Physics 315, Problem Set 9

1. In class we used space-time translation to obtain the conserved stress-energy tensor. Now you will obtain the generators of Lorentz transformations.

   Under a Lorentz transformation, a four-vector x^μ becomes $\Lambda^\mu{}_\nu x^\nu$. Summation is implied, μ and ν are space-time indices 0,1,2,3, and Λ satisfies

   $$g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta} \quad ,$$

   where $g_{\mu\nu}$ is the space-time metric. A real scalar field $\phi(x)$ behaves in the following way under a Lorentz transformation:

   $$\phi(x) \to \phi(\Lambda x)$$

   (a) Show that for an infinitesimal transformation $\Lambda^\mu_\alpha = \delta^\mu_\alpha + \epsilon^\mu_\alpha$ (terms in ϵ are small),

   $$\epsilon_{\alpha\beta} + \epsilon_{\beta\alpha} = 0 \quad .$$

   How many independent components does $\epsilon_{\alpha\beta}$ possess?

   (b) What is the infinitesimal change in $\phi(x)$ when

   $$\phi(x) \to \phi(\Lambda x) \sim \phi(x^\mu + \epsilon^{\mu\nu} x_\nu) \quad ?$$

   (c) The Lagrange density is itself a scalar,

   $$\mathcal{L}(x) \to \mathcal{L}(\Lambda x) \sim \mathcal{L}(x^\mu + \epsilon^{\mu\nu} x_\nu) \quad .$$

   Use this information, along with the antisymmetry of $\epsilon_{\mu\nu}$ and Noether's theorem to show that the conserved current is

   $$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - \epsilon^{\mu\nu} x_\nu \mathcal{L} \quad .$$

   (d) Show that this can be re-written as

   $$J^\mu = \epsilon_{\alpha\beta} x^\beta T^{\mu\alpha}$$

   and find $T^{\mu\alpha}$.

   (e) Show that there are six conserved currents $M^{\mu\alpha\beta} = x^\beta T^{\mu\alpha} - x^\alpha T^{\mu\beta}$, where $\partial_\mu M^{\mu\alpha\beta} = 0$. Show that $T^{\alpha\beta}$ is symmetric.

   (f) The associated conserved quantities are therefore

   $$\mathcal{J}^{\alpha\beta} = \int d^3x \left(x^\beta T^{\alpha 0} - x^\alpha T^{\beta 0} \right) \quad .$$

   Show that the $\mathcal{J}^{ij}$, where $i, j = 1, 2, 3$, generate rotations by considering $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$. Find the associated expression for $\mathcal{J}^{ij}$, then find the commutator between $\mathcal{J}^{ij}$ (which is a function of time) and $\phi(\vec{x}, t)$, recalling that $[\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\delta^3(\vec{x} - \vec{x}')$. Finally, calculate

   $$\exp[i/2 \; \epsilon_{ij}\mathcal{J}^{ij}] \; \phi(\vec{x}, t) \; \exp[-i/2 \; \epsilon_{ij}\mathcal{J}^{ij}]$$

   to leading order in ϵ and interpret.

(g) Interpret $\mathcal{J}^{0i}$ by finding its expression for $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2$, evaluating $\int d^3x T^{i0}$ by plugging in the modal expansions for $\pi(x)$ and $\phi(x)$, showing that you get something proportional to $\vec{k}$ times a number operator (summed over all $\vec{k}$'s; ignoring the infinite constant this is just the momentum operator) and explain the result.

2. The Lagrange density for free photons (which we will derive later) is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad ,$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

(a) Find the Euler-Lagrange equations appropriate for this $\mathcal{L}$.

(b) Show that the definition of $F^{\mu\nu}$ gives the expected relationship between electric and magnetic fields and the four-vector potential A_μ.

(c) Apply the Euler-Lagrange equations to $\mathcal{L}$ to find the inhomogenous source free Maxwell equations.