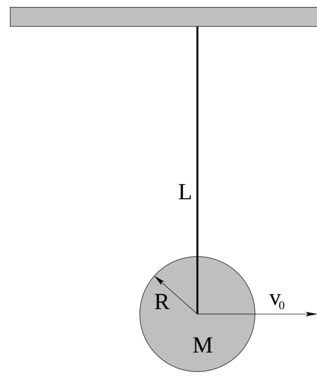
a) Determine the ***equation of motion*** for the system, solving for $\alpha = \frac{d^2\theta}{dt^2}$.

b) Determine the angular frequency of oscillation and write down the harmonic motion solution $\theta(t)$ for the disk.

c) Find the maximum speed v of the disk.

d) Is this larger or smaller than it would have been if the disk had been a point mass M at the end of the rod? Why?

Problem 643. problems-1/oscillation-pr-physical-pendulum-disk-on-stick-v0-only.tex



A physical pendulum is constructed from a thin rod of negligible mass inserted into a uniform disk of mass M and radius R. The rod has length L from the pivot point to the center of the disk. At time $t = 0$ the disk is sitting in its ***equilibrium position*** $\boldsymbol{\theta = 0}$ and is given as sharp blow so that it has an ***initial speed of*** $\boldsymbol{v_0}$ ***to the right***. The resulting oscillation is "small": you may make the ***small angle approximation*** where appropriate.

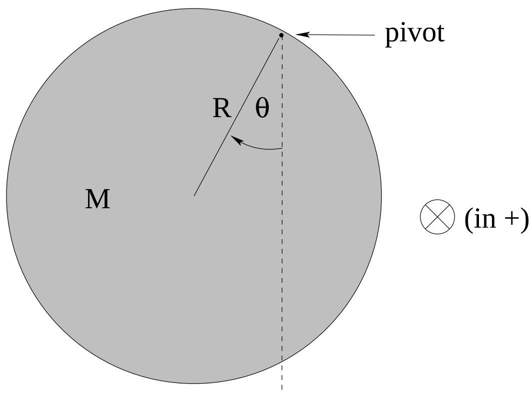
a) Draw the situation at a time that the pendulum has swung through an arbitrary angle θ. Determine the ***equation of motion*** for the system, solving for $\alpha = \frac{d^2\theta}{dt^2}$.

b) Determine the angular frequency of oscillation and write down the ***harmonic motion solution*** $\theta(t)$ for the disk. (Hint: What is the maximum angular velocity of the pendulum?)

c) Find the maximum angle $\theta_{\max}$ that the disk reaches.

d) Is angle $\theta_{\max}$ larger or smaller than it would have been if the ball had been a point mass M at the end of the rod started with the same initial velocity? Why?

Problem 644. problems-1/oscillation-pr-physical-pendulum-disk.tex



A disk of mass M and radius R is pivoted at the rim and hung from a wall as shown above. Gravity acts on the disk, pulling its center of mass down (as usual).

a) What is the moment of inertia of the disk ***about this pivot***?

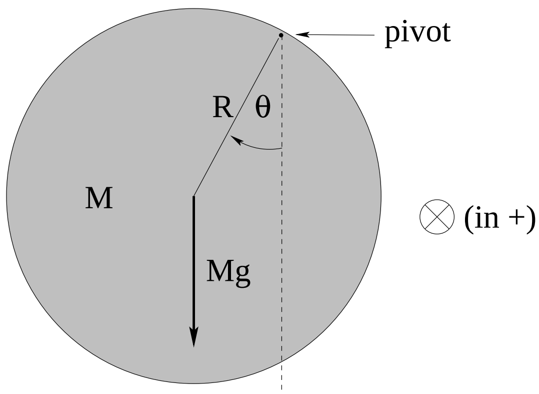
b) Assuming that it is at the ***small*** angle $+\theta$ drawn to the left, find the *differential equation of motion* for this system.

c) Identify the angular frequency and use it to write an expression for T, the period of oscillation of the disk.

d) Write down $\theta(t)$ for the disk, ***assuming that it starts at time $t = 0$ with angular position $\theta(0) = 0$ and angular velocity $\Omega(0) = \Omega_0$***.

$I =$ $\boxed{}$ $T =$ $\boxed{}$ $\theta(t) =$ $\boxed{}$

Problem 645. problems-1/oscillation-pr-physical-pendulum-disk-soln.tex



A disk of mass M and radius R is pivoted at the rim and hung from a wall as shown above. Gravity acts on the disk, pulling its center of mass down (as usual).

a) What is the moment of inertia of the disk ***about this pivot***?

b) Assuming that it is at the ***small*** angle $+\theta$ drawn to the left, find the *differential equation of motion* for this system.

c) Identify the angular frequency and use it to write an expression for T, the period of oscillation of the disk.

d) Write down $\theta(t)$ for the disk, ***assuming that it starts at time $t = 0$ with angular position $\theta(0) = 0$ and angular velocity $\Omega(0) = \Omega_0$***.

$I = \boxed{\frac{3}{2}MR^2}$ $T = \boxed{\sqrt{\frac{6R}{g}}\pi}$ $\theta(t) = \boxed{\Omega_0\sqrt{\frac{3R}{2g}}\sin(\sqrt{\frac{2g}{3R}}t)}$

Solution: Use the parallel axis theorem to find:

$$I = I_{\rm cm} + MR^2 = \frac{3}{2}MR^2$$

Next, use N2 for torque about the pivot:

$$\tau = -MgR\sin\theta = \frac{3}{2}MR^2\frac{d^2\theta}{dt^2} = I\alpha$$

Make the small angle approximation and rearrange/cancel to get:

$$\boxed{\frac{d^2\theta}{dt^2} + \left(\frac{2g}{3R}\right)\theta = 0}$$

where the circled term is ω^2. From this:

$$\omega^2 = \frac{4\pi^2}{T^2} = \frac{2g}{3R} \qquad \Rightarrow \qquad T = \sqrt{\frac{6R}{g}}\pi$$

Finally, **Note Well** that these are one of our “special” initial conditions. For these I.C.s:

$$\theta(t) = \frac{\Omega_0}{\omega}\sin(\omega t)$$

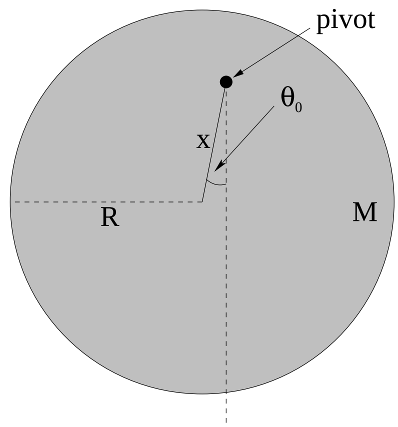
so that:

$$\Omega(t) = \Omega_0 \cos(\omega t)$$

Substituting in ω, this is thus:

$$\theta(t) = \Omega_0 \sqrt{\frac{3R}{2g}} \sin(\sqrt{\frac{2g}{3R}} t)$$

Problem 646. problems-1/oscillation-pr-physical-pendulum-disk-with-hole.tex



A uniform disk of mass M and radius R has a hole drilled in it a distance $0 \leq x < R$ from its center. It is then hung on a (frictionless) pivot, pulled to the side through a *small* angle θ_0, and released from rest to oscillate harmonically.

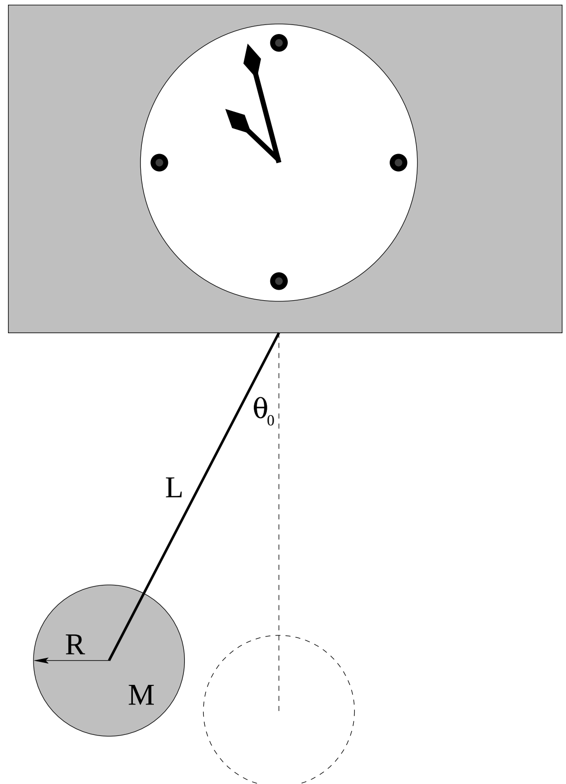
a) What is the moment of inertia of the disk around this pivot?

b) Write down the differential equation of motion for this physical pendulum. Circle ω^2.

c) Find the period of the physical pendulum as a function of (possibly) x, M, R, and g.

d) Write down the solution to the equation of motion, $\theta(t)$.

Problem 647. problems-1/oscillation-pr-physical-pendulum-grandfather-clock.tex



A Grandfather clock's pendulum is constructed from a thin rod of negligible mass inserted into a uniform **disk** of mass $M = 1.0$ kg and radius $R = 5.0$ cm. The rod has a length L from the pivot point to the center of the disk that can be adjusted from 0.20 m to 0.30 m in length so that the clock keeps the correct time. When the clock runs, its pendulum oscillates through a maximum angle of $\theta_0 = 0.05$ radians, which is a "small angle". Use $g = 10$ m/sec^2 and neglect drag.

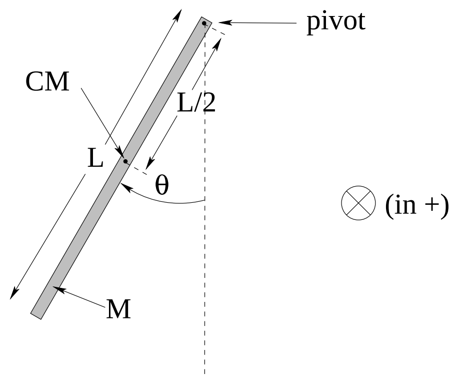
a) **Algebraically** determine the (differential) equation of motion for the system, making the small angle approximation to put it in the form of a simple harmonic oscillator equation.

b) Write down the algebraic function that describes $\theta(t)$, the angle that the pendulum makes as a function of time, assuming it starts from rest at $\theta(0) = \theta_0$ at $t = 0$.

c) The clock keeps correct time when the period of its pendulum is $T = 1$ second. What should L be (to **2** significant digits) so that this is true. (Use the algebraic form for ω^2 from your answer to part a to solve for L.)

d) Suppose one replaces the disk at the end with an identical mass concentrated in a very small (point-like) sphere. Will the clock run fast or slow?

Problem 648. problems-1/oscillation-pr-physical-pendulum-rod.tex



A rod of mass M and length L is pivoted at one end, a distance $L/2$ from the center as shown above. Gravity acts on the rod, pulling it down (as usual) at its center of mass.

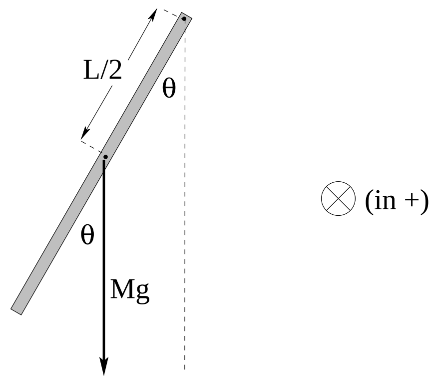
a) What is the moment of inertia of the rod about this pivot?

b) Find the *differential equation of motion* for this system.

c) Write an expression for T, the period of oscillation of the rod.

d) Write down $\theta(t)$ for the rod, assuming that it starts at time $t = 0$ with angular position $\theta(t = 0) = 0$ and angular velocity $\frac{d\theta}{dt} = \Omega(t = 0) = \Omega_0$. (Assume that the resulting oscillation is through a "small angle".)

Problem 649. problems-1/oscillation-pr-physical-pendulum-rod-soln.tex



A rod of mass M and length L is pivoted at one end, a distance $L/2$ from the center as shown above. Gravity acts on the rod, pulling it down (as usual) at its center of mass.

a) What is the moment of inertia of the rod about this pivot?

$$I = \frac{1}{3}ML^2$$

b) Find the *differential equation of motion* for this system.

$$\tau = -\frac{MgL}{2}\sin(\theta) = I\alpha = \frac{1}{3}ML^2\frac{d^2\theta}{dt^2}$$

or

$$\frac{d^2\theta}{dt^2} + \frac{3g}{2L}\sin(\theta) = 0$$

and finally, the small angle approximation gives you:

$$\frac{d^2\theta}{dt^2} + \frac{3g}{2L}\theta = 0$$

c) Write an expression for T, the period of oscillation of the rod.

$$\omega^2 = \frac{4\pi^2}{T^2} = \frac{3g}{2L}$$

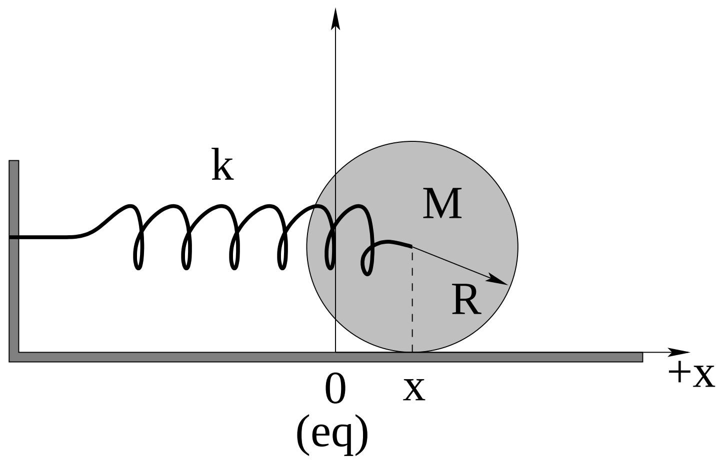
so

$$T = 2\pi\sqrt{\frac{2L}{3g}}$$

d) Write down $\theta(t)$ for the rod, assuming that it starts at time $t = 0$ with angular position $\theta(t = 0) = 0$ and angular velocity $\frac{d\theta}{dt} = \Omega(t = 0) = \Omega_0$. (Assume that the resulting oscillation is through a "small angle".)

$$\theta(t) == \frac{\Omega_0}{\omega}\sin(\omega t)$$

Problem 650. problems-1/oscillation-pr-rolling-wheel-and-spring.tex



A spring with spring constant k is attached to a wall and to the axle of a wheel of radius R, mass M, and moment of inertia $I = \beta MR^2$ that is sitting on a rough floor. The wheel is stretched a distance A from its equilibrium position and is released at rest at time $t = 0$. The rough floor provides enough static friction that, for this value of A, the wheel ***rolls without slipping***.

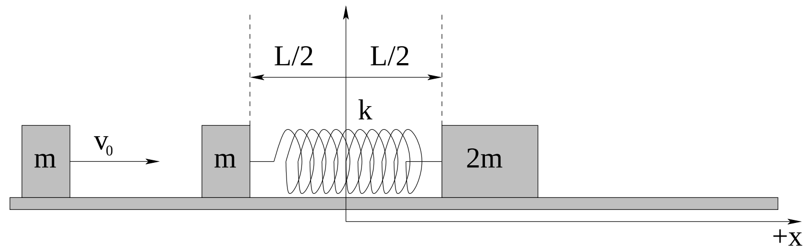
a) When the displacement of the wheel from its equilibrium position is x and the speed of center of mass of the wheel is v, what is its total mechanical energy?

b) What is the maximum velocity v_{max} of the wheel?

c) What is the angular frequency ω of the **oscillation**, of the center of mass of the wheel as it rolls back and forth? (Note that this is *not* the angular velocity Ω of the rolling wheel!)

d) **Challenge!** Find the largest amplitude A_{max} that the wheel can have before it starts to slip.

Problem 651. problems-1/oscillation-pr-three-block-inelastic-collision.tex



Two blocks of mass m and $2m$ are resting on a frictionless table, connected by an ideal (massless) spring with spring constant k at its equilibrium length L. A third block of mass m is moving to the right with speed v_0 as shown. It collides with and sticks to the block of mass m connected to the spring (forming a new "block" of mass $2m$ on the left hand end of the spring).

We wish to find the position of both the left and the right hand blocks as functions of time. This is a challenging problem and will require several steps of work. **Hints:** Think about ***what is conserved*** both during the collision and during the subsequent motion of the blocks. Try to visualize this motion. Finally, the motion of the blocks is simplest in the ***center of mass frame***.

The following questions will guide you through the work:

a) Let the origin of the laboratory frame be the location of the center of mass of the system at the instant of collision. Write an expression for $x_{\rm cm}(t)$, the position of the center of mass as a function of time.

b) What is the total kinetic energy of the system ***immediately after*** the collision?

c) What is the kinetic energy of the system at the instant (some time later) that the blocks are travelling with the same speed? (This is the kinetic energy of the center of mass motion alone.)

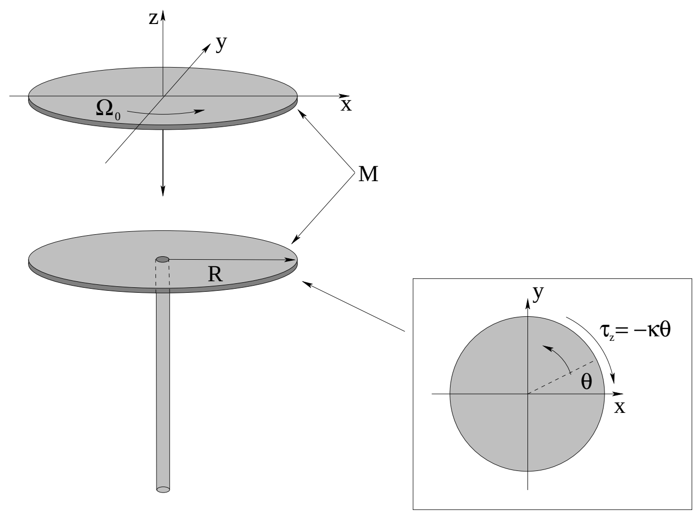
d) At this instant, the total compression of the spring is maximum with some magnitude $x_{\rm max}$. Find $x_{\rm max}$.

e) Write expressions for $x'_l(t)$ and $x'_r(t)$, the position of the left hand and right hand blocks *relative* to the center of mass of the system.

f) Add these functions to $x_{\rm cm}(t)$ to find $x_l(t)$ and $x_r(t)$, the position of the two blocks as a function of time.

g) Differentiate these solutions to find $v_l(t)$ and $v_r(t)$, and verify that your answer obeys the initial condition $v_l(0) = v_0/2$, $v_r(0) = 0$. Your overall solution should describe an "inchworm" crawl of the spring as first one mass momentarily moves at speed $v_0/2$ with the other momentarily at rest, then vice versa.

Problem 652. problems-1/oscillation-pr-torsional-oscillator-collision.tex



The ***torsional oscillator*** above consists of a disk of mass M and radius R connected to a stiff supporting rod. The rod acts like a ***torsional spring***, exerting a restoring torque:

$$\tau_z = -\kappa\theta$$

if it is twisted through an angle θ counterclockwise around the z axis (see inset above). κ is the positive torsional spring constant. This torque will make any object with a moment of inertia that is symmetrically attached to the rod ***rotationally oscillate*** around the z axis of the rod as shown.

A second identical disk also of mass M and radius R, rotating around their mutual axis at an angular speed Ω_0, is dropped gently onto the ***stationary*** first disk from above and sticks to it (so that they rotate together after the collision). At the instant of this angular collision, the disks have zero angular displacement (i.e. are at the equilibrium angle, $\theta_0 = 0$)[1].

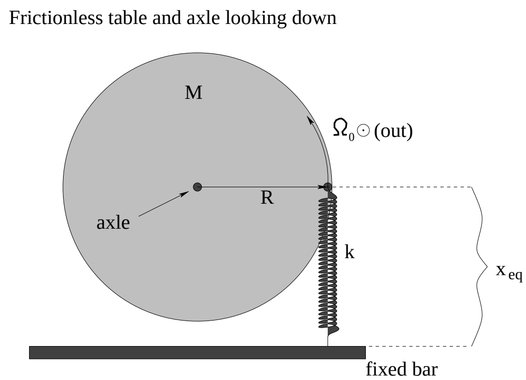
a) Find the ***final*** angular speed Ω_f of the two disks moving together immediately *after* the collision (and before the disks have time to rotate).

b) Find the energy that was ***lost*** in this (inelastic) rotational collision.

c) From the torque equation given above, find the differential ***equation of motion*** for $\theta(t)$ for the two disks moving together *after* the collision. Identify ω^2 (the angular frequency of the ***oscillator*** after the collision) in this equation, and write down the solution $\theta(t)$ in terms of Ω_f, κ, M and R. You do not have to substitute your answer to a) for Ω_f.

[1] Note that I'm using a capital omega $\Omega = d\theta/dt$ to help you keep track of the angular speed Ω of the *disks* and angular frequency ω of the *oscillator* separately below. If you cannot remember the moment of inertia of a disk in terms of M and R, use the symbol I_d for the moment of inertia of a single disk where appropriate in your answers (and lose 2 points).

Problem 653. problems-1/oscillation-pr-torsional-oscillator-spring-hard.tex



In the figure above, a disk of radius R and mass M is mounted on a nearly frictionless axle. A massless spring with spring constant k is attached to a point on its circumference so that it is in equilibrium as shown. The disk is then lightly struck at time $t = 0$ so that it is given a small instantaneous counterclockwise angular velocity of ω_0 while it is still at the equilibrium position, and it subsequently oscillates approximately harmonically through a small maximum angle θ_0. Note: $I_{\text{disk}} = \frac{1}{2}MR^2$ about its center of mass.

a) Find the angular *frequency* ω_f of its oscillation, assuming that the axle is frictionless and exerts no torque on the disk. (**Note well** that this is not the same thing as the initial angular velocity of the disk!)

b) Find the angle θ_0 through which it will rotate before (first) coming momentarily to rest in this frictionless case.

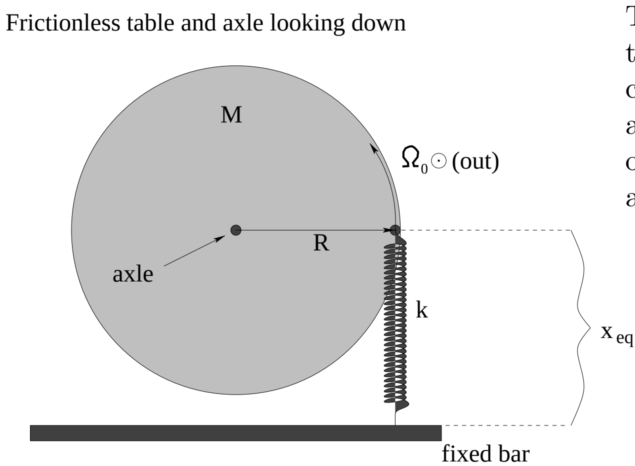
c) Suppose that the axle exerts a weak "drag" torque on the disk when the disk rotates. Do you expect the frequency of oscillation to be larger, smaller, or the same as ω_f once drag is taken into account? (Note that you do not have to derive an answer, but you should justify it on intuitive grounds.)

d) Draw a *qualitatively correct* graph of $\theta(t)$, the angle the disk has rotated through (relative to equilibrium) as a function of time when drag/friction is included as in c).

(Continued workspace on next page)

(Continuation of oscillator problem)

Problem 654. problems-1/oscillation-pr-torsional-oscillator-spring-init-omega0.tex



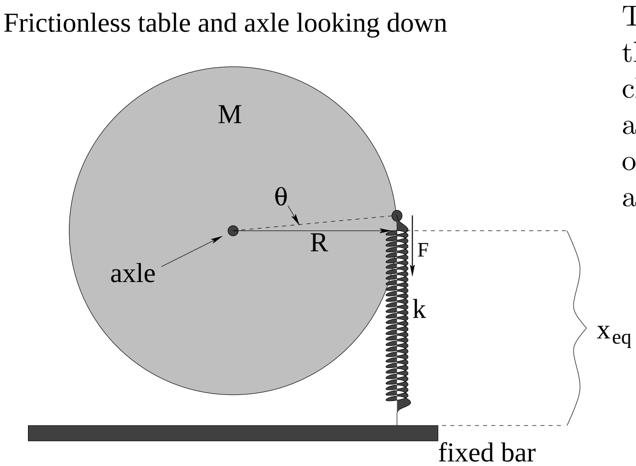
A disk of radius R and mass M is mounted on a frictionless axle. A massless spring with spring constant k is attached to a point on its circumference so that it is in equilibrium as shown. The disk is then lightly struck at time $t = 0$ so that it is given a ***small*** instantaneous counterclockwise angular velocity of Ω_0 while it is still at the equilibrium position, and it subsequently oscillates approximately harmonically through a ***small*** maximum angle θ_0.

a) Find the angular *frequency* ω of its oscillation. You may want to obtain the differential equation of motion first.

b) Find the angle θ_0 through which it will rotate before (first) coming momentarily to rest.

c) Write down (or find) $\theta(t)$, the angle the disk rotates through (relative to equilibrium) as a function of time.

Problem 655. problems-1/oscillation-pr-torsional-oscillator-spring-init-omega0-soln.tex



A disk of radius R and mass M is mounted on a frictionless axle. A massless spring with spring constant k is attached to a point on its circumference so that it is in equilibrium as shown. The disk is then lightly struck at time $t = 0$ so that it is given a ***small*** instantaneous counterclockwise angular velocity of Ω_0 while it is still at the equilibrium position, and it subsequently oscillates approximately harmonically through a ***small*** maximum angle θ_0.

a) Find the angular *frequency* ω of its oscillation. You may want to obtain the differential equation of motion first.

b) Find the angle θ_0 through which it will rotate before (first) coming momentarily to rest.

c) Write down (or find) $\theta(t)$, the angle the disk rotates through (relative to equilibrium) as a function of time.

Solution: For a), start with N2 for rotation, using the spring force as being approximately perpendicular to R throughout in the torque. At an arbitrary time, when the angle of rotational displacement is θ (out, positive) as shown:

$$\tau = -RF = -R \times k(R\theta) = I\alpha = \frac{1}{2}MR^2\frac{d^2\theta}{dt^2}$$

(where we've used $s = R\theta$ and $F = ks$ as the magnitude of the spring force at the angle θ). We simplify and put it in standard SHOE form:

$$\frac{d^2\theta}{dt^2} + \frac{2k}{M}\theta = 0 \quad \Rightarrow \quad \boxed{\omega = +\sqrt{\frac{2k}{M}}}$$

For b), we note that this is one of our special cases so that:

$$\theta(t) = \theta_0 \sin(\omega t) \quad \Rightarrow \quad \Omega(t) = \omega\theta_0\cos(\omega t) = \Omega_0\cos(\omega t) \quad \Rightarrow \quad \theta_0 = \frac{\Omega_0}{\omega}$$

or

$$\boxed{\theta_0 = \sqrt{\frac{M}{2k}}\Omega_0}$$

This instantly gives us c):

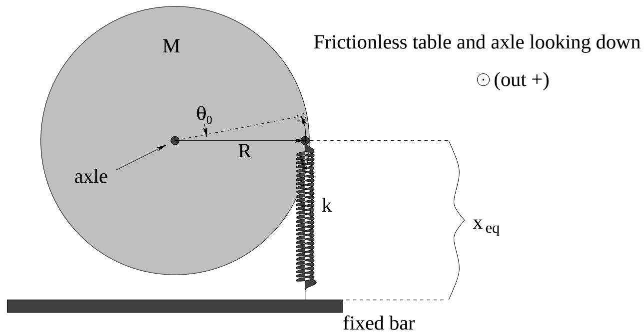
$$\boxed{\theta(t) = \sqrt{\frac{M}{2k}}\Omega_0 \sin\left(\sqrt{\frac{2k}{M}}t\right)}$$

If you differentiate this with respect to time, it obviously gives you precisely that:

$$\Omega(t) = \Omega_0 \cos\left(\sqrt{\frac{2k}{M}}t\right)$$

as expected/required.

Problem 656. problems-1/oscillation-pr-torsional-oscillator-spring-init-theta0.tex



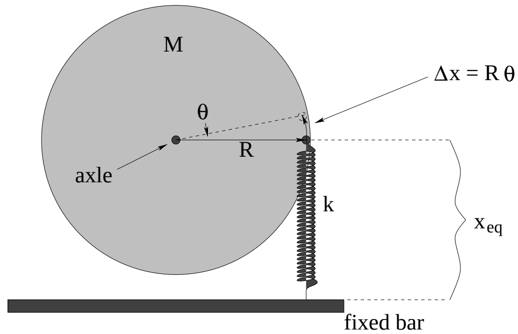
In the figure above, a **disk** of radius R and mass M is mounted on a vertical frictionless axle. A massless spring with spring constant k is attached to a point on its circumference so that it is in equilibrium as shown. The disk is then rotated through a ***small*** angle θ_0 and is released, from rest, at time $t = 0$. It subsequently oscillates approximately harmonically. (Use out of the page for positive θ.)

a) Find the angular *frequency* ω of its oscillation. You may want to obtain the differential equation of motion first.

b) Write down (or find) $\theta(t)$, the angle the disk rotates through (relative to equilibrium) as a function of time.

c) Find the maximum rotational angular *velocity* Ω_0 of the disk as it rotates.

Problem 657. problems-1/oscillation-pr-torsional-oscillator-spring-init-theta0-soln.tex



a) The magnitude of the spring force acting on the disk is (for small angles θ) is:

$$F = k\Delta x = kR\theta$$

(pulling/pushing ***back to equilibrium*** for direction). This exerts a torque:

$$\tau = -kR^2\theta = \frac{1}{2}MR^2\alpha = \frac{1}{2}MR^2\frac{d^2\theta}{dt^2}$$

This can be rearranged into SHOE:

$$\frac{d^2\theta}{dt^2} + \frac{2k}{M}\theta = 0$$

where:

$$\boxed{\omega = \sqrt{\frac{2k}{M}}}$$

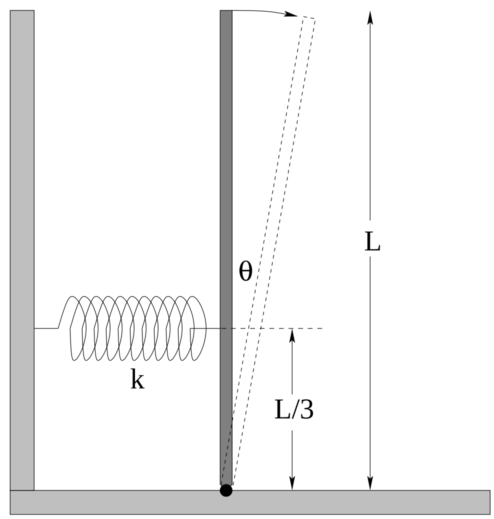
b) Then:

$$\boxed{\theta(t) = \theta_0 \cos(\sqrt{\frac{2k}{M}}t)}$$

c) And:

$$\boxed{\Omega(t) = -\omega\theta_0\sin(\omega t) \implies \Omega_0 = \omega\theta_0 = \sqrt{\frac{2k}{M}}\theta_0}$$

Problem 658. problems-1/oscillation-pr-vertical-bar-and-spring-2.tex



A uniform vertical bar of mass M and length L is pivoted at the bottom. A spring with spring constant k is attached a height $L/3$ over the pivot. This spring is strong enough that the bar will oscillate harmonically about the vertical if it is tipped over to a ***small angle*** θ and released.

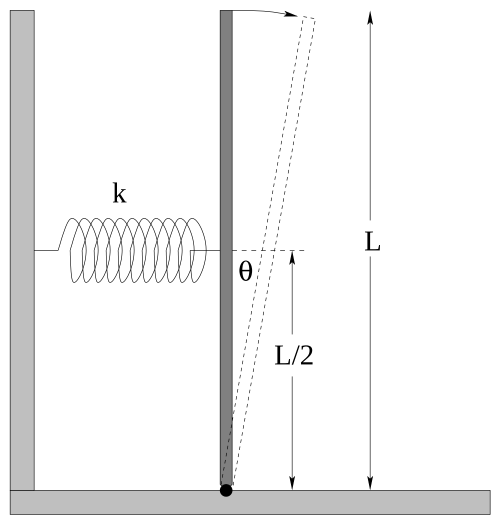
Find:

a) The total torque (magnitude and direction, where $\boldsymbol{\theta}$ ***is positive into the page*** as shown) due to ***both*** gravity and the spring as a function of θ.

b) What is the angular frequency ω of the bar as it oscillates? Recall that the moment of inertia of a uniform bar is $\frac{1}{3}ML^2$.

c) What is the smallest value that k can have such that the bar is in stable equilibrium in the vertical position? [If the spring constant is ***larger*** than this smallest value of k, the spring can sustain oscillations of the bar and does not fall over if perturbed from equilibrium.]

Problem 659. problems-1/oscillation-pr-vertical-bar-and-spring.tex



A uniform vertical bar of mass M and length L is pivoted at the bottom. A spring with spring constant k is attached a height $L/2$ over the pivot. This spring is strong enough that the bar will oscillate harmonically about the vertical if it is tipped over to a ***small*** angle θ and released.

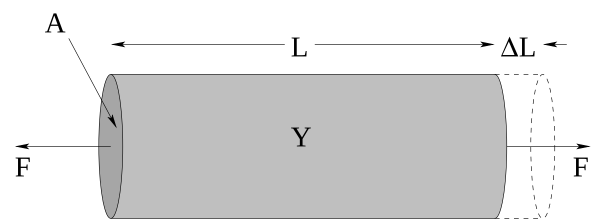
Find:

a) The total torque (magnitude and direction, where **into the page is positive** $\boldsymbol{\theta}$ as shown) due to both gravity and the spring as a function of θ.

b) What is the angular frequency ω of the bar as it oscillates? Recall that the moment of inertia of a uniform bar is $\frac{1}{3}ML^2$.

c) What is the smallest value that k can have such that the bar is in stable equilibrium in the vertical position? [If the spring constant is ***larger*** than this smallest value of k, the spring can sustain oscillations of the bar and does not fall over if perturbed from equilibrium.]

Problem 660. problems-1/oscillation-pr-youngs-modulus.tex



A cylindrical bar of material with cross-sectional area A, unstressed length L, and a Young's Modulus Y is subjected to a force F that stretches the bar as shown. The bar behaves like an elastic "spring", pulling back with a force $F = -k\Delta L$.

a) Show that the "spring constant" of the bar is $k = AY/L$.

b) Show that the energy stored in the bar when it is stretched by length ΔL is $U = \frac{1}{2}F\Delta L$. This will be easiest if you assume that the bar is a "spring" with the spring constant k determined in part a).

Chapter 12

Waves on a String

12.1 Waves on a String

12.1.1 Multiple Choice Problems

Problem 661. problems-1/waves-mc-fixed-and-free-fundamental-review.tex

Consider a vibrating string of length $L = 2$ m. It is found that there are successive standing wave resonances at 50 Hz and 70 Hz. Then the standing wave with the lowest possible frequency (i.e. the first mode or fundamental mode) has frequency:

a) 10 Hz.

b) 20 Hz.

c) 30 Hz.

d) 40 Hz.

e) 50 Hz.

and the string has (check one):

☐ Either both ends fixed or both ends free

☐ One end (either end) fixed and the other free

Problem 662. problems-1/waves-mc-fixed-and-free-fundamental-review-soln.tex

Let's do this one with verbal/conceptual reasoning and not get lost in algebra. The difference between two successive frequencies is clearly 20 Hz. The frequencies themselves are not integer multiples of 20 Hz. Therefore this must be the ***odd harmonic*** series corresponding to a string fixed at one end (either end!) and free at the other. We count backwards in our heads by 20's: 70, 50, 30, 10 – and conclude that the fundamental frequency must be 10 Hz. Note that the length of the string is irrelevant, although it might be important in some other problem. You will often have more information than you need in any given problem! Make sure that you know how to pick out what *is* important!

Hence:

a) **10 Hz**

and

☒ One end (either end) fixed and the other free

Problem 663. problems-1/waves-mc-unknown-fixed-and-free-bcs-fundamental.tex

You are given the following information resulting from measurements of the resonant modes of a string of length L with unknown boundary conditions. You are told that two ***successive*** resonant frequencies are $f_i = 125$ Hz and $f_{i+1} = 175$ Hz where ***mode*** index i counts the frequencies from the bottom. Select the true statement from the following list:

a) The fundamental frequency is 25 Hz, the string is definitely fixed at both ends, and 125 Hz is the $m = 5$ fifth harmonic (fifth ***multiple*** of the fundamental frequency).

b) The fundamental frequency is 25 Hz, the string is definitely free at both ends, and 125 Hz is the $m = 5$ fifth harmonic.

c) The fundamental frequency is 25 Hz, the string is definitely fixed at one end and free at the other, and 125 Hz is the $m = 3$ third harmonic.

d) The fundamental frequency is 25 Hz, the string is definitely fixed at one end and free at the other, and 125 Hz is the $m = 5$ fifth harmonic.

e) None of the above are correct. We cannot tell whether the string is fixed at both ends or free at both ends and/or what harmonic 125 Hz is from this data.

Problem 664. problems-1/waves-mc-unknown-fixed-and-free-bcs-fundamental-soln.tex

The term "harmonic" refers to ***integer multiples of the fundamental frequency only!*** That is, strings fixed or free at both ends allow ***all integer multiples*** of the fundamental (lowest) frequency – "all harmonics". Strings that are fixed at one end and free at the other support only ***odd integer multiples*** of the fundamental frequency – "odd harmonics".

The interval between successive frequencies is 50 Hz. If we count backwards *subtracting* 50 Hz from 175 Hz we get the series 175, 125, 75, 25. This tells us that the fundamental frequency is $f_1 = 25$ Hz. We observe:

$$25 = 1 * f_1$$

$$75 = 3 * f_1$$

$$125 = 5 * f_1$$

$$175 = 7 * f_1$$

Only odd multiples of f_1 occur, so we know the string is fixed at one end and free at the other. 125 is then the ***fifth multiple*** of f_1, which according to our definition above is the ***fifth harmonic***.

a) The fundamental frequency is 25 Hz, the string is definitely fixed at both ends, and 125 Hz is the $m = 5$ fifth harmonic (fifth ***multiple*** of the fundamental frequency).

b) The fundamental frequency is 25 Hz, the string is definitely free at both ends, and 125 Hz is the $m = 5$ fifth harmonic.

c) The fundamental frequency is 25 Hz, the string is definitely fixed at one end and free at the other, and 125 Hz is the $m = 3$ third harmonic.

d) The fundamental frequency is 25 Hz, the string is definitely fixed at one end and free at the other, and 125 Hz is the $m = 5$ fifth harmonic.

e). None of the above are correct. We cannot tell whether the string is fixed at both ends or free at both ends and/or what harmonic 125 Hz is from this data.

Problem 665. problems-1/waves-mc-unknown-fixed-bcs-fundamental-1.tex

You are given the following information resulting from measurements of the resonant frequencies of a string of length L with unknown boundary conditions. You are told that two ***successive*** frequencies are $f_i = 200$ Hz and $f_{i+1} = 250$ Hz for some index i that just counts the frequencies observed from the lowest one (principle harmonic) and is not necessarily a harmonic index. Select the true statement from the following list:

- ☐ The fundamental frequency is 50 Hz, the string is definitely free at both ends, and $i = 4$.
- ☐ The fundamental frequency is 100 Hz, the string might be fixed at both ends ***or*** free at both ends, and $i + 1 = 2.5$.
- ☐ The fundamental frequency is 25 Hz, the string might be fixed at both ends ***or*** free at both ends, and $i = 8$.
- ☐ The fundamental frequency is 50 Hz, the string might be fixed at both ends ***or*** free at both ends and $i = 4$.
- ☐ The fundamental frequency is 50 Hz, the string is definitely fixed at one end and free at the other, and $i + 1 = 5$.

Problem 666. problems-1/waves-mc-unknown-fixed-bcs-fundamental-1-soln.tex

You are given the following information resulting from measurements of the resonant frequencies of a string of length L with unknown boundary conditions. You are told that two ***successive*** frequencies are $f_i = 200$ Hz and $f_{i+1} = 250$ Hz for some index i that just counts the frequencies observed from the lowest one (principle harmonic) and is not necessarily a harmonic index. Select the true statement from the following list:

- ☐ The fundamental frequency is 50 Hz, the string is definitely free at both ends, and $i = 4$.
- ☐ The fundamental frequency is 100 Hz, the string might be fixed at both ends ***or*** free at both ends, and $i + 1 = 2.5$.
- ☐ The fundamental frequency is 25 Hz, the string might be fixed at both ends ***or*** free at both ends, and $i = 8$.
- ☒ The fundamental frequency is 50 Hz, the string might be fixed at both ends ***or*** free at both ends and $i = 4$.
- ☐ The fundamental frequency is 50 Hz, the string is definitely fixed at one end and free at the other, and $i + 1 = 5$.

Solution: The two frequencies are both integer multiples of $f_1 = 50$ Hz. In particular $i = 4 \Rightarrow f_4 = 4 \times 50 = 4f_1$. An integer series implies that the string is either fixed *or* free at both ends, but not fixed at one end *and* free at the other.

Problem 667. problems-1/waves-mc-unknown-fixed-bcs-fundamental.tex

You are given the following information resulting from measurements of the resonant modes of a string of length L with unknown boundary conditions. You are told that two ***successive*** resonant modes have frequencies of $f_m = 350$ Hz and $f_{m+1} = 400$ Hz for some mode index m. Select the true statement from the following list:

a) The fundamental frequency is 50 Hz, the string is definitely fixed at both ends, and $m = 7$.

b) The fundamental frequency is 50 Hz, the string is definitely free at both ends, and $m = 7$.

c) The fundamental frequency is 50 Hz, the string is definitely fixed at one end and free at the other, and $m = 4$.

d) The fundamental frequency is 50 Hz, the string might be fixed ***or*** free at both ends, and $m = 7$.

e) The fundamental frequency is 100 Hz, the string might be fixed ***or*** free at both ends, and $m = 3$.

12.1.2 Short Answer Problems

Problem 668. problems-1/waves-sa-breaking-guitar-string.tex

A certain guitar string is tuned to vibrate at the (principle harmonic) frequency f when its tension is adjusted to T. The string will break at a tension $3T$.

a) Can you double the frequency of the string by increasing the tension (only) without breaking the string?

b) What is the maximum frequency that you *can* make the string have, in terms of f, without (quite) breaking the string?

Problem 669. problems-1/waves-sa-breaking-guitar-string-soln.tex

A certain guitar string is tuned to vibrate at the (principle harmonic) frequency f when its tension is adjusted to T. The string will break at a tension $3T$.

a) For a string fixed at both ends:

$$f_1 = \frac{v}{2L} = \sqrt{\frac{T}{4\mu L}} = \left(\frac{1}{\sqrt{4\mu L}}\right)\sqrt{T}$$

so to double f_1 we have to multiply T by a factor of 4. This would break the string.

b)

$$f_{\max} = \sqrt{3} f_1$$

Problem 670. problems-1/waves-sa-heavy-to-light.tex

One end of a heavy rope is tied to a lighter rope as shown in the figure. An upright wave pulse is incident from then left and travels to the right reaching the junction between the ropes at time $t = 0$, so that, for time $t > 0$, there are two pulses - a transmitted pulse in the light rope and a reflected pulse in the heavy rope.

Compare the transmitted and reflected pulses ***to the incident pulse*** by filling in the table below (each answer is "relative to the same property of the incident pulse"):

	Transmitted	**Reflected**
speed (greater, lesser, equal)		
orientation (upright, inverted)		
energy (greater, lesser, equal)		

Problem 671. problems-1/waves-sa-heavy-to-light-soln.tex

One end of a heavy rope is tied to a lighter rope as shown in the figure. An upright wave pulse is incident from then left and travels to the right reaching the junction between the ropes at time $t = 0$, so that, for time $t > 0$, there are two pulses - a transmitted pulse in the light rope and a reflected pulse in the heavy rope.

Compare the transmitted and reflected pulses ***to the incident pulse*** by filling in the table below (each answer is "relative to the same property of the incident pulse"):

	Transmitted	**Reflected**
speed (greater, lesser, equal)	greater	same
orientation (upright, inverted)	upright	upright
energy (greater, lesser, equal)	lesser	lesser

Solution: Use:

$$v = \sqrt{\frac{T}{\mu}}$$

For the first answer plus the rules that transmitted waves are always erect, reflected waves invert only when going from light to heavy, and that incident energy has to be *split* between the reflected and transmitted pulses.

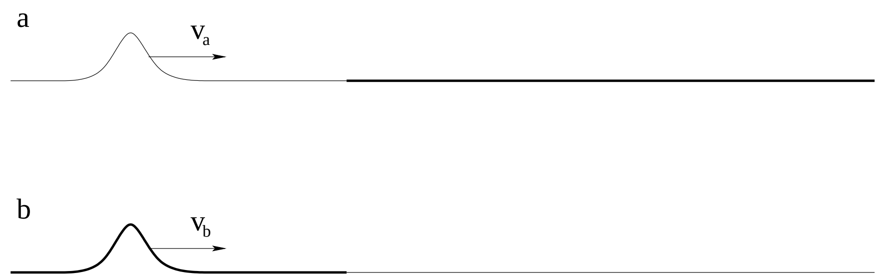
Problem 672. problems-1/waves-sa-reflected-wave-energy.tex

A string of some mass density μ is smoothly joined to a string of greater mass density and the combined string is stretched to a uniform tension T_{en} (the same in both wires). The speed of a wave pulse on the thinner wire is ***twice the speed of a pulse*** on the thicker wire. A wave pulse reflected from the thin-to-thick junction has ***half the amplitude*** of the original pulse. Assuming no loss of energy in the wire:

a) What fraction of the incident energy is reflected at the junction?

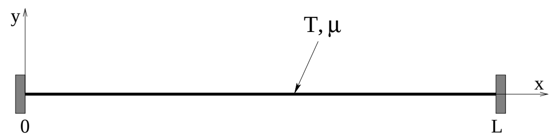
b) Is the reflected pulse upside down or right side up?

Problem 673. problems-1/waves-sa-reflection-transmission-at-junction.tex



Two combinations of two strings with different mass densities are drawn above that are connected in the middle. In both cases the string with the greatest mass density is drawn darker and thicker than the lighter one, and the strings have the same tension T in both a and b. A wave pulse is generated on the string pairs that is travelling from left to right as shown. The wave pulse will arrive at the junction between the strings at time t_a (for a) and t_b (for b). Sketch reasonable *estimates* for the transmitted and reflected wave pulses onto the a and b figures at time $2t_a$ and $2t_b$ respectively. Your sketch should correctly represent things like the relative speed of the reflected and transmitted wave and any changes you might reasonably expect for the amplitude and appearance of the pulses.

Problem 674. problems-1/waves-sa-string-fixed-both-ends.tex



A string of mass density μ is stretched to a tension T and fixed at $x = 0$ and $x = L$. The transverse string displacement is measured in the y direction. All answers should be given ***in terms of these quantities*** or new quantities (such as v) you ***define in terms of these quantities***.

Write down $k_n, \omega_n, f_n, \lambda_n$ for the first three modes supported by the string. Sketch them in on the axes below, labelling nodes and antinodes. You do not have to derive them, although of course you may want to justify your answers to some extent for partial credit in case your answer is carelessly wrong.

Problem 675. problems-1/waves-sa-two-string-densities-frequency.tex

Two identical strings of length L, fixed at both ends, have an identical tension T, but have different mass densities. One string has a mass density of μ, the other a mass density of 16μ

When plucked, the first string produces a (principle harmonic) tone at frequency f_1. What is the frequency of the tone produced by the second string?

a) $f_2 = 4f_1$

b) $f_2 = 2f_1$

c) $f_2 = f_1$

d) $f_2 = \frac{1}{2}f_1$

e) $f_2 = \frac{1}{4}f_1$

Problem 676. problems-1/waves-sa-wave-energies.tex

A wave on a string with mass density μ travels to the right ($+x$) according to the formula:

$$y(x, t) = A\sin(kx - \omega t)$$

Suppose this wave has an average energy per unit length E_0. Identify all of the changes one can make to this wave that will produce a wave with a average energy per unit length of $4E_0$. In all cases the changes indicated are the ***only*** changes in the string or wave formula.

a) Change $A \to 2A$ and change $\mu \to 2\mu$.

b) Change $A \to 2A$.

c) Change $\mu \to 2\mu$.

d) Change $\mu \to 2\mu$ and $k \to 2k$.

e) Change $\mu \to 2\mu$ and $\omega \to 2\omega$.

f) Change $\omega \to 2\omega$.

g) Change $k \to 2k$.

Problem 677. problems-1/waves-sa-wave-facts.tex

Answer the five short questions below:

a) Suppose you are given string A with mass density μ that is stretched until it has tension T_A. You are given a second string B with the ***same*** mass density stretched to ***twice the tension***, $T_B = 2T_A$.

What is the speed of a wave v_B on string B relative to v_A, the speed on string A?

$$v_B = \boxed{} \times v_A$$

b) Suppose you are given string A with mass density μ_A that is stretched until it has tension T. You are given a second string B with ***four times the mass density*** of A $\mu_B = 4\mu_A$ but at the ***same*** tension.

What is the speed of a wave v_B on string B relative to v_A, the speed on string A?

$$v_B = \boxed{} \times v_A$$

c) Suppose you are given string A with mass density μ that is stretched until it has tension T_A. You are given a second (identical) string B with mass density μ that is stretched to ***twice the tension***, $T_B = 2T_A$. Both strings are carrying a ***travelling harmonic wave at the same frequency***.

What is the wave number k_B on string B in terms of the wave number k_A on string A?

$$k_B = \boxed{} \times k_A$$

d) Suppose you are given string A with mass density μ_A that is stretched until it has tension T_A. You are given a second string B with four times the mass density of A $\mu_B = 4\mu_A$ but at the same tension. Both strings are carrying waves with the same wavelength λ.

What is the (regular) frequency f_B on string B in terms of the frequency f_A on string A?

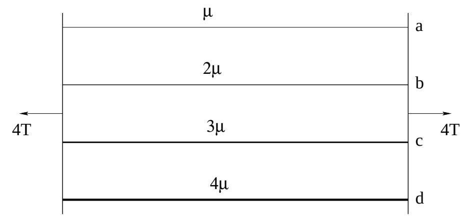
$$f_B = \boxed{} \times f_A$$

e) Suppose you are given string A with mass density μ_A that is stretched until it has tension T_A. You are given a second string B with four times the mass density of A $\mu_B = 4\mu_A$ and a tension four times the tension of A $T_B = 4T_A$. Both strings carry a wave with the same frequency f.

What is the wavelength λ_B in terms of the wavelength λ_A?

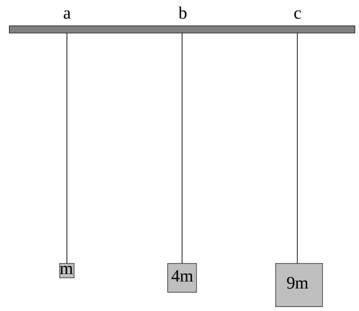
$$\lambda_B = \boxed{} \times \lambda_A$$

Problem 678. problems-1/waves-sa-wave-speed-vary-mu.tex



In the figure above, the neck of a stringed instrument is schmatized. Four strings of different *thickness* and the same length are stretched in such a way that the tension in each is about the same (T) for a total of $4T$ between the end bridges – if this were not so, the neck of the guitar or ukelele or violin would tend to bow towards the side with the greater tension. If the velocity of a wave on the first (lightest) string is v_1, what is the speed of a wave of the other three in terms of v_1?

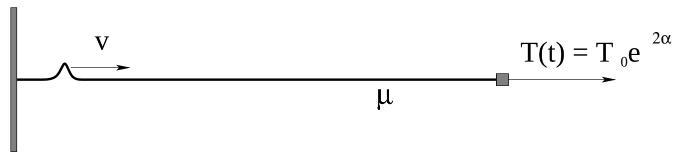
Problem 679. problems-1/waves-sa-wave-speed-vary-T.tex



Three strings of length L (not shown) with the same mass per unit length μ are suspended vertically and blocks of mass m, $4m$ and $9m$ are hung from them. The total mass of each string $\mu L \ll m$ (the strings are *much* lighter than the masses hanging from them). If the speed of a wave pulse on the first string (a) is v_0, what is the speed of the same wave pulse on the second (b) and third (c) strings?

12.1.3 Regular Problems

Problem 680. problems-1/waves-pr-accelerating-wave-pulse.tex



A wave pulse is started on a string with mass density μ with an applied tension that *increases* like $T_0 e^{2\alpha t}$.

a) Find the initial velocity of the wave pulse at time $t = 0$.

b) Find the acceleration of the wave pulse as a function of time.

Problem 681. problems-1/waves-pr-accelerating-wave-pulse-soln.tex

a) Find the initial velocity of the wave pulse at time $t = 0$.

Start with:

$$v = \sqrt{\frac{T_0}{\mu} e^{2\alpha t}} = \sqrt{\frac{T_0}{\mu}} e^{\alpha t} = v_0 e^{\alpha t}$$

where:

$$v(t = 0) = v_0 = \sqrt{\frac{T_0}{\mu}}$$

b) Find the acceleration of the wave pulse as a function of time.

$$a = \frac{dv}{dt} = \alpha v_0 e^{\alpha t}$$

Problem 682. problems-1/waves-pr-construct-transverse-travelling-wave-1.tex

A very long string aligned with the x-axis is being shaken at the ends in such a way that there is a ***travelling harmonic wave*** on it. y is the vertical direction perpendicular to the string in the direction of the string's displacement. Given the following data (**note units**):

Amplitude $A = 1$ cm
Wavelength $\lambda = 0.5$ m
Period $T = 0.001$ sec

a) Write down the formula for a transverse wave travelling ***in the $-x$ direction*** (that is, to the ***left***) corresponding to these numerical parameters. You may use π in your answer as a symbol as needed.

$y(x, t) =$ ☐

It might help you to fill in the following boxes before writing down the answer:

$k =$ ☐ $\omega =$ ☐

b) What is the speed of the wave on the string in terms of the givens?

$v =$ ☐

c) Suppose one wished to ***double the power*** transmitted by the string by changing ***only one of*** A, T, λ and nothing else. Enter $\times$ in the provided boxes if the stated *relative* change to one of the parameters of the wave would accomplish this ***assuming no other change to the other wave parameters is made***. Be careful! Some changes might affect more than one component of the formula for transmitted power! There can be zero or more than one box checked in the correct answer(s).

☐ Change the amplitude to $A' = \frac{\sqrt{2}}{2} A$ ☐ Change the amplitude to $A' = 2\ A$
☐ Change the period to $T' = \frac{\sqrt{2}}{2} T$ ☐ Change the period to $T' = 0.5\ T$
☐ Change the wavelength to $\lambda' = 0.5\,\lambda$ ☐ Change the wavelength to $\lambda' = 2.0\,\lambda$

Problem 683. problems-1/waves-pr-construct-transverse-travelling-wave-1-soln.tex

A very long string aligned with the x-axis is being shaken at the ends in such a way that there is a ***travelling harmonic wave*** on it. y is the vertical direction perpendicular to the string in the direction of the string's displacement. Given the following data (**note units**):

Amplitude $A = 1$ cm
Wavelength $\lambda = 0.5$ m
Period $T = 0.001$ sec

a) Write down the formula for a transverse wave travelling ***in the $-x$ direction*** (that is, to the ***left***) corresponding to these numerical parameters. You may use π in your answer as a symbol as needed.

$$y(x,t) = \boxed{A\sin(4\pi x + 2000\pi t)}$$

It might help you to fill in the following boxes before writing down the answer:

$$k = \boxed{\frac{2\pi}{0.5} = 4\pi} \qquad \omega = \boxed{\frac{2\pi}{0.001} = 2000\pi}$$

b) What is the speed of the wave on the string in terms of the givens?

$$v = \boxed{\frac{\lambda}{T} = 2000 \text{ m/sec}}$$

c) Suppose one wished to ***double the power*** transmitted by the string by changing ***only one of*** A, T, λ and nothing else. Enter $\times$ in the provided boxes if the stated *relative* change to one of the parameters of the wave would accomplish this ***assuming no other change to the other wave parameters is made***. Be careful! Some changes might affect more than one component of the formula for transmitted power! There can be zero or more than one box checked in the correct answer(s).

- ☐ Change the amplitude to $A' = \frac{\sqrt{2}}{2}A$
- ☐ Change the amplitude to $A' = 2\ A$
- ☐ Change the period to $T' = \frac{\sqrt{2}}{2}T$
- ☐ Change the period to $T' = 0.5\ T$
- ☐ Change the wavelength to $\lambda' = 0.5\,\lambda$
- ☒ Change the wavelength to $\lambda' = 2.0\ \lambda$

Solution: Most of this is just remembering definitions: $k = 2\pi/\lambda$, $\omega = 2\pi/T$, $v = \lambda/T$, plus:

$$y(x,t) = A\sin(kx + \omega t) \qquad (+ \text{ for wave to the } \textbf{\textit{left}}, -x \text{ direction})$$

$$P = \frac{1}{2}\mu\omega^2 A^2 \sin^2(kx+\omega t)v = \frac{1}{2}\mu\frac{4\pi^2}{T^2}\frac{\lambda}{T}A^2\sin^2(kx+\omega t)$$

The only "tricky" part is to note that P scales like one over T ***cubed*** because T occurs in both ω and v (as students were warned might happen in the problem text). The A' answers are both obviously wrong. The first T' answer *would* work if it scaled like $1/T'^2$, but it doesn't. λ, however, only occurs once on top so doubling it doubles the power as desired.

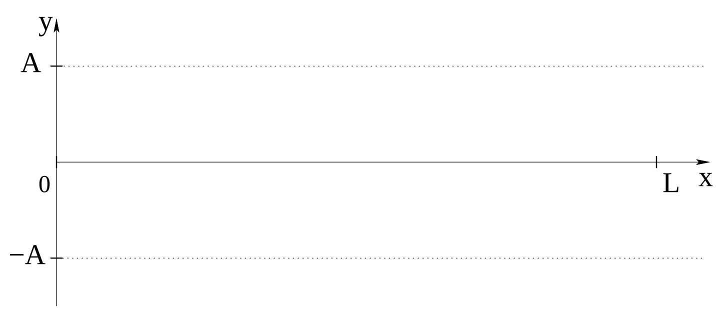
Problem 684. problems-1/waves-pr-fixed-both-ends.tex

A string with mass density μ and under tension T vibrates in the y-direction. The string is ***fixed at both ends*** at $x = 0$ and $x = L$. Answer all questions in terms of these givens.

a) What are the ***two lowest frequencies*** f_1 and f_2 that a standing wave can have for this string?

b) Write down an equation for $y(x, t)$, the y-displacement of the string as a function of position x along the string and time t for the ***standing wave*** corresponding to the ***second lowest frequency*** $\boldsymbol{f_2}$ (the second mode) that you just computed. Assume that the standing wave has a maximum vertical displacement of $y = A$.

c) On the graph below, plot the y-displacement for the second mode versus horizontal position x at an instant when the the string achieves its maximum displacement. Indicate the positions on the x-axis of any nodes or antinodes.



Problem 685. problems-1/waves-pr-fixed-both-ends-soln.tex

A string with mass density μ and under tension T vibrates in the y-direction. The string is ***fixed at both ends*** at $x = 0$ and $x = L$. Answer all questions in terms of these givens.

a) What are the ***two lowest frequencies*** f_1 and f_2 that a standing wave can have for this string?

   One or two complete "sausages" in L so:

$$\lambda_1 = 2L, \quad \lambda_2 = L$$

   plus use

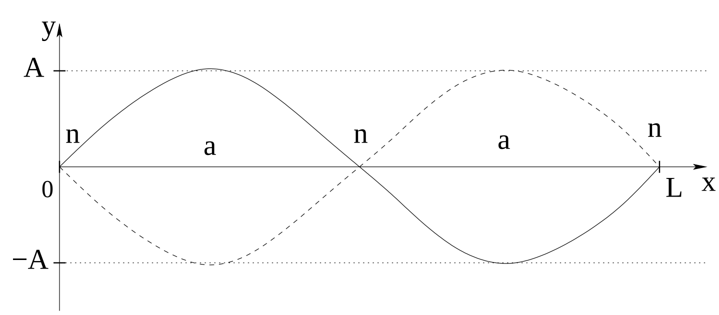
$$v = \sqrt{\frac{T}{\mu}} = f_i \lambda_i$$

b) Write down an equation for $y(x, t)$, the y-displacement of the string as a function of position x along the string and time t for the ***standing wave*** corresponding to the ***second lowest frequency f_2*** (the second mode) that you just computed. Assume that the standing wave has a maximum vertical displacement of $y = A$.

   It is fixed at the left ($x = 0$) so we need to use $\sin(kx)$ instead of $\cos(kx)$ or a phase. We know $k_2 = 2\pi/\lambda_2$ with $\lambda_2 = L$ from above. We make $\omega_2 = k_2 v$ out of k. Putting it together:

$$y(x, t) = A \sin(k_2 x) \cos(\omega_2 t) = A \sin\left(\frac{2\pi}{L} x\right) \cos\left(\frac{2\pi}{L}\sqrt{\frac{T}{\mu}} t\right)$$

c) On the graph below, plot the y-displacement for the second mode versus horizontal position x at an instant when the the string achieves its maximum displacement. Indicate the positions on the x-axis of any nodes or antinodes.



Problem 686. problems-1/waves-pr-fixed-one-end.tex

A string with mass density μ and under tension T vibrates in the y-direction. The string is ***fixed*** at $x = 0$ and ***free*** at $x = L$. Answer all questions in terms of these givens.

a) What are the ***two lowest frequencies*** f_1 and f_2 that a standing wave can have for this string?

b) Write down an equation for $y(x, t)$, the y-displacement of the string as a function of position x along the string and time t for the ***standing wave*** corresponding to the ***second lowest frequency*** $\boldsymbol{f_2}$ (the second mode) that you just computed. Assume that the standing wave has a maximum vertical displacement of $y = A$.

c) On the graph below, plot the y-displacement for the second mode versus horizontal position x at an instant when the the string achieves its maximum displacement. Indicate the positions on the x-axis of any nodes or antinodes.

Problem 687. problems-1/waves-pr-fixed-one-end-soln.tex

A string with mass density μ and under tension T vibrates in the y-direction. The string is ***fixed*** at $x = 0$ and ***free*** at $x = L$. Answer all questions in terms of these givens.

a) What are the ***two lowest frequencies*** f_1 and f_2 that a standing wave can have for this string?

Half or one and a half "sausages" in L so:

$$\lambda_1 = 4L, \quad \lambda_2 = 4L/3$$

plus use

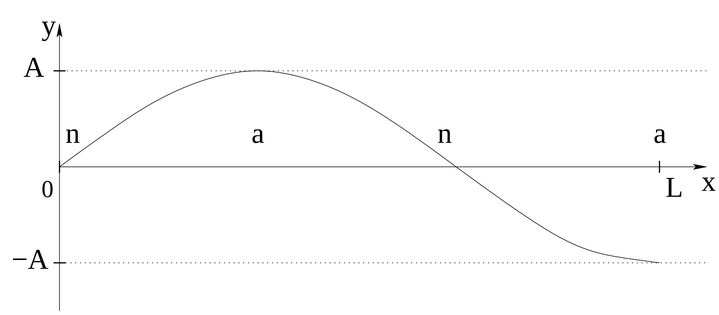
$$v = \sqrt{\frac{T}{\mu}} = f_i \lambda_i$$

b) Write down an equation for $y(x,t)$, the y-displacement of the string as a function of position x along the string and time t for the ***standing wave*** corresponding to the ***second lowest frequency*** $\boldsymbol{f_2}$ (the second mode) that you just computed. Assume that the standing wave has a maximum vertical displacement of $y = A$.

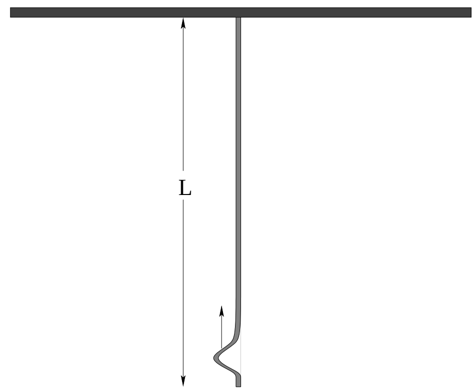
It is fixed on the left at $x = 0$, therefore we need to use $\sin(kx)$ instead of $\cos(kx)$ or a phase. We find $k_2 = 2\pi/\lambda_2$ using the result above, or $k_2 = 3\pi/2L$. Finally, we form $\omega_2 = k_2 v$ using v in terms of the givens, to get:

$$y(x,t) = A\sin(k_2 x)\cos(\omega_2 t) = A\sin\left(\frac{3\pi}{2L}x\right)\cos\left(\frac{3\pi}{2L}\sqrt{\frac{T}{\mu}}t\right)$$

c) On the graph below, plot the y-displacement for the second mode versus horizontal position x at an instant when the the string achieves its maximum displacement. Indicate the positions on the x-axis of any nodes or antinodes.



Problem 688. problems-1/waves-pr-speed-on-hanging-string.tex



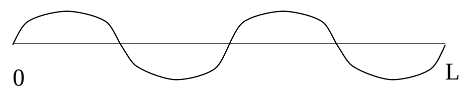
A string of total length L with a mass density μ is shown hanging from the ceiling above.

a) Find the tension in the string as a function of y, the distance *up* from its bottom end. Note that the string is not massless, so each small bit of string must be in static equilibrium.

b) Find the velocity $v(y)$ of a small wave pulse cast into the string at the bottom that is travelling upward.

c) Find the amount of time it will take this pulse to reach the top of the string, reflect, and return to the bottom. Neglect the size (width in y) of the pulse relative to the length of the string.

Problem 689. problems-1/waves-pr-standing-wave-mode-energy.tex



A string of total mass M and total length L is fixed at both ends, stretched so that the speed of waves on the string is v. It is plucked so that it harmonically vibrates in its $n = 4$ mode:

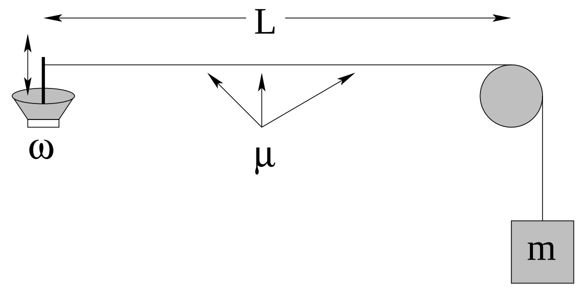
$$y(x,t) = A_4 sin(k_4 x) cos(\omega_4 t).$$

Find (derive) the instantaneous total kinetic energy in the string in terms of M, L, $n = 4$, v and A_4 (although it will simplify matters to use k_4 and ω_4 **once you define them**).

Remember (FYI):

$$\int_0^{n\pi} \sin^2(u) du = \int_0^{n\pi} \cos^2(u) du = \frac{n\pi}{2}$$

Problem 690. problems-1/waves-pr-string-and-hanging-mass.tex



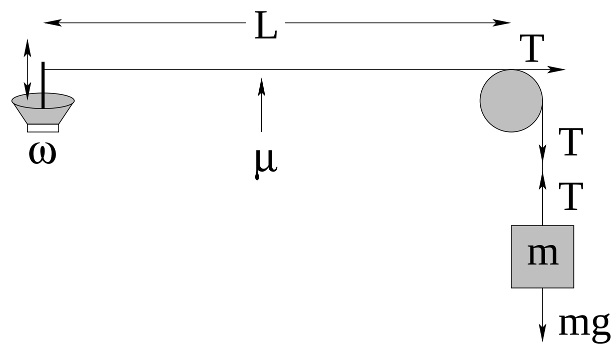
In the figure above, a string of length L and mass density μ is run over a pulley and maintained at some tension by a stationary hanging mass m. The string is driven with ***tiny*** oscillations at a tunable frequency ω by a speaker attached to one end as shown (assume a ***node*** at this end). You may neglect the weight of the string compared to the weight of the mass m.

a) For a given mass m, write an expression for the velocity of waves on the string.

b) Find the frequency of the ***third harmonic*** of the string (expressed in terms of the givens).

c) What is the wavelength of the sound wave produced by the string vibrating at this (third harmonic) frequency? You may express your answer algebraically in terms of $v_{\rm a}$ (the speed of sound in air).

Problem 691. problems-1/waves-pr-string-and-hanging-mass-soln.tex



In the figure above, a string of length L and mass density μ is run over a pulley and maintained at some tension by a stationary hanging mass m. The string is driven with ***tiny*** oscillations at a tunable frequency ω by a speaker attached to one end as shown (assume a ***node*** at this end). You may neglect the weight of the string compared to the weight of the mass m.

Solutions:

a) For a given mass m, write an expression for the velocity of waves on the string.

$$\boxed{v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{mg}{\mu}}}$$

b) Find the frequency of the ***third harmonic*** of the string (expressed in terms of the givens).

$$\lambda_3 = \frac{2L}{3} \qquad v = f_3\lambda_3 \Rightarrow \boxed{f_3 = \frac{v}{\lambda_3} = \frac{3}{2L}\sqrt{\frac{mg}{\mu}}}$$

c) What is the wavelength of the sound wave produced by the string vibrating at this (third harmonic) frequency? You may express your answer algebraically in terms of v_{a} (the speed of sound in air).

$$v_a = f_3\lambda_{3,a} \Rightarrow \boxed{\lambda_{3,a} = \frac{v_a}{f_3} = \frac{2L}{3}\sqrt{\frac{\mu}{mg}} \times v_a}$$

Problem 692. problems-1/waves-pr-travelling-wave-analysis.tex

A travelling wave on a string of mass $\mu = 0.01$ kg/meter is given by the expression:

$$y(x,t) = 2.0\sin(0.02\pi x + 2\pi t) \qquad \text{(meters)}$$

Answer the following questions about this wave. All of the arithmetic *should* be doable without a calculator, but if you have any doubt feel free to leave arithmetical expressions of the algebra unevaluated.

a) What is the amplitude of this wave?

b) What is its wavelength?

c) What is its period?

d) What is the ***velocity*** of this wave (include direction!)?

e) Write an *algebraic* expression for the kinetic energy per unit length in the string as a function of time.

Problem 693. problems-1/waves-pr-travelling-wave-analysis-soln.tex

A travelling wave on a string of mass $\mu = 0.01$ kg/meter is given by the expression:

$$y(x,t) = 2.0\sin(0.02\pi x + 2\pi t) \qquad \text{(meters)}$$

Don't forget units! Also, don't give answers with absurd numbers of digits – all of these quantities are give with no MORE than 2 significant digits (A) and both ω and k have only one.

a) $A = 2.0$ meters

b) $\lambda = 2\pi/(0.02\pi) = 100$ meters

c) $T = 2\pi/(2\pi) = 1$ second

d) $v_x = -\omega/k = \lambda/T = 100$ meters/second (to the ***left***, see + sign).

e)
$$\frac{\Delta K_{\rm avg}}{\Delta x} = \frac{1}{4}\mu\omega^2 A^2 = 0.5 \cdot 0.01 \cdot 4\pi^2 \cdot 4 = 4 \times 10^{-2}\pi^2 \text{ joules/meter}$$

Chapter 13

Sound

13.1 Sound

13.1.1 Multiple Choice Problems

Problem 694. problems-1/sound-mc-car-horn-decibels.tex

You are stuck in freeway traffic and need to get home. So does the driver next to you – she starts blowing the horn of her car, which you hear as a sound with a sound level of 90 dB. Not to be outdone, the driver behind you, in front of you, and to the other side of you all lean on their horn as well, so that now you are hearing all **four** horns (which reach your ears with equal intensities) at once. The sound level you *now* hear is:

a) 93 db

b) 96 dB

c) 180 dB

d) 360 dB

e) Unchanged.

Problem 695. problems-1/sound-mc-car-horn-decibels-soln.tex

Rule of thumb: Each doubling of intensity is a change of +3 dB. This is because $\log_{10}(2) \approx 0.3$. Two doublings is +6 dB, hence:

a) 93 db

b) 96 dB

c). 180 dB

d). 360 dB

e). Unchanged.

Problem 696. problems-1/sound-mc-exam-noise-in-dB.tex

200 students are taking an examination in a room, and the sounds of pens scratching on paper, sighs, groans, and muttered imprecations has created a more or less continuous sound level of this noise of 60 dB. Assuming each student contributes equally to this noise and nothing else changes or adds to it, what will the sound level in the room be when only 50 students are left?

a) 50 dB

b) 15 dB

c) 66 dB

d) 54 dB

e) 57 dB

Problem 697. problems-1/sound-mc-exam-noise-in-dB-soln.tex

200 students are taking an examination in a room, and the sounds of pens scratching on paper, sighs, groans, and muttered imprecations has created a more or less continuous sound level of this noise of 60 dB. Assuming each student contributes equally to this noise and nothing else changes or adds to it, what will the sound level in the room be when only 50 students are left?

Rule of thumb (for fast problem solving, conceptual reasoning): Doubling/halving intensity is the same as $\pm\mathbf{3}$ dB. If we assume that 1/4 as many students make 1/4 as much total noise (intensity), which is two halvings, we should ***subtract*** 2x3 = 6 dB to get:

a) 50 dB

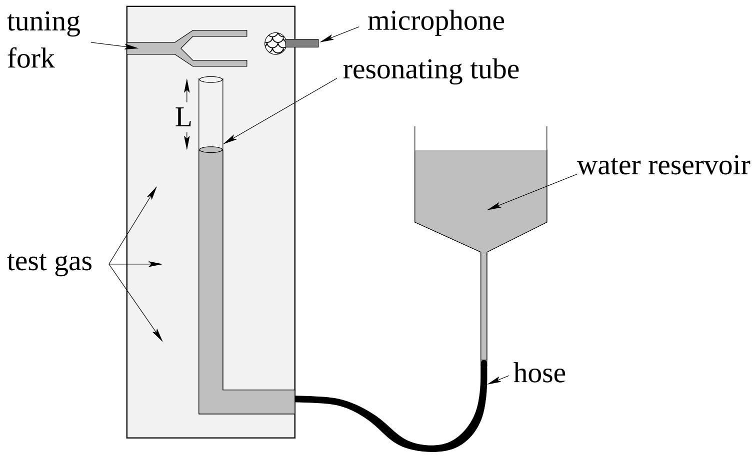
b) 15 dB

c) 66 dB

d) 54 dB

e). 57 dB

Problem 698. problems-1/sound-mc-measuring-speed-of-sound.tex



A simple method for measuring the speed of sound in a reservoir filled with gas is to hold a tuning fork at a fixed, known frequency above a tube connected with a flexible hose to a reservoir such that the height of the water in the ***open at the top*** tube can easily be varied. The sound one detects with a microphone is then the loudest when the tuning fork is in ***resonance*** with standing wave modes in the tube.

If you hold a 2000 Hz tuning fork above the tube when it is completely full and then lower the reservoir slowly to drop the water level in the tube, you hear the fork resonate most loudly when the water is $L = 2.5$, 7.5, and 12.5 cm beneath the end of the tube. The speed of sound in the gas is therefore:

☐ 50 m/sec ☐ 100 m/sec ☐ 200 m/sec ☐ 500 m/sec ☐ 750 m/sec

Problem 699. problems-1/sound-mc-measuring-speed-of-sound-soln.tex

The difference in fluid height between any two neighboring resonances is $\Delta L = \lambda/2$ for the wavelength corresponding to f of the tuning fork. Hence:

$$v = f\lambda = 2f\Delta L$$

with $\Delta L = 5$ cm $= 0.05$ m. $v = 2 * 2000 * 0.05$ or

c) 200 m/sec

Problem 700. problems-1/sound-mc-shooting-a-gun-dB.tex

A 30-06 rifle makes a bang that peaks at 170 decibels 1 meter away from the muzzle. If you are standing 100 meters away (approximately) what sound level do you hear in decibels?

- ☐ 120 dB
- ☐ 130 dB
- ☐ 140 dB
- ☐ 150 dB
- ☐ 160 dB

Problem 701. problems-1/sound-mc-shooting-a-gun-dB-soln.tex

A 30-06 rifle makes a bang that peaks at 170 decibels 1 meter away from the muzzle. If you are standing 100 meters away (approximately) what sound level do you hear in decibels?

The intensity is lower by a factor of $\left(\frac{1}{100}\right)^2 = 10^{-4}$ so the sound level drops by $10 \log_{10} 10^{-4} = 40$ dB, and $170 - 40 = 130$ so:

- ☐ 120 dB
- ☒ 130 dB
- ☐ 140 dB
- ☐ 150 dB
- ☐ 160 dB

Problem 702. problems-1/sound-mc-siren-in-dB.tex

A siren radiates sound energy uniformly in all directions. When you stand a distance 100 m away from the siren you hear a sound level of 90 dB. If you move to a distance of 10 m from the siren, the sound level is:

a) 90 dB, no change.

b) 100 dB.

c) 110 dB.

d) 120 dB.

e) 130 dB.

Problem 703. problems-1/sound-mc-siren-in-dB-soln.tex

Intensity scales like $1/r^2$. Reducing r by a factor of 10 should increase instensity by a factor of 100.

To compute the change in decibel level, we use the following math. Suppose the original intensity is I_1 such that:

$$\beta_1 = 90 \text{ dB} = 10 \log_{10}\left(\frac{I_1}{I_0}\right)$$

where I_0 is the usual threshold of hearing (10^{-12} Watts/m^2). We need to find the sound level corresponding to $I_2 = 100 I_1$. Thus:

$$\begin{aligned}
\beta_2 &= 10 \log_{10}\left(\frac{100 I_1}{I_0}\right) \\
&= 10 \log_{10}(100) + 10 \log_{10}\left(\frac{I_1}{I_0}\right) \\
&= 20 + \beta_1 = 20 + 90 = 110 \text{ dB} \qquad (13.1)
\end{aligned}$$

so:

c) 110 dB

Note well: we used the important property of logs:

$$\log(a * b) = \log(a) + \log(b)$$

(true for any log base). **Make sure that you know this!** With a tiny bit of practice, you can do these computations in your head and answer questions like this quickly and with confidence.

Problem 704. problems-1/sound-mc-sound-level-to-pressure-1.tex

You measure the intensity level of a single frequency sound wave produced by a loudspeaker with a calibrated microphone to be 80 dB. At that intensity, the *peak* pressure in the sound wave at the microphone is $P_0 + P_a$, where P_a is the baseline atmospheric pressure and P_0 is the pressure over that associated with the wave. The loudspeaker's amplitude is turned up until the measured intensity level is 120 dB. What is the peak pressure of the sound wave now?

a) $4P_0 + P_a$

b) $10P_0 + P_a$

c) $40P_0 + P_a$

d) $100P_0 + P_a$

e) $100(P_0 + P_a)$

Problem 705. problems-1/sound-mc-sound-level-to-pressure-2.tex

You measure the sound level of a single frequency sound wave produced by a loudspeaker with a calibrated microphone to be ***80 dB***. At that intensity, the *peak* pressure in the sound wave at the microphone is $P_0 + P_a$, where P_a is the baseline atmospheric pressure and P_0 is the pressure over that associated with the wave. The loudspeaker's amplitude is turned up until the measured sound level is ***100 dB***. What is the peak pressure of the sound wave now?

a) $4P_0 + P_a$

b) $10P_0 + P_a$

c) $40P_0 + P_a$

d) $100(P_0 + P_a)$

e) $100P_0 + P_a$

Problem 706. problems-1/sound-mc-sound-level-to-pressure-2-soln.tex

An increase in sound level of 20 dB is equivalent to multiplying the intensity by a factor $10^{20/10} = 100$. The intensity, in turn is proportional to pressure according to:

$$I \propto P_0^2$$

where P_0 is the ***overpressure*** compared to the baseline atmospheric pressure P_a, for example in an expression like:

$$P(x,t) = P_a + P_0 \sin(kz - \omega t)$$

We therefore need the overpressure (only) to be multiplied by a factor of 10 to produce an increase in intensity by a factor of 100, and:

b) $10P_0 + P_a$

13.1.2 Short Answer Problems

Problem 707. problems-1/sound-sa-alarm-clocks.tex

Wal Mart had a special on alarm clocks, and you bought ten of them just to make sure that you will wake up in time for your physics final exam. Each alarm clock produces an incoherent sound level in your ears of 90 dB when you place the clock on the nightstand one meter from your head. Ignore reflection of sound energy from walls, etc and treat the clocks like point sound sources.

a) If you put 4 clocks on the nightstand one meter from your head, you will hear a sound level of (approximately to the nearest integer): $\boxed{}$ dB

b) If you put 8 clocks on the dresser 2 meters from your head, you will hear a sound level of: $\boxed{}$ dB

c) If you put all 10 clocks in the far corner of the room 4 meters from your head, you will hear a sound level of: $\boxed{}$ dB

Problem 708. problems-1/sound-sa-beats.tex

Two identical strings of length L have mass μ and are fixed at both ends. One string has tension T. The other has tension $1.21T$. When plucked, the first string produces a tone at frequency f_1. What is the *beat* frequency produced if the second string is plucked at the same time, producing a tone f_2? Are the beats likely to be audible if f_1 is 500 Hz?

Problem 709. problems-1/sound-sa-decibels-sun-human-body.tex

Sunlight reaches the surface of the earth with *roughly* 1000 Watts/meter^2 of intensity. What is the "intensity level" of a *sound* wave that carries as much energy per square meter, in **decibels**? In table 15-1 in Tipler and Mosca, what kind of sound sources produce this sort of intensity? Bear in mind that the Sun is *150 million kilometers away* where sound sources capable of reaching the same intensity are typically only a few meters away. Hmmm, seems as though the Sun produces a *lot* of (electromagnetic) energy compared to terrestrial sources of (sound) energy.

While you are at it, the human body produces energy at the rate of roughly 100 Watts. *Estimate* the fraction of this energy that goes into my lecture when I am speaking in a loud voice in front of the class (loud enough to be heard as loudly as normal conversation ten meters away).

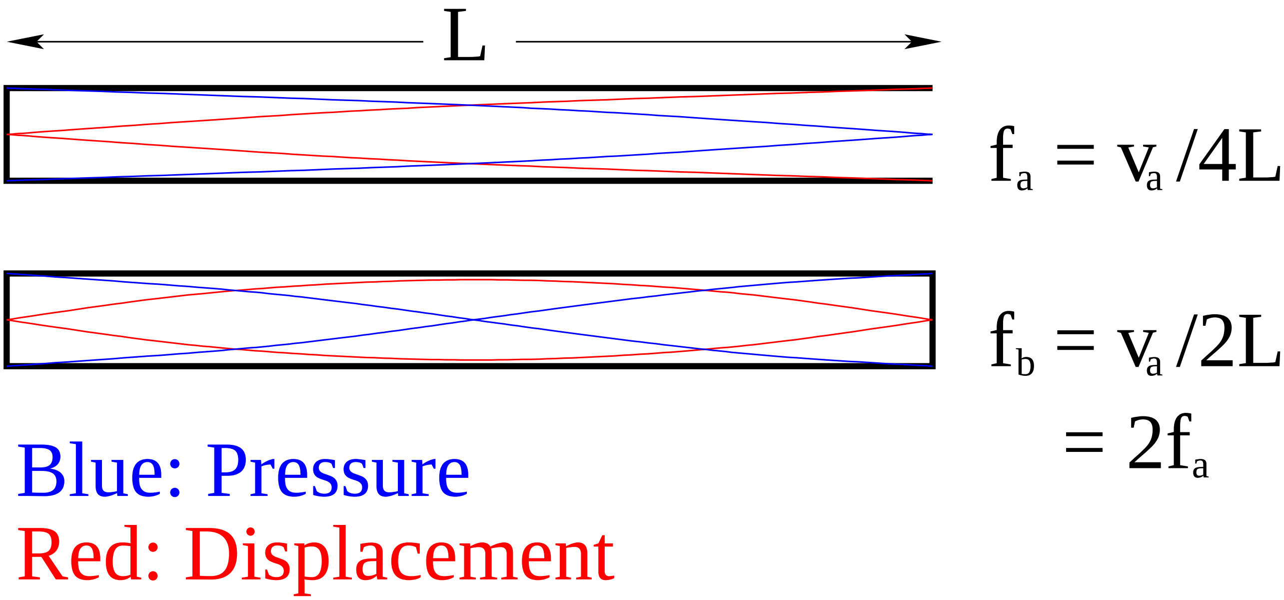
Problem 710. problems-1/sound-sa-principle-harmonic-series.tex

Two pipes used in ***different*** musical instruments have the ***same length L***, but the fundamental frequency (frequency of the principal harmonic, $m = 1$) of one is ***twice that of the other***. Explain how this could be, illustrating your answer with a drawing of two pipes and the principle modes such that this is true. Make sure you indicate which pipe has the higher frequency and which pipe has the lower one, and whether your diagram is illustrating pressure or displacement standing waves!

Problem 711. problems-1/sound-sa-principle-harmonic-series-soln.tex

Two pipes used in ***different*** musical instruments have the ***same length L***, but the fundamental frequency (frequency of the principal harmonic, $m = 1$) of one is ***twice that of the other***. Explain how this could be, illustrating your answer with a drawing of two pipes and the principle modes such that this is true. Make sure you indicate which pipe has the higher frequency and which pipe has the lower one, and whether your diagram is illustrating pressure or displacement standing waves!

Solution: This simply means that one pipe is (say) closed at both ends, and the other is closed at one end and open at the other.



The diagram says it all. Really.

Problem 712. problems-1/sound-sa-scaling-thunder-dB.tex

Lightning strikes one kilometer away, and the resulting thunderclap has an intensity of 5×10^{-3} Watts/meter^2. What is the intensity level in decibels? If one is instead 10 kilometers away, approximately how many decibels lower would the intenstity level be?

Problem 713. problems-1/sound-sa-scaling-time-thunder-dB.tex

You see the flash of lightning and ***three*** seconds later you hear a thunderclap with a peak sound level of 120 dB. A few minutes later you see a second flash of lightning and ***twelve*** seconds later you hear the thunderclap.

a) Approximately what peak sound level do you hear in the second (presumably "identically produced") thunderclap?

   Second thunderclap is: $\boxed{}$ dB

b) Roughly – to the nearest kilometer – how far away are the two lightning flashes?

   First (three seconds): $\boxed{}$ km

   Second (twelve seconds): $\boxed{}$ km

Problem 714. problems-1/sound-sa-scaling-time-thunder-dB-soln.tex

You see the flash of lightning and ***three*** seconds later you hear a thunderclap with a peak sound level of 120 dB. A few minutes later you see a second flash of lightning and ***twelve*** seconds later you hear the thunderclap.

We use two simple/conceptual rules here. First, our simple estimator for time-to-distance for sound waves is that every three seconds in the delay between seeing and hearing is 1 km, every five seconds is 1 miles. Second, sound intensity drops off like $\boldsymbol{1/r^2}$. Third, every halving of intensity ***subtracts 3 dB*** from the initial sound level.

Note well that in this problem, you have to answer part b) ***first*** in order to answer part a). This was mean of me (although inadvertent) but it does emphasize an important point: You need to ***READ THE WHOLE PROBLEM*** and ***THINK FOR A FEW SECONDS*** before starting your answer. The problem might not have had part b) in it at all!

a) Approximately what peak sound level do you hear in the second (presumably "identically produced") thunderclap?

$$\frac{1^2 \text{ km}^2}{4^2 \text{ km}^2} = 1/16$$

so we expect it to have 1/16th of the intensity of the first one. That is four halvings, each subtracts 3 dB, and 120 - 12 = 108 dB:

Second thunderclap is: **108** dB

b) Roughly – to the nearest kilometer – how far away are the two lightning flashes?

First (three seconds): **1** km

Second (twelve seconds): **4** km

Problem 715. problems-1/sound-sa-sound-speed-decibels.tex

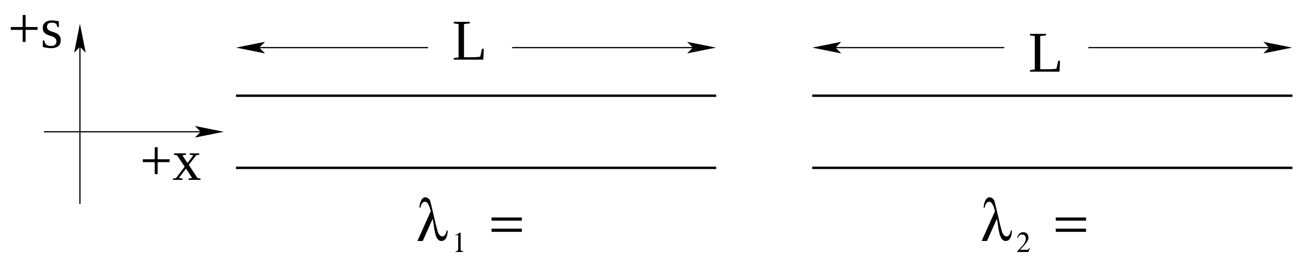
(12 points) Some short questions about sound:

a) Show that doubling the intensity of a sound wave corresponds to an increase in its intensity level or loudness by about 3 dB.

b) I sometimes work as a timer at my son's swim meets. We are told to start our watches when we see a light flash on the starter's console, not when we hear the starting horn. If I am timing a lane on the far side of the pool some 17 meters away from the starter and start when the sound of the horn reaches me, how much will the times I measure (on average) change? Will the swimmer have an advantage or a disadvantage relative to a swimmer timed by someone that starts on the flash of light?

c) Suppose I turn the knob on my surround-sound amplifier and decrease the loudness where I'm listening by 6 dB. By roughly what fraction has the *amplitude* of oscillation of the speakers changed?

Problem 716. problems-1/sound-sa-tube-open-both-ends-1.tex



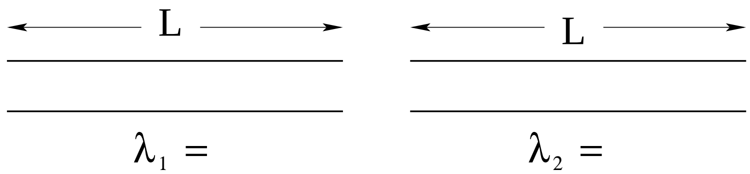
A tube open at both ends used as a "panpipe" musical instrument. It has length $L = 34$ centimeters.

a) Sketch the ***first two displacement modes*** (or harmonics) in the provided tubes.

b) Label the nodes and antinodes, and underneath each tube indicate the wavelength of the mode/harmonic.

c) What is the ***frequency*** of the ***second harmonic*** of the tube (an actual number, please, in Hertz or cycles per second).

Problem 717. problems-1/sound-sa-tube-open-both-ends-2.tex



A tube open at both ends used as a "panpipe" musical instrument. It has length $L = 34$ centimeters.

a) Sketch the ***first two displacement modes*** (or harmonics) in the provided tubes.

b) Label the nodes and antinodes, and underneath each tube indicate the wavelength of the mode/harmonic.

c) What is the ***frequency*** of the ***principle harmonic*** of the tube (an actual number, please, in Hertz or cycles per second).

Problem 718. problems-1/sound-sa-two-wave-speeds-frequency.tex

Two identical pipes, both closed at both ends, are filled with two ***different*** gases. In the first gas, the speed of sound is $v_1 = \sqrt{B_1/\rho_1}$, in the second the speed of sound is $v_2 = \sqrt{B_2/\rho_2} = 2v_1$. Both pipes are driven by speakers in resonance with their ***fundamental harmonic frequency***, f_1 and f_2 respectively.

If f_1 is the fundamental frequency in the first pipe, what is the fundamental frequency f_2 in the second pipe?

☐ $f_2 = 4f_1$ ☐ $f_2 = 2f_1$ ☐ $f_2 = f_1$ ☐ $f_2 = \frac{1}{2}f_1$ ☐ $f_2 = \frac{1}{4}f_1$

Problem 719. problems-1/sound-sa-two-wave-speeds-frequency-soln.tex

Two identical pipes, both closed at both ends, are filled with two ***different*** gases. In the first gas, the speed of sound is $v_1 = \sqrt{B_1/\rho_1}$, in the second the speed of sound is $v_2 = \sqrt{B_2/\rho_2} = 2v_1$. Both pipes are driven by speakers in resonance with their ***fundamental harmonic frequency***, f_1 and f_2 respectively.

If f_1 is the fundamental frequency in the first pipe, what is the fundamental frequency f_2 in the second pipe?

☐ $f_2 = 4f_1$ ☒ $f_2 = 2f_1$ ☐ $f_2 = f_1$ ☐ $f_2 = \frac{1}{2}f_1$ ☐ $f_2 = \frac{1}{4}f_1$

Solution: The two pipes are *identical*, and hence have the same *length* (say) L. Since they are closed at both ends, $\lambda_1 = \lambda_2 = 2L$. But we *also* know that:

$$v_1 = f_1\lambda_1 \quad \text{and} \quad v_2 = f_2\lambda_2$$

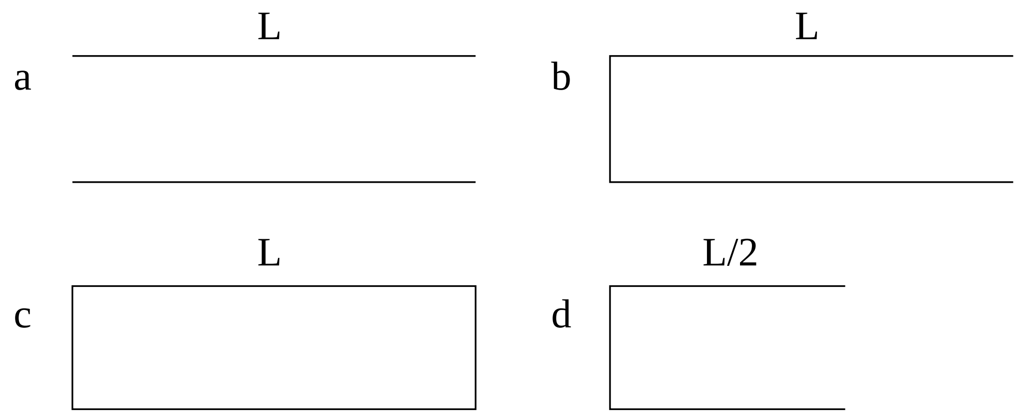
so that:

$$v_2 = 2v_1 = 2f_1\lambda_1 = f_2\lambda_2 = f_2\lambda_1 \quad \Rightarrow \quad \boxed{f_2 = 2f_1}$$

This is a simple scaling argument – v scales with f and λ, λ is the same, so f scales *identically* to v.

13.1.3 Ranking Problems

Problem 720. problems-1/sound-ra-sound-resonances-pressure.tex

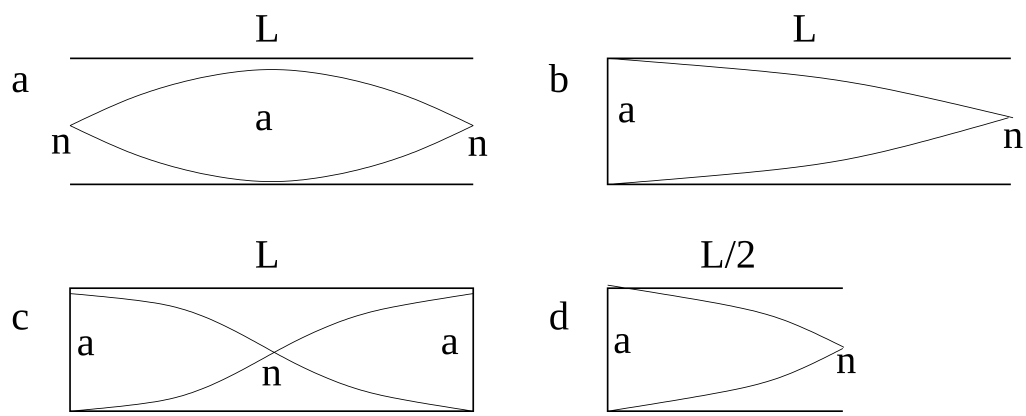


a) Rank the fundamental harmonic resonant ***frequencies*** ($n = 1$) of the four open/closed pipes drawn above, where equality is a possible answer. An answer might be (but probably isn't) $f_a < f_b = f_c < f_d$.

b) Draw into each pipe a representation of a the ***pressure wave*** associated with each resonance.

c) Label the nodes (in your representation of the waves) with an **N** and antinodes with an **A**.

Problem 721. problems-1/sound-ra-sound-resonances-pressure-soln.tex



Another problem where you are better off doing b) and c) before a). ***Read the whole problem first*** before starting to solve it!

a) Clearly the fundamental wavelengths are: $\lambda_a = 2L$, $\lambda_b = 4L$, $\lambda_c = 2L$, and $\lambda_d = 2L$. The frequencies are all given by:

$$f_i = \frac{v_a}{\lambda_i}$$

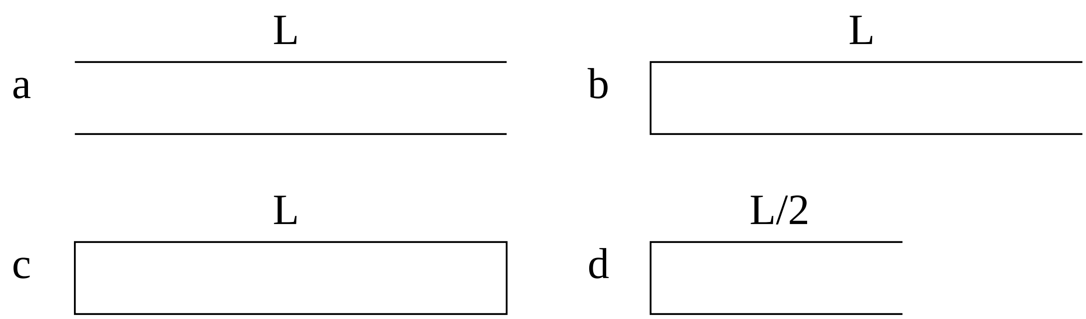
where v_a is the (constant) speed of sound in air. Hence larger wavelengths are lower frequencies and we get:

$$f_b < f_a = f_c = f_d$$

b) See above.

c) See above.

Problem 722. problems-1/sound-ra-sound-resonances.tex



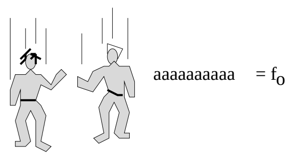
a) Rank the fundamental harmonic resonant ***frequencies*** ($n = 1$) of the four open/closed pipes drawn above, where equality is a possible answer. An answer might be (but probably isn't) $f_a < f_b = f_c < f_d$.

b) Draw into each pipe a representation of a the ***displacement wave*** associated with each resonance.

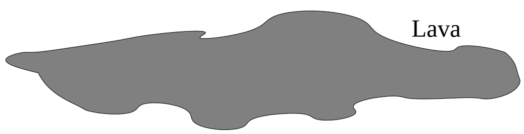
c) Label the nodes (in your representation of the waves) with an **N** and antinodes with an **A**.

13.1.4 Regular Problems

Problem 723. problems-1/sound-pr-bill-and-ted-double-doppler-1.tex

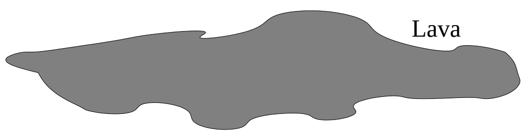


Bill and Ted are falling at a constant speed (terminal velocity) into hell, and are screaming at a frequency f_0. They hear their own voices reflecting back to them from the puddle of molten rock that lies below at a frequency of $2f_0$. How fast are they falling in meters per second?



Problem 724. problems-1/sound-pr-bill-and-ted-double-doppler-1-soln.tex

Bill and Ted are falling at a constant speed (terminal velocity) into hell, and are screaming at a frequency f_0. They hear their own voices reflecting back to them from the puddle of molten rock that lies below at a frequency of $2f_0$. How fast are they falling in meters per second?



Solution: Their emitted scream is *doubly* Doppler shifted. They are falling towards the molten lava as a moving source. The sound reflects there (no further frequency change in the already-Doppler-shifted reflected scream) and they then *fall into the reflected sound* as a moving *receiver*. Hence:

$$f' = 2f_0 = \frac{1 + \frac{v}{v_a}}{1 - \frac{v}{v_a}} f_0$$

Then it is just algebra. Canceling out the (irrelevant) f_0, and substituting the symbol $\alpha = v/v_a$ for simplicity:

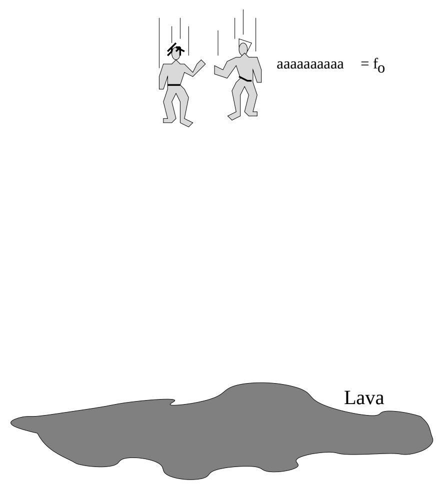
$$2 = \frac{1+\alpha}{1-\alpha} \quad \Rightarrow \quad 2 - 2\alpha = 1 + \alpha \quad \Rightarrow \quad 1 = 3\alpha$$

or:

$$\alpha = \frac{v}{v_a} = \frac{1}{3} \quad \Rightarrow \quad \boxed{v = \frac{1}{3} v_a \approx 114 \text{ m/sec}}$$

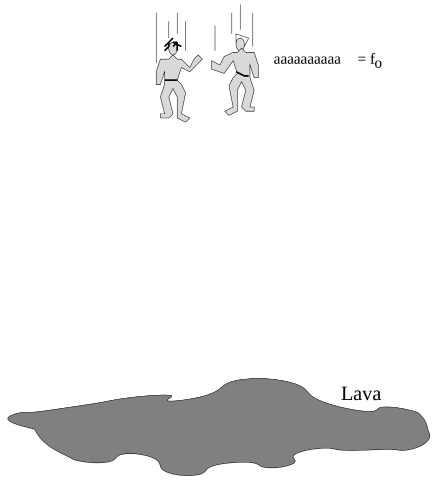
Note that since the question asks for an actual speed in meters per second, one must know that $v_a = 343$ m/sec to complete the problem. Since estimation is encouraged, any answer from (say) 111 to 115 m/sec is acceptable, as long as the student shows their numerical estimate for v_a and divide it by 3.

Problem 725. problems-1/sound-pr-bill-and-ted-double-doppler-2.tex



Bill and Ted are falling at a constant speed (terminal velocity) into hell, and are screaming at a frequency f_0. They hear their own voices reflecting back to them from the puddle of molten rock that lies below at a frequency of $1.5f_0$. How fast are they falling relative to the speed of sound?

Problem 726. problems-1/sound-pr-bill-and-ted-double-doppler-2-soln.tex



Bill and Ted are falling at a constant speed (terminal velocity) into hell, and are screaming at a frequency f_0. They hear their own voices reflecting back to them from the puddle of molten rock that lies below at a frequency of $1.5f_0$. How fast are they falling relative to the speed of sound?

Let's define $r = v/v_a$ to be the ratio of their speed to the speed of sound. Then:

$$1.5f_0 = \frac{1+r}{1-r}f_0 \Rightarrow 1.5(1-r) = 1+r = 1.5 - 1.5r$$

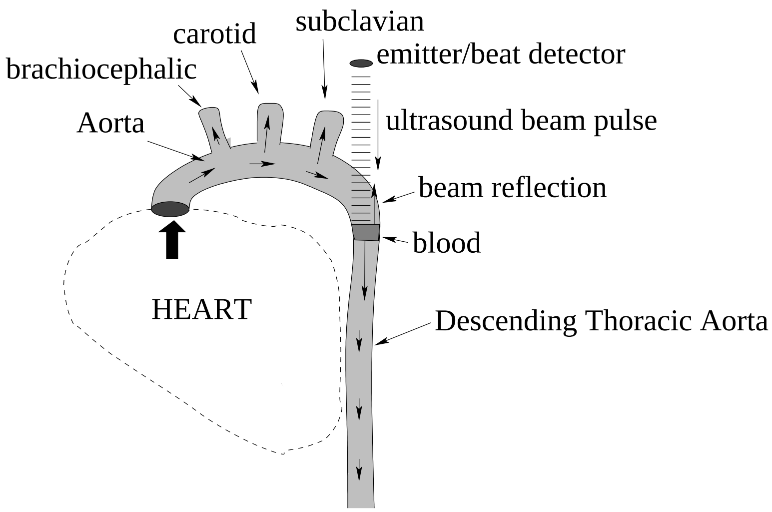
Rearranging:

$$(1.5-1) = (1.5+1)r \Rightarrow r = \frac{0.5}{2.5} = 0.2$$

So they are travelling at:

$$r = 0.2 \text{ times the speed of sound}$$

Problem 727. problems-1/sound-pr-doppler-electrocardiograph.tex



During a cardiac cycle, blood is ejected by the heart into the aorta with a typical peak speed around 0.5 m/sec for a healthy adult. However, in a patient with an obstruction, the peak speed can be much higher. The peak blood speed can be detected noninvasively using a pulsed ultrasound beam.

Let us model this process as a simple highly directional ultrasound beam of frequency f_0 that is being directed through a patient's descending thoracic aorta parallel to the artery as shown. We will assume that the ultrasound beam is reflected off of just one small (shaded) section of the flowing blood fluid that is travelling at a speed v in the direction shown the same way it would be reflected off of a moving object. Use $v_{\rm us}$ for the speed of ultrasonic sound in blood or living tissue.

patient

a) Write an *expression* for the the frequency f we expect the detector to detect in terms of f_0, $v_{\rm us}$, and v. Is f higher than or lower than the beam frequency f_0?

$$f =$$

b) The detector *measures* f, but we wish to know v. Solve for $v/v_{\rm us}$ in terms of f_0, and f.

$$\frac{v}{v_{\rm us}} =$$

The next two questions involve actual numbers. Suppose $f_0 = 2 \times 10^6$ Hz and $v_{\rm us} = 1.5 \times 10^3$ m/sec.

c) What is the wavelength of the incident beam? $\lambda =$ $\boxed{}$

d) **Extra Credit (2 points):** If a beat detector detects a beat frequency of $\Delta f = 8 \times 10^3$ Hz between the incident and reflected ultrasound beams, find the blood speed and then determine whether the patient is likely to have an obstructed descending thoracic aorta based upon information provided above. (The speed of the blood is expected to be much smaller than that of the ultrasound so that beats can be detected comparing the outgoing to the incoming doppler shifted wave.)

Problem 728. problems-1/sound-pr-doppler-moving-receiver-derive.tex

A microphone mounted on a cart is moved directly toward a harmonic source at a speed of $v_r = 34$ m/sec. The harmonic source is emitting sound waves at a frequency of $f_0 = 1000$ Hz.

a) **Derive** an expression for the frequency of the waves picked up by the moving microphone.

b) What is that frequency?

Problem 729. problems-1/sound-pr-doppler-moving-receiver-derive-soln.tex

The key to this derivation is to realize that the receiver and wavefronts are moving towards one another so that after the receiver receives a wavefront, it ***meets the next one in less time.*** Let's let $t = 0$ be the time a wavefront hits the receiver. The next wavefront is at that instant $\lambda_0 = v_a T_0$ away, where $f_0 = 1/T_0$ relates the frequency of the source to its period.

A time t later, the wavefront has moved a distance $v_a t$ to the right. The receiver, in the meantime, has moved a distance $v_r t$ to the left. Suppose that at the time $t = T'$, the receiver meets the ***next*** wavefront. Then:

$$\lambda_0 = v_a T_0 = v_a T' + v_r T'$$

This can easily be rearranged (omitted) into:

$$f' = \frac{1}{T'} = (1 + v_r/v_1)\,\frac{1}{T_0} = (1 + v_r/v_1)\,f_0$$

Numerically, (and using the fact that $v_a \approx 340$ m/sec, something you should know!) this is $f' = (1 + 0.1)f_0 = 1.1f_0 \approx 1100$ Hz.

Problem 730. problems-1/sound-pr-doppler-moving-source-derive.tex

A speaker mounted on a cart is moved directly toward a stationary microphone at a speed $v_s = 34$ m/sec. It is emitting harmonic sound waves at a source frequency of $f_0 = 1000$ Hz. $v_a = 340$ m/sec is the speed of sound in air.

a) ***Derive*** an algebraic expression for the frequency f' of the waves picked up by the stationary microphone, beginning with a suitable picture of the wave fronts. Limited partial credit will be awarded for ***just*** correctly remembering it if you cannot derive it.

b) What is the frequency f' in Hz? You should be able to do the arithmetic without a calculator.

Problem 731. problems-1/sound-pr-doppler-moving-source-derive-soln.tex

The heart of the derivation is that the cart moves ***into*** the waves as they are emitted, shortening the wavelength. In a single period of the wave source T_0, the unshifted wavelength is $\lambda_0 = v_a T_0$. The cart moves forward a distance $v_s T_0$ in that much time. The resulting wavelength is:

$$\lambda' = \lambda_0 - v_s T_0 = v_a T_0 - v_s T_0 = (v_a - v_s) T_0$$

The shifted frequency picked up by the receiver is then just (skipping a bit of algebra that you should be able to do):

$$f' = \frac{v_a}{\lambda'} = \frac{1}{1 - v_s/v_a} f_0$$

Numerically, $f' = (1/(1-0.1))\, f_0 = (1/0.9) f_0 \approx 1.111 f_0 \approx 1100$ Hz, near enough.

Problem 732. problems-1/sound-pr-doppler-moving-source.tex

A speaker mounted on a cart is moved directly toward a stationary microphone at a speed $v_s = 34.00$ m/sec. It is emitting harmonic sound waves at a source frequency of $f_0 = 1000$ Hz. $v_a = 340.0$ m/sec is the speed of sound in air.

a) What is the frequency f' of the waves picked up by the microphone in Hz? You should be able to do the arithmetic without a calculator.

b) Suppose a second source with the same frequency f_0 was located an identical distance to the right of the microphone receiver that is *also* moving towards the receiver at this same speed. What would be the frequency of the ***beats*** recorded by the microphone?

c) Suppose the source on the right was *receding* from the microphone at this same speed. In that case, what would the beat frequency observed by the microphone be?

Problem 733. problems-1/sound-pr-doppler-moving-source-soln.tex

A speaker mounted on a cart is moved directly toward a stationary microphone at a speed $v_s = 34.00$ m/sec. It is emitting harmonic sound waves at a source frequency of $f_0 = 1000$ Hz. $v_a = 340.0$ m/sec is the speed of sound in air.

a) What is the frequency f' of the waves picked up by the microphone in Hz? You should be able to do the arithmetic without a calculator.

b) Suppose a second source with the same frequency f_0 was located an identical distance to the right of the microphone receiver that is *also* moving towards the receiver at this same speed. What would be the frequency of the ***beats*** recorded by the microphone?

c) Suppose the source on the right was *receding* from the microphone at this same speed. In that case, what would the beat frequency observed by the microphone be?

Solution: For a) using the "approaching moving source doppler shift" formula:

$$f' = \frac{1}{1 - \frac{v_s}{v_a}} f_0 = \frac{1}{1 - 0.1} f_0 \approx 1.111 f_0$$

or

$$\boxed{f' \approx 1111 \text{ Hz}}$$

For part b), as the source on the right is *also* approaching the receiver, its doppler shift is the *same*. The *difference* in frequencies is then *zero* and:

$$\boxed{f_b = |f_l - f_r| = 1111 - 1111 = 0}$$

The main point is that there is no "vectorocity" to the doppler shift so the answer $f_b = 2222$ Hz is just wrong – the two waves (if sufficiently coherent) might produce a standing wave pattern and the receiver might get more or less energy depending on where it was in the pattern, but the pattern would not be *time dependent*.

For part c):

$$f_r = \frac{1}{1 + \frac{v_s}{v_a}} f_0 = \frac{1}{1 + 0.1} f_0 \approx .9091 f_0$$

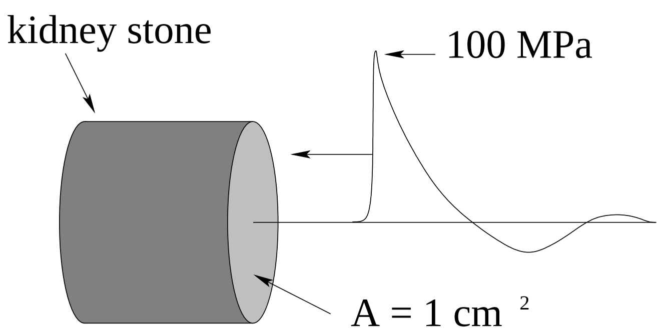
(to three places, the most we can use as significant digits) or

$$\boxed{f' \approx 909 \text{ Hz}}$$

Hence:

$$\boxed{f_b = |f_l - f_r| = 1111 - 909 = 202 \text{ Hz}}$$

Problem 734. problems-1/sound-pr-lithotripsy-decibels.tex



Modern ***lithotripsy*** machines create a focused acoustical shock wave (SW) pulse with an overpressures that range from $P_0 = 4 \times 10^7$ to over 10^8 Pascals[1]. A harmonic wave in water with this amplitude would have an intensity $I \sim P_0^2 \times 10^4$ when P_0 is expressed in ***atmospheres*** of pressure and I is the usual ***watts per square meter***. Although this expression will not be exact for a non-harmonic shock wave pulse, it should give the right order of magnitude for the average intensity in the initial peak.

a) Estimate I for an acoustical pulse with a peak amplitude of 10^8 Pascals. Algebra first! Careful with the units!

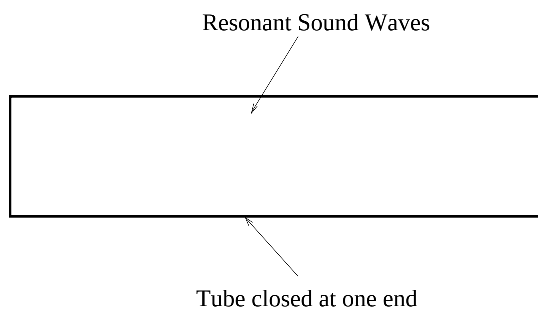
b) Express this intensity in decibels. Use the usual reference intensity for sound waves (the threshold of hearing).

c) Estimate the "instantaneous" peak force (rise time on the order of nanoseconds) exerted by the shock wave overpressure on the front face of a cylindrical kidney stone with an area of 1 square centimeter.

d) Assuming that this primary pulse lasts for $\Delta t = 10$ nanoseconds (or 10^{-8} seconds), what is the total impulse imparted to the front face of the kidney stone by this force?

[1] This dynamic pressure is comparable to the *static* pressure in the deep ocean trenches ten kilometers beneath the surface, where even "incompressible" water compresses by around 4 or 5%.

Problem 735. problems-1/sound-pr-standing-waves-organ-pipe.tex

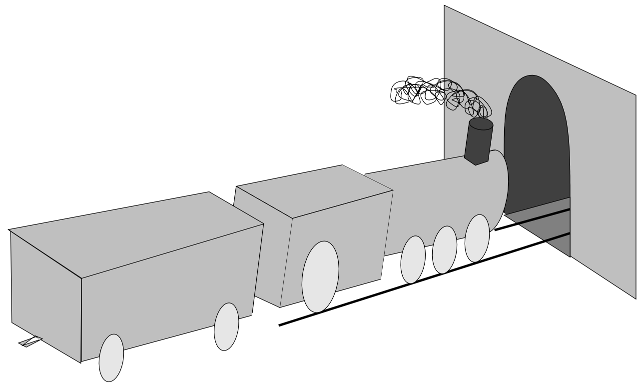


An organ pipe is made from a brass tube closed at one end as shown. The pipe is 3.4 meters long. When driven it produces a sound that is a mixture of the first and fifth harmonic.

a) What are the frequencies of these harmonics?

b) Sketch the displacement wave amplitudes for the fifth harmonic mode (only) in on the figure, indicating the nodes and antinodes.

Problem 736. problems-1/sound-pr-train-double-doppler-shift-2.tex



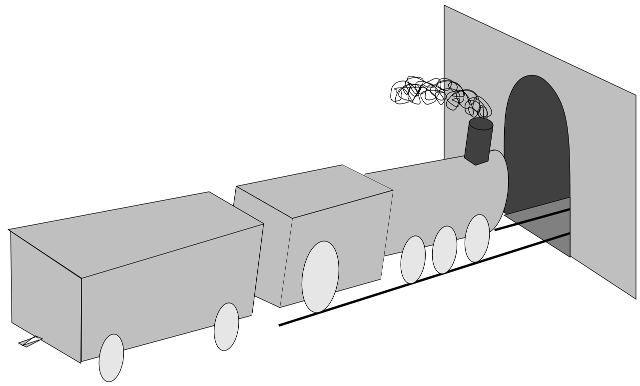
A train approaches a tunnel in a sheer cliff at speed v_{train}. The train blows a whistle of frequency 1000 Hz. A listener on the train hears a beat frequency of 10 Hz between the original whistle and the reflected sound.

a) What is the frequency of the reflected wave as heard by the passengers on the train?

b) Find the speed of the train relative to the speed of sound in air:

$$\frac{v_{\text{train}}}{v_{\text{air}}} =$$

Problem 737. problems-1/sound-pr-train-double-doppler-shift.tex



A train approaches a tunnel in a sheer cliff. The train is moving at 34 m/s, and it blows a horn of frequency 900 Hz. The speed of sounds is 340 m/s.

a) What frequency would a listener at the base of the cliff hear?

b) What frequency do the train passengers hear from the echo (the reflection from the cliff face)?

Problem 738. problems-1/sound-pr-train-double-doppler-shift-soln.tex

Bear in mind, I ***usually*** will not give you $v_a = 340$ m/sec! Be sure you know this number.

a) The train approaches the listener, so:

$$\begin{aligned} f_1 &= \frac{1}{1-\frac{v_t}{v_a}} f_0 \\ &= \frac{1}{1-0.1} 900 \\ &= \frac{900}{0.9} = 1000\text{Hz} \end{aligned} \tag{13.2}$$

b) Now the passengers are a moving receiver approaching the reflected "source", so:

$$\begin{aligned} f_2 &= (1+\frac{v_t}{v_a}) f_1 \\ &= 1.1 * 1000 \\ &= 1100\text{Hz} \end{aligned} \tag{13.3}$$

c) The beat frequency is just the ***difference*** in the frequency received by the moving train and the frequency it emits:

$$f_b = |f_2 - f_0| = 200\text{Hz}$$

Chapter 14

Newtonian Gravitation

14.1 Newtonian Gravitation

14.1.1 Multiple Choice Problems

Problem 739. problems-1/gravitation-mc-drag-changes-orbit.tex

A satellite in a low-Earth (circular) orbit will slowly lose energy to frictional drag forces ***while remaining in an approximately circular orbit***. What happens to its orbit radius and speed?

a) Its orbit radius increases and its speed increases;

b) Its orbit radius increases and its speed decreases;

c) Its orbit radius decreases and its speed increases;

d) Its orbit radius decreases and its speed decreases;

e) There is no enough information to determine the change to its orbit radius and speed.

Briefly explain or justify your answer.

Problem 740. problems-1/gravitation-mc-drag-changes-orbit-soln.tex

Frictional drag can only ***remove mechanical energy from the system, converting it into heat*** under these circumstances. This means that the mechanical energy of the still-circular orbit has to ***decrease***.

For circular orbits (recall or prove as necessary):

$$U = -\frac{GMm}{r}$$

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r}$$

$$E_{\rm tot} = -\frac{GMm}{2r}$$

For $E_{\rm tot}$ to decrease – become more negative – r has to become *smaller*, and its kinetic energy has to *increase* as r decreases. Hence:

a) Its orbit radius increases and its speed increases;

b) Its orbit radius increases and its speed decreases;

c) **Its orbit radius decreases and its speed increases;**

d). Its orbit radius decreases and its speed decreases;

e). There is no enough information to determine the change to its orbit radius and speed.

Problem 741. problems-1/gravitation-mc-kepler-and-scaling.tex

True or False:

a) Kepler's law of equal areas implies that gravity varies inversely with the square of the distance. **T** **F**

b) The planet closest to the sun on average (smallest semimajor axis) has the shortest period of revolution about the sun. **T** **F**

c) The acceleration of an apple near the surface of the earth, compared to the acceleration of the moon as it orbits the earth, is in the ratio of R_m/R_e, where R_m is the radius of the moon's orbit and R_e is the radius of the earth. **T** **F**

Problem 742. problems-1/gravitation-mc-kepler-and-scaling-soln.tex

a) **F** Any radial force law will do it.

b) **T** Kepler's 3rd law says so.

c) **F** Ratio should be squared.

Problem 743. problems-1/gravitation-mc-period-of-missing-planet.tex

The Kepler project is surveying the night sky for stars with planets (and so far over 5000 "exoplanets" have been discovered, with more being found every day). Suppose the Kepler telescope discovers that a gas giant similar to Jupiter (the easiest kind of planet to detect) is orbiting a particular star at a distance of 4 astronomical units (AU – the radius of the Earth's orbit around the Sun). The period of the planet's orbit is determined to be 16 Earth years. What would the period of a possible Earth-like planet that was orbiting that star at 1 AU be?

All answers below are in Earth years:

☐ 1/2 ☐ 1 ☐ 2 ☐ $\sqrt{2}/2$ ☐ 3

Problem 744. problems-1/gravitation-mc-period-of-missing-planet-soln.tex

The Kepler project is surveying the night sky for stars with planets (and so far over 5000 "exoplanets" have been discovered, with more being found every day). Suppose the Kepler telescope discovers that a gas giant similar to Jupiter (the easiest kind of planet to detect) is orbiting a particular star at a distance of 4 astronomical units (AU – the radius of the Earth's orbit around the Sun). The period of the planet's orbit is determined to be 16 Earth years. What would the period of a possible Earth-like planet that was orbiting that star at 1 AU be?

All answers below are in Earth years:

☐ 1/2 ☐ 1 ☒ 2 ☐ $\sqrt{2}/2$ ☐ 3

Solution: Kepler's third law tells us

$$R^3 = CT^2$$

We need to determine C for this star. From the data:

$$C = 4^3/16^2 = 64/256 = 1/4$$

in units of $\text{AU}^3/\text{year}^2$. Hence the earth-like planet at 1 AU would have to have a period of:

$$T_e^2 = 1^3/C = 4 \text{ year}^2 \quad \Rightarrow \quad \boxed{T_e = 2 \text{ years}}$$

Problem 745. problems-1/gravitation-mc-scaling-moon-orbit.tex

Planet Bongo has a moon, Mongo, that orbits it in a circular orbit much like the Moon orbits the Earth. You are told that

$$M_{\text{Bongo}} = 3M_{\text{Earth}} \qquad R_{\text{Bongo}} = 2R_{\text{Earth}} \qquad r_{\text{Mongo}} = 2r_{\text{Earth}}$$

where M is each planet's mass, R is its planetary radius, and r is the orbital radius of the respective moon about the center of its planet.

a) Compared to the speed of the Moon, the moon Mongo's speed is:

   A) larger;
   B) the same;
   C) smaller;
   D) unknown, as there is not enough information to decide;

b) Find the ratio of the period of the circular motion between the two:

$$\frac{T_{\text{Mongo}}}{T_{\text{Moon}}} = \boxed{}$$

Problem 746. problems-1/gravitation-mc-scaling-moon-orbit-soln.tex

Planet Bongo has a moon, Mongo, that orbits it in a circular orbit much like the Moon orbits the Earth. You are told that

$$M_{\rm Bongo} = 3M_{\rm Earth} \qquad R_{\rm Bongo} = 2R_{\rm Earth} \qquad r_{\rm Mongo} = 2r_{\rm Earth}$$

where M is each planet's mass, R is its planetary radius, and r is the orbital radius of the respective moon about the center of its planet.

For starters, the R radii of the planets themselves is irrelevant, as long as $r \gg R$.

All of the answers below come from Newton's Second Law, Newton's Law of Gravitation, and the usual formulae for centripetal acceleration in a circular orbit and the relationship between period and angular speed:

$$F_c = \frac{GMm}{r^2} = \frac{mv^2}{r} = m\omega^2 r = m\frac{4\pi^2}{T^2}r$$

We will use these in ***ratios*** that emphasize the ***scaling*** and let us cancel out as much of the irrelevant pieces as possible. We'll use v_e as the speed of the moon about Earth, and v_M as the speed of Bongo's moon, Mongo:

a) Compared to the speed of the Moon, the moon Mongo's speed is:

$$v_e^2 = \frac{GM_e}{r_e}$$
$$v_M^2 = \frac{GM_B}{r_M} = \frac{G3M_e}{2r_e} = \frac{3}{2}v_e^2$$

A) larger;
B) the same;
C) smaller;
D) unknown, as there is not enough information to decide;

b) Find the ratio of the period of the circular motion between the two:

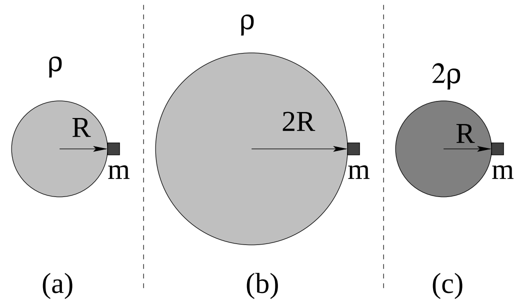
$$T_e^2 = \frac{4\pi^2}{GM_e}r_e^3$$
$$T_M^2 = \frac{4\pi^2}{GM_B}r_M^3 = \frac{4\pi^2}{G3M_e}(2r_e)^3 = \frac{8}{3}\frac{4\pi^2}{GM_e}r_e^3$$

Take the ratio and lots of stuff cancels! In fact, everything cancels but some pure dimensionless numbers:

$$\frac{T_M^2}{T_e^2} = \frac{8}{3}$$
$$\frac{T_M}{T_e} = \sqrt{\frac{8}{3}}$$

$$\frac{T_{\rm Mongo}}{T_{\rm Moon}} = \sqrt{\frac{8}{3}}$$

Problem 747. problems-1/gravitation-mc-surface-gravity-scaling.tex



In the figure above, a small mass m is sitting on the surface of three planets. The density and radius of the planets are as shown:

a) ρ, R

b) $\rho, 2R$

c) $2\rho, R$

If the force on m due to gravity for the first planet is F_a, find and express F_b and F_c **in terms of $\boldsymbol{F_a}$**.

$F_b =$

$F_c =$

Problem 748. problems-1/gravitation-mc-surface-gravity-scaling-soln.tex

$M = 4\pi\rho R^3/3$, so:

$$F_a = \frac{GMm}{R^2} = \frac{4\pi G\rho R}{3}$$

This means that the force scales ***linearly with $\boldsymbol{R}$*** and ***linearly with $\boldsymbol{\rho}$***. Hence:

$F_b = 2F_a$

$F_c = 2F_a$

14.1.2 Short Answer Problems

Problem 749. problems-1/gravitation-sa-escape-condition.tex

Answer the following short questions about escaping from a planet's gravitational field at its surface to "infinity". The answer to each is best given as an equation or short derivation or by a single sentence that correct captures the concept involved and explains or answers the question.

a) What is the ***condition*** for an object sitting on a planetary surface to escape to infinity?

b) Use the condition from part a) to derive (in a couple of lines of algebra) the escape ***speed*** from a planet of mass M and radius R. This is the smallest speed the object be moving with to escape to infinity.

c) Does it matter what direction the object leaves the surface (that is, does it have to leave travelling straight up or can it leave at an angle) as long as its path doesn't intersect the surface itself?

d) Assume that the planet is Earth, with mass M_e and radius R_e. Show that the escape speed from Earth can be written $v_e = \sqrt{2gR_e}$ where g is the usual gravitational field (acceleration) near the surface of the Earth.

Problem 750. problems-1/gravitation-sa-escape-condition-soln.tex

a) The escape condition is fundamentally $E_{\rm tot} = U + K = 0$, so that the object can reach $r \to \infty$ and arrive there at rest. This ($E_{\rm tot} \approx 0$) is also a good assumption to make for *any* object that falls to an attractor from far far away, for example for a falling asteroid or comet, for the purposes of estimation.

Hence:

$$E_{\rm tot} = \frac{1}{2}mv_e^2 - \frac{GMm}{R} = 0$$

b)

$$v_e = \sqrt{\frac{2GM}{R}}$$

is both the (minimum) escape speed and a good estimate for the speed of e.g. a falling asteroid as it enters the Earth's atmosphere.

c) No. This is an energy condition, and does not depend on direction, as long as one doesn't run into something (like the planet itself) along the way!

d) Use $g = \dfrac{GM_e}{R_e^2}$ as follows:

$$v_e = \sqrt{\frac{2GM_e}{R_e} \times \frac{R_e}{R_e}} = \sqrt{2\left(\frac{GM_e}{R_e^2}\right) R_e} = \sqrt{2gR_e}$$

Problem 751. problems-1/gravitation-sa-circular-orbit-vs-escape.tex

It is very costly (in energy) to lift a payload from the surface of the earth into a circular orbit, but once you are there, it only costs you that same amount of energy again to get from that circular orbit to anywhere you like – if you are willing to wait a long time to get there. Science Fiction author Robert A. Heinlein succinctly stated this as: “By the time you are in orbit, you're halfway to anywhere.”

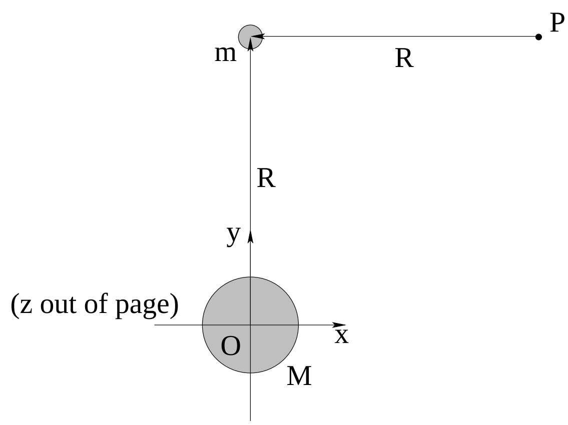
Prove this by comparing the total energy of a mass:

a) On the ground. Neglect its kinetic energy due to the rotation of the Earth.

b) In a (very low) circular orbit with at radius $R \approx R_E$ – assume that it is still more or less the same distance from the center of the Earth as it was when it was on the ground.

c) The orbit with minimal escape energy (that will arrive, at rest, “at infinity” after an infinite amount of time).

Problem 752. problems-1/gravitation-sa-force-and-torque.tex



In the figure above, a mass M is located at the origin, and a mass m is located at $(0, R)$ as drawn. The z-axis in the figure comes ***out of the page***. All ***vector*** answers below may be indicated in any of the permissible ways.

a) Find the gravitional force acting on mass little m.

b) Find the torque around the origin O.

c) Find the torque on mass m relative to the pivot P. Draw and label an arrow symbol onto the figure above to explicitly indicate its direction

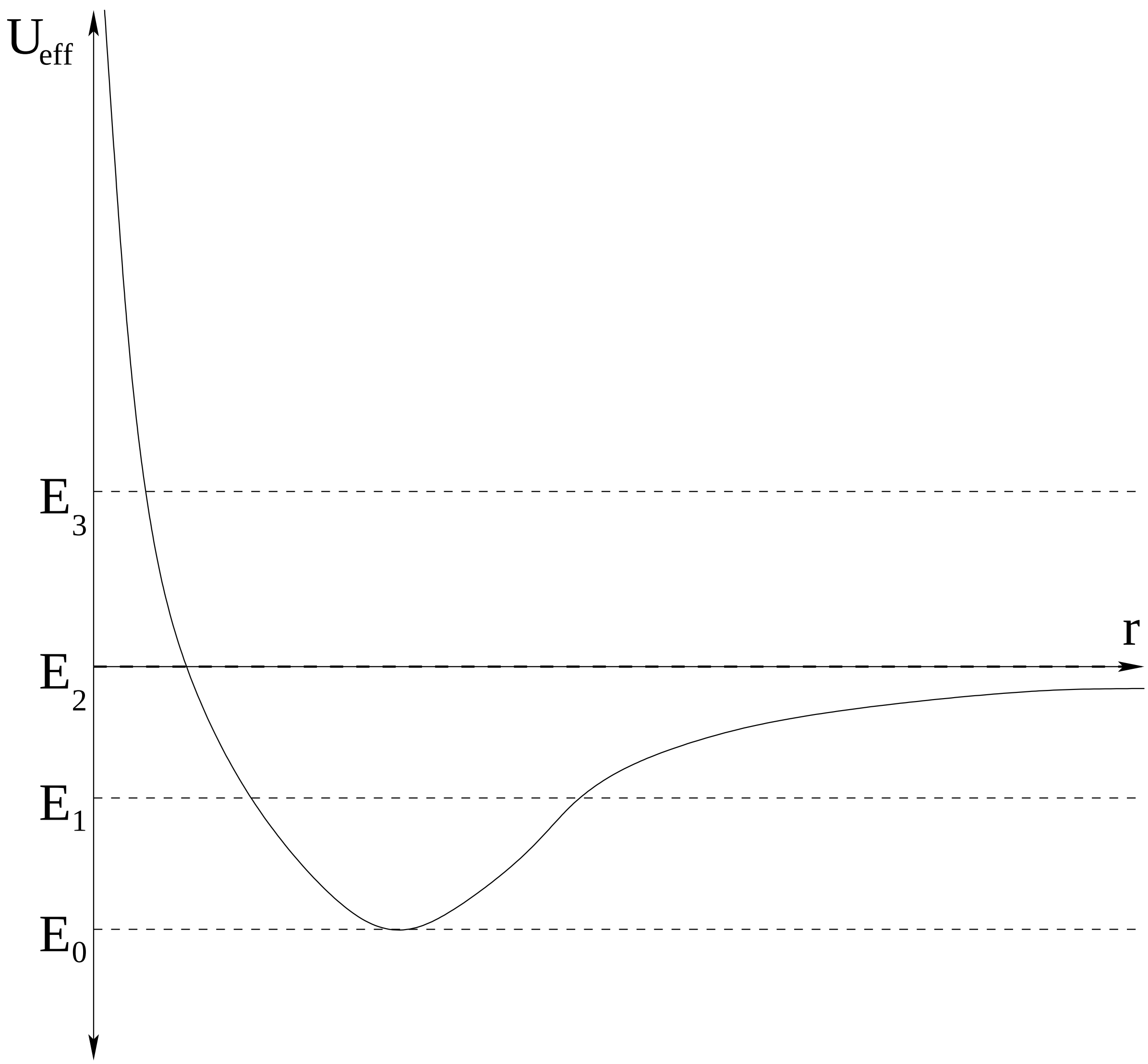
Problem 753. problems-1/gravitation-sa-force-and-torque-soln.tex

a) $\vec{\boldsymbol{F}} = -\dfrac{GMm}{R^2}\hat{\boldsymbol{y}}$

b) $\vec{\boldsymbol{\tau}} = 0$

c) $\vec{\boldsymbol{\tau}} = \dfrac{GMm}{R}\hat{\boldsymbol{z}}$

Problem 754. problems-1/gravitation-sa-identify-four-orbits.tex



The *effective radial potential* of a planetary object of mass m in an orbit around a star of mass M is:

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

The total energies E_0, E_1, E_2, E_3 of four orbits are drawn as dashed lines on the figure above for $G = 1$, $M = 100$, $m = 1$ and $L = 5$ (in some system of units). Name the kind of orbit (circular, elliptical, parabolic, hyperbolic) each energy represents and mark its turning point(s) in on the graph.

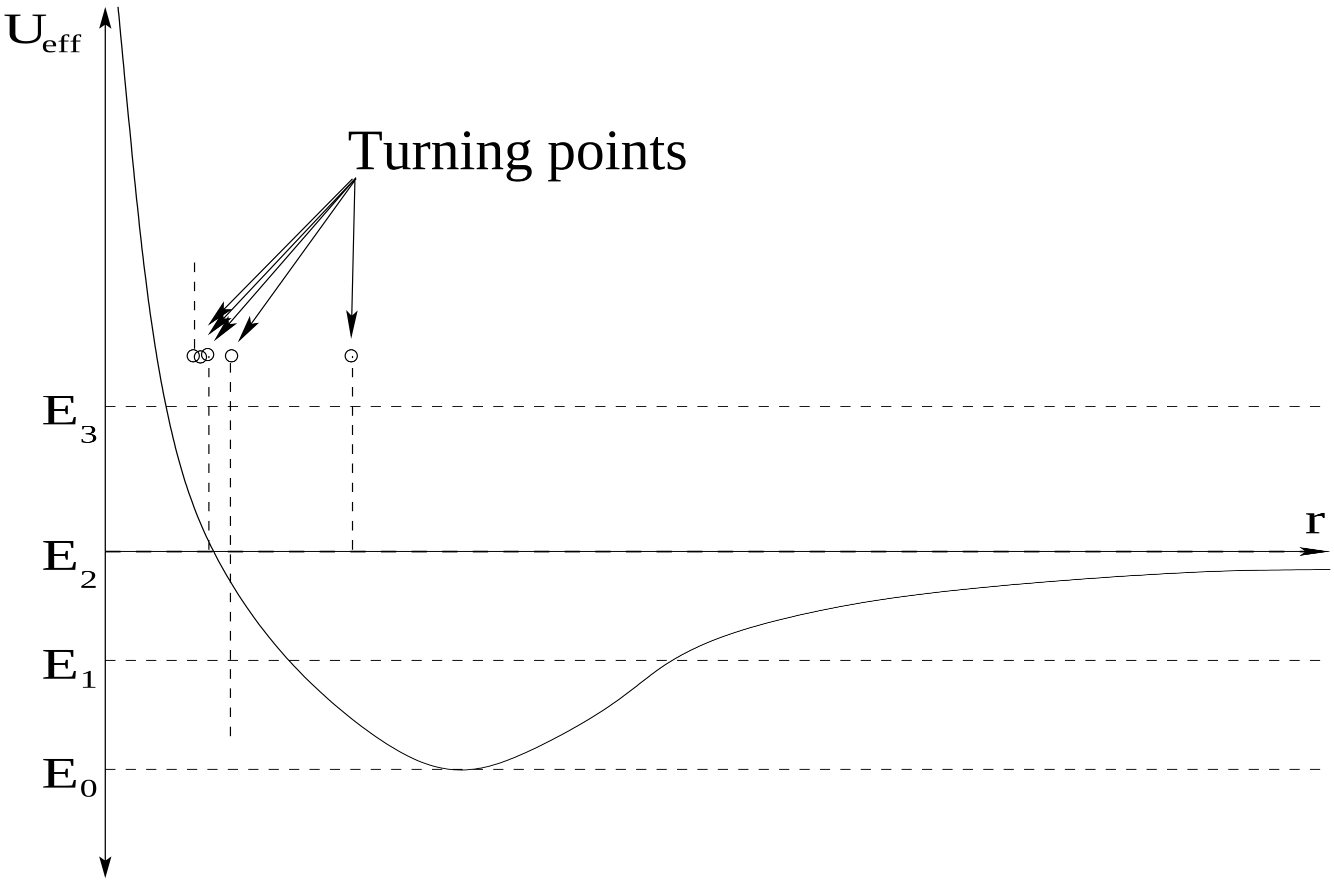
E_0 $\boxed{}$

E_1

E_2

E_3

Problem 755. problems-1/gravitation-sa-identify-four-orbits-soln.tex



E_0 $\boxed{\text{circular}}$

E_1 $\boxed{\text{elliptical}}$

E_2 $\boxed{\text{parabolic}}$

E_3 $\boxed{\text{hyperbolic}}$

Problem 756. problems-1/gravitation-sa-kepler-3-circular-orbits.tex

In your homework, you studied several different cases of a mass m in a **circular orbit** around (or inside) another mass M, with different radial force laws. Suppose you are given a radial force law of the form:

$$\vec{F} = -\frac{A}{r^n}\hat{r}$$

Prove that (for circular orbits in particular):

$$r^{n+1} = CT^2$$

where T is the period of the orbit and r is the radius of the circle, and find the constant C. ($A = GMm$, $n = 2$ then leads to Kepler's third law, and $A = GMm/R^3$, $n = -1$ leads to the relation you derived for the mass in the tunnel through the death star).

Problem 757. problems-1/gravitation-sa-kepler-3-circular-orbits-soln.tex

In your homework, you studied several different cases of a mass m in a **circular orbit** around (or inside) another mass M, with different radial force laws. Suppose you are given a radial force law of the form:

$$\vec{\boldsymbol{F}} = -\frac{A}{r^n}\hat{\boldsymbol{r}}$$

Prove that (for circular orbits in particular):

$$r^{n+1} = CT^2$$

where T is the period of the orbit and r is the radius of the zcircle, and find the constant C. ($A = GMm$, $n = 2$ then leads to Kepler's third law, and $A = GMm/R^3$, $n = -1$ leads to the relation you derived for the mass in the tunnel through the death star).

Newton's Second Law (for this new force law) becomes:

$$F = \frac{A}{r^n} = \frac{mv^2}{r} = m\omega^2 r = m\frac{4\pi^2}{T^2}r$$

or (simply rearranging):

$$r^{n+1} = \frac{A}{4\pi^2 m}T^2$$

(where we cannot overtly cancel m, but it is probably part of the given "A" as indicated above).

Hence:

$$C = \frac{A}{4\pi^2 m}$$

(and as a check, if $A = GMm$, $C = \dfrac{GM}{4\pi^2}$ as usual).

Problem 758. problems-1/gravitation-sa-period-of-saturn.tex

The earth's orbit is "one astronomical unit" (AU) in radius (this turns out to be about 150 million kilometers). The period of its orbit is one year. The mean radius of Saturn's orbit is (roughly) 10 AU. What is its "year" (period of revolution around the sun) in years? (You may express your answer as a power of a rational fraction without a calculator.)

Problem 759. problems-1/gravitation-sa-period-of-saturn-soln.tex

The earth's orbit is "one astronomical unit" (AU) in radius (this turns out to be about 150 million kilometers). The period of its orbit is one year. The mean radius of Saturn's orbit is (roughly) 10 AU. What is its "year" (period of revolution around the sun) in years? (You may express your answer as a power of a rational fraction without a calculator.)

Solution: Use Kepler's Third Law:

$$\frac{T_s^2}{T_e^2} = \frac{R_s^3}{R_e^3}$$

in years and AU respectively:

$$T_s^2 = \frac{10^3}{1^3} \times 1^2 = 1000$$

so

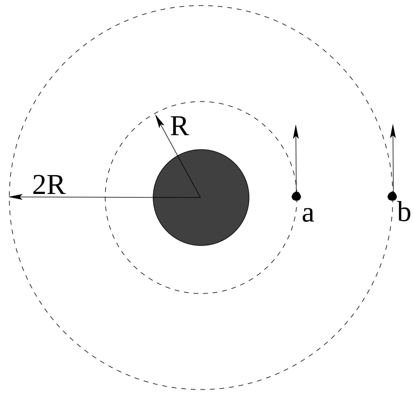
$$\boxed{T_s = \sqrt{1000} \approx 31.5 \text{ years}}$$

Problem 760. problems-1/gravitation-sa-speed-of-jupiter.tex

The Earth's approximately circular orbit about the Sun is "one astronomical unit" (AU) in radius (this turns out to be about 150 million kilometers). The mean radius of Jupiter's approximately circular orbit is (roughly) 5 AU. What is the average speed of Jupiter $v_{jupiter}$ in terms of the average speed of the Earth v_{earth} as it moves around the Sun?

$$\boldsymbol{v_{jupiter}} = \boxed{}$$

Problem 761. problems-1/gravitation-sa-two-orbits-scaling.tex



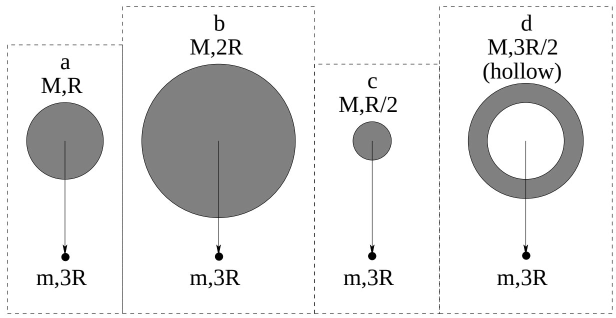
Two satellites are in circular orbits around the earth, one at radius R and the other at $2R$.

a) **Circle** the satellite that is moving **faster**.

b) How *much* faster is it moving? (Express the faster satellite's speed in terms of the speed of the slower satellite.)

14.1.3 Ranking Problems

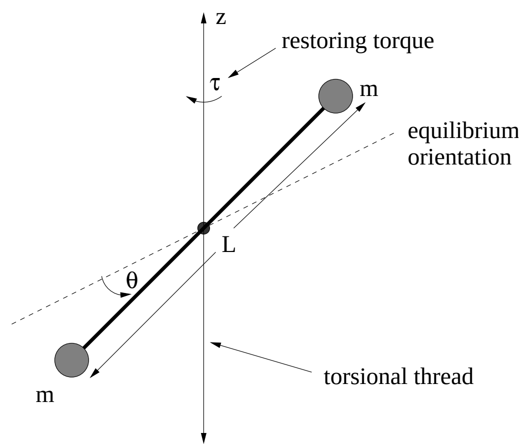
Problem 762. problems-1/gravitation-ra-four-planets.tex



(6 points) Four planets of mass M are drawn to scale above, each exerting a gravitational force of magnitude F_i (for $i = a, b, c, d$) on the small mass m at the position $3R$ from the center of each planet as shown. Rank the F_i from least to greatest including possible equalities. Indicate *why* you are answering the way that you answer in words or an equation or two.

14.1.4 Regular Problems

Problem 763. problems-1/gravitation-pr-cavendish-torsional-oscillator.tex



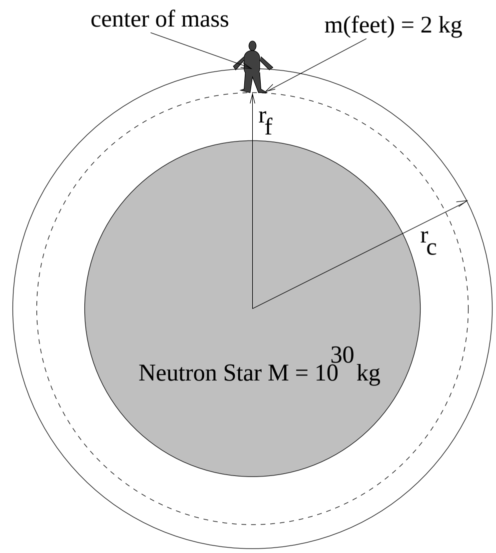
In the Cavendish experiment, the gravitational force is measured between two big masses M (not shown) acting on two small masses m on a rod of length L (assumed to be of negligible mass in this problem, although it isn't really) attached to a thin thread such that it makes a torsional pendulum (as drawn above). The twisting thread exerts a restoring torque of magnitude $\tau = -\kappa\theta$ on the rod connecting the small masses, where theta is measured from the equilibrium angle of the rod as shown.

In the experiment the two large masses are placed symmetrically so that they exert a torque on the small mass arrangement aligned with the torsional thread. The two small masses twist the thread toward the big masses until the gravitational torque is balanced by the torque of the thread. *If* κ is known, a measurement of the angle of deviation θ_0 suffices to determine the gravitional torque, hence the gravitational force, hence the gravitational constant G.

There's only one catch – one needs κ, and most spools of thread don't come labeled with their torsional response properties.

Show and tell how you can do a simple experiment to measure κ with nothing but an accurate stop watch, a measurement of the mass(es) m, and a measurement of the length L of the connecting rod. (Describe the experiment and derive the relation between the quantity you choose to measure and the desired result, κ).

Problem 764. problems-1/gravitation-pr-dangerous-tides.tex

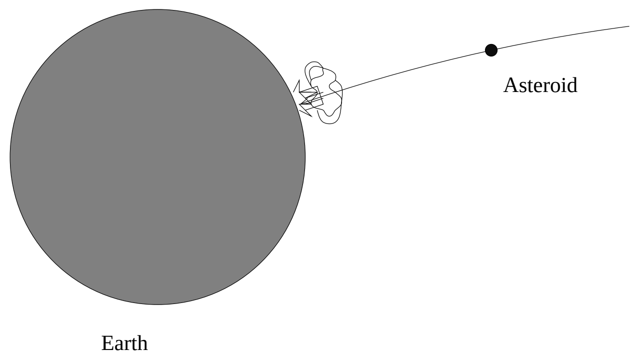


Tides can be dangerous. You are a scientist in orbit around a neutron star with a mass $M = 10^{30}$ kg and a radius of 8 km. Your center of mass moves in a perfect circle 10 km around the center of the star. You have just enough angular momentum that your feet always point "down" toward the center of the star and your head points away. Your feet are therefore also in a circular trajectory around the center of the star, but they cannot also be in orbit (free fall).

Assuming that your feet have a mass of approximately 2 kg and are located approximately 1 meter closer to the star than your center of mass, how much force do your legs have to provide to keep your feet from falling off? Do they fall off?

Hints: Proceed by finding the centripetal acceleration/force of your center of mass in terms of the gravitational field/force of the star at that location. Repeat this for your feet separately, assuming that they have the same angular frequency of circular motion as your center of mass but are in a (much!) stronger gravitational field. The difference in the force required to keep the feet in a circular orbit (the total centripetal force) and the actual gravitational force must be provided by your legs. Also, the binomial expansion might well be useful here...

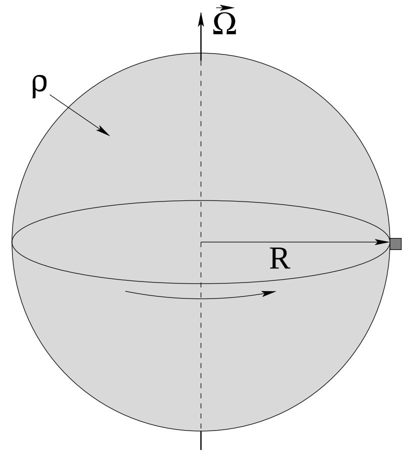
Problem 765. problems-1/gravitation-pr-dinosaur-killer-asteroid.tex



Estimate the total energy released when a spherical "Dinosaur Killer" asteroid with a density $\rho = 10$ kg/m^3 and radius $R = 1000$ meters falls onto the surface of the earth from "outer space" (far away). Obviously your answer should be justified by a good physical argument.

Note that this is a *lot* of energy – more than enough to wipe out all life within perhaps 1000 km of the point of impact (or more) and to change the climate of the planet.

Problem 766. problems-1/gravitation-pr-equatorial-weight.tex



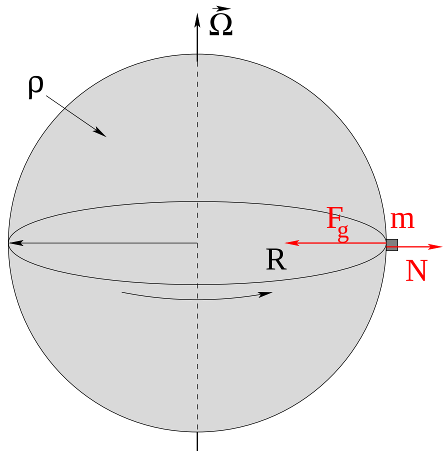
In the figure above, a planet with uniform mass density ρ and radius r rotates at a constant angular velocity $\vec{\boldsymbol{\Omega}}$ around its N-S axis. A small block of mass m is located on the planet's equator and at the instant shown is at rest relative to the surface (meaning that it too is rotating around the axis with constant angular velocity $\vec{\boldsymbol{\Omega}}$). Express all answers in terms of G, ρ, m and Ω as needed or appropriate.

a) Draw the forces acting on the block into the picture, assuming that the planet's rotation is slow enough for the block to remain on the surface.

b) Derive an expression for the apparent weight of the block as a function of Ω (the magnitude of the angular velocity).

c) If the planet's rotational speed is very slowly increased, at some point the normal force will go to zero. Find T_0, the period of the planet's rotation when this occurs.

Problem 767. problems-1/gravitation-pr-equatorial-weight-soln.tex



In the figure above, a planet with uniform mass density ρ and radius r rotates at a constant angular velocity $\vec{\boldsymbol{\Omega}}$ around its N-S axis. A small block of mass m is located on the planet's equator and at the instant shown is at rest relative to the surface (meaning that it too is rotating around the axis with constant angular velocity $\vec{\boldsymbol{\Omega}}$). Express all answers in terms of G, ρ, m and Ω as needed or appropriate.

a) Draw the forces acting on the block into the picture, assuming that the planet's rotation is slow enough for the block to remain on the surface.

b) Derive an expression for the apparent weight of the block as a function of Ω (the magnitude of the angular velocity).

c) If the planet's rotational speed is very slowly increased, at some point the normal force will go to zero. Find T_0, the period of the planet's rotation when this occurs.

Solution: There are only two forces: gravity and the normal force, drawn above.

The magnitude of the force of gravity acting on m is (from NLG):

$$F_g = \frac{G\left(\frac{4\pi R^3 \rho}{3}\right)m}{R^2} = \frac{4\pi G\rho R m}{3}$$

N2 for the mass m in the centripetal direction is:

$$F_g - N = ma_c = m\Omega^2 R$$

The apparent weight of the block is the normal force:

$$N = F_g - ma_c = \frac{4\pi G\rho R m}{3} - m\Omega^2 R = \boxed{m\left(\frac{4\pi G\rho R}{3} - \Omega^2 R\right) = mg'}$$

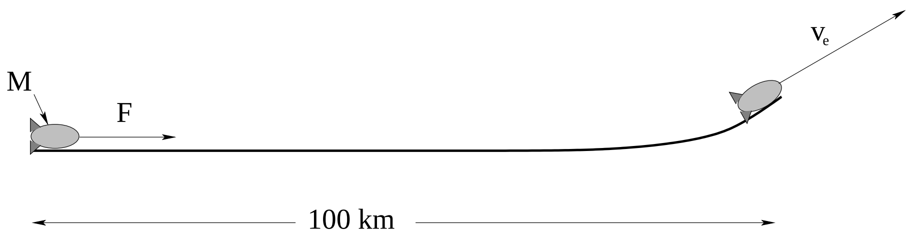
where the quantity in parentheses identified as g' is "g" in the accelerated/rotating frame.

The apparent weight will go to zero when g' goes to zero:

$$\frac{4\pi G\rho R}{3} - \Omega^2 R = 0 \quad \Rightarrow \quad \Omega = \sqrt{\frac{4\pi G\rho}{3}} = \frac{2\pi}{T_0} \quad \Rightarrow \quad \boxed{T_0 = \sqrt{\frac{3\pi}{G\rho}}}$$

independent of the radius of the planet!

Problem 768. problems-1/gravitation-pr-escape-velocity-linear-accelerator.tex



One way to reduce the cost of lifting mass into orbit is to use a linear accelerator to drive a payload up to escape velocity (or thereabouts) and then let it go. This way one doesn't have to lift the fuel used to lift the fuel used to lift the ... (almost all the fuel used in a rocket is used to lift fuel, not payload).

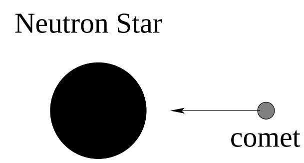
Assume that fusion energy has been developed and electricity is cheap, and that high temperature superconductors have made such a mass driver feasible. Your job is to do a first estimate of the design parameters.

A proposed plan for the mass driver is shown above. The track is 100 kilometers long and slopes gently upwards. The payload capsule has a mass of 2×10^3 kg (two metric tons). The head of the track is high in the Andes, $R = 6375$ kilometers from the center of the earth.

a) Neglecting air resistance, find the escape velocity for the capsule. Although bound orbits will not require quite as much energy, air resistance will dissipate some energy. Either way, this is a reasonable estimate of the velocity the driver must be able to produce.

b) Assuming that the capsule is started from rest and that a constant tangential force accelerates it, find the tangential force necessary to achieve escape velocity at the end of the track. Note: Ignore the normal force that the track must exert to divert it so that it departs at an upward angle.) From this find the acceleration of the capsule, in multiples of g. Is this acceleration likely to be tolerable to humans?

Problem 769. problems-1/gravitation-pr-escape-velocity-neutron-star.tex



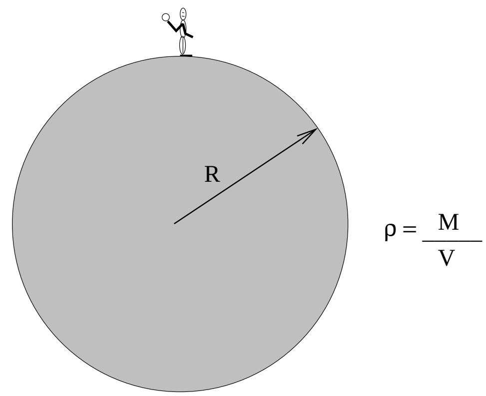
Cool stuff about gravity. A neutron star has a mass $M = 10^{30}$ kg and a radius R of 8 km. Answer the following problems **algebraically** using the variables M, m, G, R **first**, then (if you have a calculator handy or can do the arithmetic by hand) do the arithmetic and put down numbers. You can get full credit from the algebra alone, but the number answers are pretty interesting.

a) What is the escape velocity from the surface of the neutron star? (If you do the arithmetic, express the result as a fraction of c, the speed of light: $c = 3 \times 10^8$ m/sec).

b) A comet with a mass $m = 10^{14}$ kg falls from infinity into the neutron star. What is the energy liberated as it (inelastically) hits?

c) Compare this energy to the total (rest) mass energy of the comet, mc^2.

Problem 770. problems-1/gravitation-pr-escape-velocity-of-baseball.tex



Suppose that planetary rock has an average density ρ_p. Assuming that you can throw a fastball in baseball at v_f find an expression (in terms of G, ρ, v_e and known constants) for he radius R of the largest spherical planet where you can stand on the surface and throw a baseball away ***to "infinity"*** (so that it never comes back)?

If you want to have fun or "check" your algebra, try evaluating this expression for $v_f = 40$ m/sec (nearly 90 mph) and $\rho_p = 10^4$ kg/m^3. I get around 17 km, making the planet just about 10 miles in radius. The same expression could be used to find the largest planet you could *jump off of* (assuming you have a vertical leap of 1 meter on Earth).

Problem 771. problems-1/gravitation-pr-escape-velocity-of-baseball-soln.tex

Escape *energy* is $E_{\text{tot}} = 0$, hence:

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R}$$

The mass of the planet is $M = 4\pi\rho R^3/3$, so:

$$\frac{1}{2}mv_e^2 = \frac{4\pi G\rho m R^2}{3}$$

Solving for R:

$$R^2 = \frac{3v_e^2}{8\pi G\rho}$$

or numerically, given $v_e = 40$ m/sec and $\rho = 10^4$ kg/m^3,

$$R_{\text{max}} = \sqrt{3 * 1600/(8 * 3.14 * 6.67 \times 10^{-11} * 10^4)} \approx 17 \text{ km}$$

Not so very large.

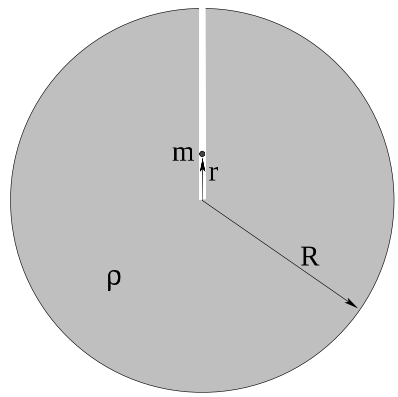
For fun, suppose that you have a vertical leap of 1 meter on the surface of the Earth (enough to slam dunk a basketball). At this point, you should be able to determine how fast you must be moving as your feet leave the ground ($\sqrt{2gH} \approx 2\sqrt{5} = 4.5$ m/sec). Use that to determine the maximum size planet you could *jump* off of. I get just about exactly $2\frac{2}{3}$ km!

Problem 772. problems-1/gravitation-pr-geosync-orbit.tex

The Duke Communications company wants to put a satellite into a circular geosynchronous orbit over the equator (this is a satellite whose period is exactly one day, so that it stays over the same point of the rotating Earth).

Ignoring perturbations like the Moon and the Sun, find the radius R_g of such an orbit as a multiple of the radius of the Earth R_e. Although *as always* you should *solve for the result algebraically first* you may wish to know some of the following data: The radius of the Moon's orbit is $R_m = 384,000$ kilometers, or $R_m = 60R_e$. The period of the Moon is $T_m = 27.3$ days compared to $T_g = 1$ day. $R_e = 6400$ kilometers. $M_e = 6 \times 10^{24}$ kilograms. One day contains 86400 seconds.

Problem 773. problems-1/gravitation-pr-half-tunnel-escape.tex



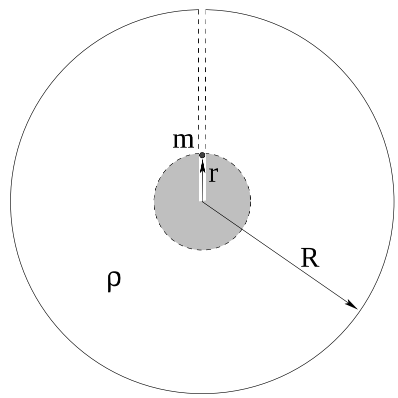
In the figure above a spherical planet of uniform density ρ and total radius R is shown. A small tunnel is drilled from the surface to the center.

a) Find the magnitude of the gravitational field $g(r)$ in the tunnel as a function of r.

b) How much work is required to lift a mass m at a constant speed from the center of this planet to the surface?

c) Suppose the mass m has reached the surface of the planet and is at rest. What upward-directed speed must you give the mass m at the surface so that the mass escapes from the planet altogether?

Problem 774. problems-1/gravitation-pr-half-tunnel-escape-soln.tex



From the shell theorem, there is ***no gravitational field or force acting on m from the mass outside of the radius r***. The field comes only from the mass *inside* the sphere of radius r (shaded above):

$$M(r) = \frac{4\pi r^3}{3}\rho$$

Hence:

a)

$$g_r = -\frac{GM(r)}{r^2} = -\frac{G4\pi\rho r}{3}$$

(radially "in" towards the center of the planet, no other components).

b) We push *against* the downward force in the direction of lift, so the work is *positive*. The force acting on m is just $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{g}}$ so:

$$W = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{\ell}} = \int_0^R F_r dr = \frac{G4\pi\rho m}{3} \int_0^R r dr = \frac{G2\pi\rho R^2 m}{3}$$

c) As alway, the escape condition is $E_{\text{tot}} = 0$. Hence:

$$E_{\text{tot}} = -\frac{GMm}{R} + \frac{1}{2}mv_e^2 = 0$$

or

$$v_e = \sqrt{\frac{2GM}{R}}$$

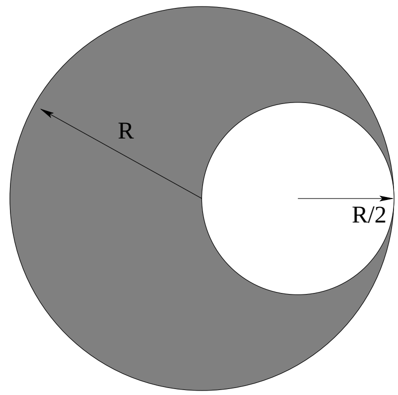
but we have to express this in terms of the ***given*** ρ!

$$M = M(R) = \frac{4\pi R^3}{3}\rho$$

so the correct answer in terms of the givens is:

$$v_e = \sqrt{\frac{G8\pi R^2\rho}{3}}$$

Problem 775. problems-1/gravitation-pr-planet-with-spherical-hole.tex



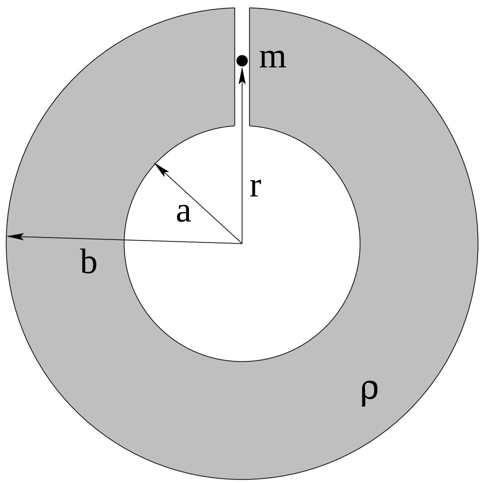
Above is pictured a spherical mass with radius R and mass density (mass per unit volume) ρ. It has a spherical hole cut out of it of radius $R/2$ as shown. Find the gravitational field in the hole in terms of G, R, and ρ, proving that it is uniform and points to the left.

Problem 776. problems-1/gravitation-pr-spherical-cow.tex

There is an old physics joke involving cows, and you will need to use its punchline to solve this problem.

A cow is standing in the middle of an open, flat field. A plumb bob with a mass of 1 kg is suspended via an unstretchable string 10 meters long so that it is hanging down roughly 2 meters away from the center of mass of the cow. Making any reasonable assumptions you like or need to, *estimate* the angle of deflection of the plumb bob from vertical due to the gravitational field of the cow.

Problem 777. problems-1/gravitation-pr-thick-shell-force.tex



A "thick" shell of mass with uniform mass density ρ, inner radius a, and outer radius b is shown. A small (frictionless) hole has been drilled at the top along the z axis, and a mass m is at a distance r from the center of the shell along the z axis so that it can be moved vertically up or down from outside of the shell to the inside by means of the tunnel.

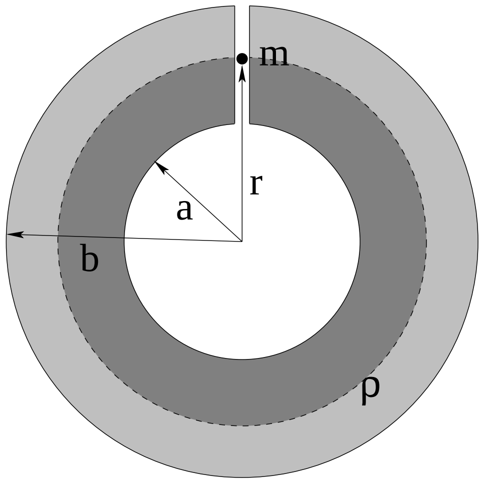
Find an expression for the magnitude of the radial force F_r acting on m when the mass is:

a) Outside of the shell of mass entirely, at some $r > b$.

b) In the tunnel, where $a < r < b$.

c) Inside the shell, at some point $r < a$.

Problem 778. problems-1/gravitation-pr-thick-shell-force-soln.tex



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Solution: We use the "shell theorem" that states that for a spherically symmetric mass distribution, we only get a contribution to the gravitational field from the mass *inside* a given radius, while outside of that radius it behaves like the field of a point mass, and inside of that radius its field is zero. (Alternatively, there is no field *inside* a spherically symmetric shell of mass.)

Either way, the field inside the thick shell is zero, so:

$$\boxed{F_r = 0 \quad \text{for } (r < a)}$$

Inside the shell itself, the mass inside radius r is (from the dark shaded portion only):

$$M(r) = \frac{4\pi\rho}{3}(r^3 - a^3)$$

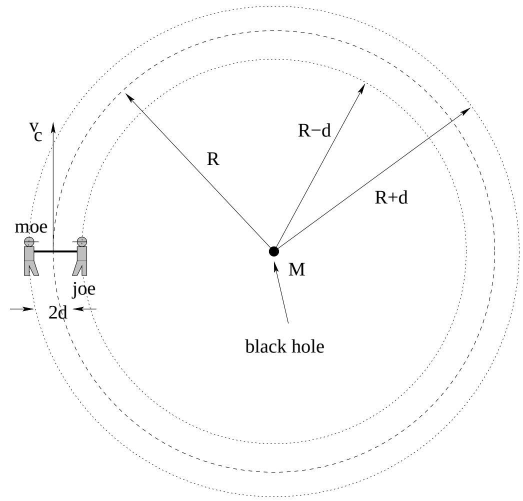
so

$$\boxed{F_r = \frac{GM(r)m}{r^2} = \frac{4\pi G\rho m}{3}\left(r - \frac{a^3}{r^2}\right) \quad \text{for } (a < r < b)}$$

Finally, outside of the shell, all of the mass behaves like a point mass at the origin):

$$\boxed{F_r = \frac{GM(b)m}{r^2} = \frac{4\pi G\rho m}{3}\frac{b^3 - a^3}{r^2} \quad \text{for } (r > b)}$$

Problem 779. problems-1/gravitation-pr-tides-moe-and-joe.tex



Moe and Joe, who have identical masses m, are in a circular orbit around a ***black hole*** about the size of a marble, which contains roughly the same mass M as the earth, in the orientation shown above. The radius of the orbit of their center of mass is R (which we'll assume is much larger than the BH). Moe and Joe and tied with a very strong rope $2d$ meters long (with $d \ll R$) that keeps them moving around the Black Hole at the *same* angular speed as their center of mass. Alas, this means that neither Joe nor Moe are actually in orbit (free fall) so the rope has to exert a force to keep them moving with their center of mass. Find:

a) The speed v_c of their center of mass in the circular orbit, as well as its angular speed ω_c, as a function of G, M, and R. This is just an ordinary circular orbit problem, don't make it overcomplicated.

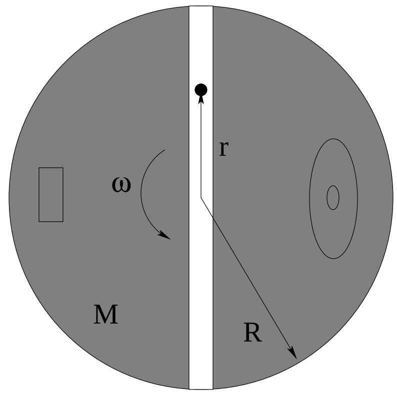
b) If Joe (closer to the BH) is moving in a circular trajectory with radius $R-d$ and the *same* angular velocity that you obtained in a) as the orbital angular velocity correct for radius R, what is the *net* force that must be exerted on Joe by the BH and the rope together?

c) What is the force exerted on Joe by the BH alone at this radius?

d) Therefore, what must the tension T be in the rope (still as a function of G, M, m, R and d)?

This "force" (opposed by the tension T) is the ***tide***.

Problem 780. problems-1/gravitation-pr-tunnel-through-death-star-orbit.tex



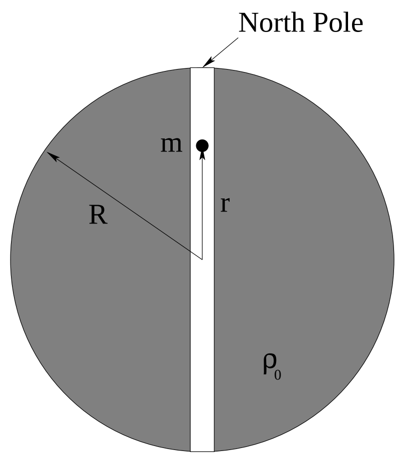
A straight, smooth (frictionless) transit tunnel is dug through a spherical asteroid of radius R and mass M that has been converted into Darth Vader's *death star*. The tunnel is in the equatorial plane and passes through the center of the death star. The death star moves about in a hard vacuum, of course, and the tunnel is open so there are no drag forces acting on masses moving through it.

a) Find the force acting on a car of mass m a distance $r < R$ from the center of the death star.

b) You are commanded to find the precise rotational frequency of the death star ω such that objects in the tunnel will orbit *at* that frequency and hence will appear to *remain at rest* relative to the tunnel at any point along it. That way Darth can Use the Dark Side to move himself along it almost without straining his midichlorians. In the meantime, he is reaching his crooked fingers towards you and you feel a choking sensation, so better start to work.

c) Which of Kepler's laws does your orbit satisfy, and why?

Problem 781. problems-1/gravitation-pr-tunnel-through-planet-oscillator.tex



A straight, smooth (frictionless) transit tunnel is dug through a planet of radius R whose mass density ρ_0 is constant. The tunnel passes through the center of the planet and is lined up with its axis of rotation (so that the planet's rotation is **irrelevant** to this problem). All the air is evacuated from the tunnel to eliminate drag forces.

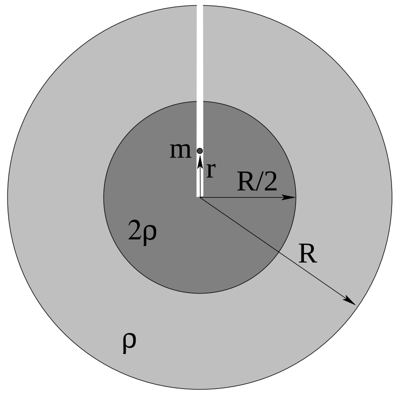
a) Find the force acting on a car of mass m a distance $r < R$ from the center of the planet.

b) Write Newton's second law for the car, and extract the differential equation of motion. From this find $r(t)$ for the car, assuming that it starts at $r_0 = R$ on the North Pole at time $t = 0$.

c) How long does it take the car to get to the center of the planet starting from rest at the North Pole? How long does it take if one starts half way down to the center? Comment.

All answers should be given in terms of G, ρ_0, R and m (or in terms of quantities you've already defined in terms of these quantities, such as ω).

Problem 782. problems-1/gravitation-pr-two-densities-difficult.tex

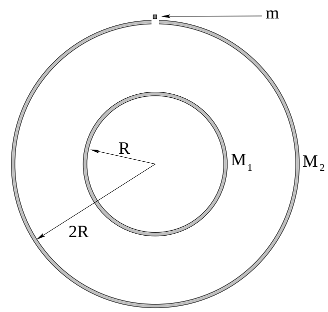


In the figure above a spherical planet of total radius R is shown that has a spherical iron core with radius $R/2$ and density 2ρ surrounded by a (liquid) rock mantle with density ρ.

a) Find the gravitational field $\vec{g}(r)$ as a function of the distance from the center.

b) Suppose a small, well-insulated tunnel were drilled all the way to the center. How much work is required to lift a mass m from the center to the surface?

Problem 783. problems-1/gravitation-pr-two-spherical-shells.tex



A hollow spherical mass shell of mass M_1 and radius R is inside another hollow spherical mass shell of mass M_2 and radius $2R$. The shells are concentric and of negligible thickness.

a) A small mass m is placed on the outer surface of the bigger shell M_2. Calculate its acceleration due to gravity g_2 in terms of the shell masses M_1 and M_2, G and R.

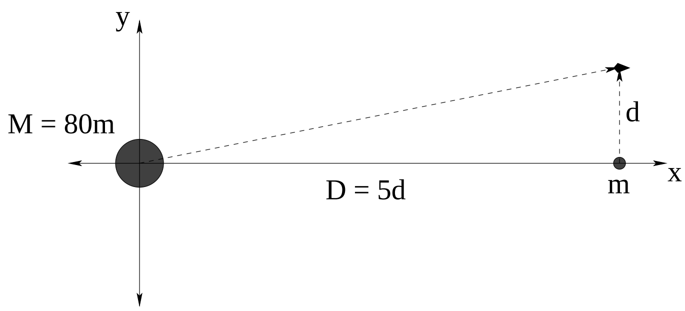
b) The mass m is placed on the outer surface of the smaller shell M_1. Its acceleration due to gravity g_1 is measured and found to be the same as the value of g_2 from part (a). Use this equality of g_1 and g_2 to express M_2 in terms of M_1, G and R.

c) With the relationship you have just derived between M_1 and M_2, compute the gravitational potential energy P_1 of a mass m on the outer surface of the bigger shell. Express P_1 in terms of G, m, M_1 and R, using the convention that the gravitation potential is defined as zero at infinite radius.

d) Compute the *change* in gravitational potential energy ΔP as the mass m moves from its position on the outer surface of M_2 to a position on the outer surface of of M_1 (being lowered through the small hole in the outer shell). Is the potential energy larger (more positive) at R or $2R$?

e) If an object is *dropped from rest* through the hole in the bigger shell, what is its speed when it hits the smaller shell? You may give this answer in terms of ΔP so that you can get it right even if you get (d) wrong.

Problem 784. problems-1/gravitation-pr-vector-field-two-masses.tex



The large mass above is the Earth, the smaller mass the Moon. Find an expression for the **vector gravitational field** acting on the spaceship on its way from Earth to Mars (swinging past the Moon at the instant drawn) in the picture above, in terms of M, m and d. Remember, magnitude *and* direction!