Duke University
Department of Physics

Physics 391 Fall Term 2010

WUN2K FOR LECTURE 4

These are notes summarizing the main concepts you need to understand and be able to apply.

- The dimensionless "correlation coefficient" for variables u and v is defined as $\rho \equiv \frac{\sigma_{uv}^2}{\sigma_u \sigma_v}$. It can take values $-1 \le \rho \le 1$; if the variables are completely uncorrelated, $\rho = 0$; $\rho = \pm 1$ corresponds to complete (linear) correlation (or anti-correlation).

- The "error matrix" is a compact way of specifying uncertainties and correlations for multiple variables $(u_1, u_2, ...)$; the elements of the error matrix are $M_{ij} \equiv \langle (u_i - \bar{u}_i)(u_j - \bar{u}_j) \rangle$. We have for $(u, v, w, ...)$, $M = \begin{bmatrix} \sigma_u^2 & \sigma_{uv}^2 & \sigma_{uw}^2 & \cdots \\ \sigma_{uv}^2 & \sigma_v^2 & \sigma_{vw}^2 & \cdots \\ \sigma_{uw}^2 & \sigma_{vw}^2 & \sigma_w^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$.
  The error matrix is symmetric, and for uncorrelated variables, it's diagonal.

- A "binormal" joint probability for two random variables u, v (with zero mean) can be written $P(u, v) = \frac{1}{2\pi\sigma_u\sigma_v}\frac{1}{\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left[\frac{1}{1-\rho^2}\left(\frac{u^2}{\sigma_u^2} + \frac{v^2}{\sigma_v^2} - 2\frac{\rho u v}{\sigma_u \sigma_v}\right)\right]}$. More generally for k variables with non-zero means, this can be written $P(u_1, u_2, ...) = \frac{1}{(2\pi)^{k/2}}\frac{1}{|M|^{1/2}} e^{-\frac{1}{2}(\mathbf{u}-\mu)^T \mathbf{M}^{-1}(\mathbf{u}-\mu)}$, where $\mathbf{M}$ is the error matrix, $\mathbf{u}$ is a column vector of variables, and μ is a column vector of their means.

- The "error ellipse" corresponding to two random variables u and v with binormal joint probability $P(u, v)$ is a contour in u-v space for which the joint probability $P(u, v)$ is $1/\sqrt{e}$ of its maximum. This ellipse will be tilted for non-zero ρ.

- Uses of error matrices (which in general make calculations by computer easy):
  - For usual error propagation, to find the uncertainty on a function of some variables $x = f(u, v, ...)$, we can write $\sigma_x^2 = \mathbf{D}^T\mathbf{M}\mathbf{D}$, where $\mathbf{D}^T$ is a row vector of derivatives, $\mathbf{D}^T = [\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, ...]$, and $\mathbf{M}$ is the error matrix.
  - For a change of variables, $u' = f_1(u, v, ..), v' = f_2(u, v, ..)$, the error matrix for the transformed variables is given by $\mathbf{M}' = \mathbf{A^T}\mathbf{M}\mathbf{A}$, where $A = \begin{bmatrix} \frac{\partial u'}{\partial u} & \frac{\partial v'}{\partial u} & \cdots \\ \frac{\partial u'}{\partial v} & \frac{\partial v'}{\partial v} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$.
  - For a function of changed variables, $z = g(u', v', ...)$, we can find the uncertainty on z according to $\sigma_z^2 = \mathbf{D}^T\mathbf{A}^T\mathbf{M}\mathbf{A}\mathbf{D}$, where in this case $\mathbf{D} = \begin{pmatrix} \frac{\partial z}{\partial u'} \\ \frac{\partial z}{\partial v'} \\ \vdots \end{pmatrix}$.