

Physics 315, Problem Set 7

1. This problem will follow the procedure of the $K \rightarrow \pi \mu \bar{\nu}_\mu$ SU(2) prediction we did in class. Only now you will look at a case involving SU(3). The Large Hadron Collider (LHC) at CERN has as part of its mission the search for new particles involving b -quarks. It is expected that particles containing two b quarks may exist at the energies the LHC can access. Fortunately for us, the b quark transforms as a singlet under flavor SU(3). The up, down, and strange quarks transform as a triplet:

$$[3]^i = \begin{pmatrix} u \\ d \\ s \end{pmatrix} .$$

So the particles with quark quantum numbers bbu , bbd , and bbs , also transform as a triplet:

$$[3_{bb}]^i = \begin{pmatrix} bbu \\ bbd \\ bbs \end{pmatrix} .$$

Particle Data Group nomenclature dictates that these be called Ξ_{bb}^0 , Ξ_{bb}^- , and Ω_{bb}^- , respectively. The superscript stands for the total charge of the particle, and the subscript denotes its double- b status. The Ξ symbol is pronounced “cascade” in the particle world. We will consider the decay path via the weak interaction that transforms as an octet under flavor SU(3): the operators $(b\bar{c})(u\bar{d}) = H(8)_2^1$ and $(b\bar{c})(u\bar{s}) = H(8)_3^1$. (Information regarding the Dirac structure and the W boson intermediate state are suppressed.) The size of these operators appears in the ratio $H(8)_3^1/H(8)_2^1 \sim \lambda \sim 0.23$. (This arises from the ratio of CKM matrix elements that was briefly mentioned in class.) We will consider decays to a 6 of SU(3), involving only one b quark.

- (a) Extract the 6 from $3 \otimes 3$ in terms of the up, down, and strange quarks (each 3 being a triplet of the up, down, and strange quarks). Write the following particles in terms of the relevant $[6_b]^{ij}$: $\Sigma_b^+ = buu$, $\Sigma_b^- = bdd$, $\Omega_b^- = bss$, $\Sigma_b^0 = (1/\sqrt{2})(bud + bdu)$, $\Xi_{b2}^0 = (1/\sqrt{2})(bus + bsu)$, and $\Xi_{b2}^- = (1/\sqrt{2})(bds + bsd)$. Remember that the b transforms as a singlet under flavor SU(3) and so just goes along for the ride.
- (b) We wish to consider the following decay: $[3_{bb}] \rightarrow [6_b] + D$, where D stands for the anti-triplet of charmed mesons, $D = (D^0, D^+, D_s^+) = (c\bar{u}, c\bar{d}, c\bar{s})$. Therefore, we need to calculate the matrix element

$$\langle [3_{bb}] | H(8) | [6_b] D \rangle .$$

To do this, find all the ways you can contract the indices in

$$[\bar{3}_{bb}]_i H(8)_k^j [6_b]^{lm} D_n$$

to form a singlet. ($[\bar{3}_{bb}]_i$ being the hermitian conjugate of $[3_{bb}]^i$). Each contraction provides the product of C-G coefficients that multiplies a reduced matrix element.

- (c) Form the decay amplitude using as many reduced matrix elements, c_α , as there are ways of obtaining singlets:

$$A = c_1[\text{singlet}]_1 + c_2[\text{singlet}]_2 + \cdots$$

Expand out the contractions and substitute in the associated particles to relate the decay amplitudes. As an example, you should find the following relationship between decay probabilities (recall that probability is amplitude squared):

$$\Gamma(\Xi_{bb}^0 \rightarrow \Sigma_b^- D^+) = 2 \Gamma(\Xi_{bb}^0 \rightarrow \Xi_{bb}^- D_s^+) \quad .$$

2. One way to form the generators of $SU(N)$ is to use the so-called “standard form” prescription, where the diagonal elements are given by $(H_m)_{jk} = \delta_{jk}/\sqrt{m(m+1)N}$ for $j < m+1$; $(H_m)_{jk} = -m\delta_{jk}/\sqrt{m(m+1)N}$ for $j = m+1$; $(H_m)_{jk} = 0$ for $j > m+1$, $m = 1, 2, \dots, (N-1)$ while the non-diagonal generators may be written as $(E_{ij})_{\alpha\beta} = N^{-1/2}\delta_{i\alpha}\delta_{j\beta}$, where $(i, j) = (1, 2, \dots, N)$ but $i \neq j$. Applying this to $SU(3)$, how do the generators compare to the F generators that are $1/2$ the Gell-Mann matrices? How do the non-diagonal generators compare to the ones you found from solving the root equation for $SU(3)$ on your last problem set? Now use this prescription to write down the weight vectors of the first fundamental representation of $SU(4)$. These are just the four three-component weight vectors of the diagonal generators of $SU(4)$.
3. In class we found and plotted the weight vectors for the decomposition of $3 \otimes \bar{3} = 8 \oplus 1$ of $SU(3)$ and for the $3 \otimes 3 = 6 \oplus \bar{3}$ of $SU(3)$. Now do the decomposition of $3 \otimes 3 \otimes 3$ for $SU(3)$. (Aside: These are the multiplets in which baryons such as the proton and neutron and their partner particles live, but that is just one application of the general form you will find). Find and plot the weight vectors associated with the irreducible representations you find. (Aside: the singlet you find in this decomposition can be used to find the color portion of the wavefunction of a baryon; baryons are made up of three quarks, red, green, and blue, combined to form a color singlet.) Identify the isospin multiplets you observe from the weight diagrams.
4. Find the associated wavefunctions for the states in the multiplets you found in Problem 3 above. We did most of the highest weight (the 10) in class. Note that if you interpret the fundamental triplet as the up, down, and strange quark you will recapture the isospin doublet of proton and neutron states you found on your midterm. But now you will also find the other particles that fill out that baryon octet in $SU(3)$.
5. $SU(4)$. Decompose the $4 \otimes 4$ into irreducible representations using (i) tensor methods and (ii) Young diagrams. For the Young diagrams, write the results in the “standard arrangement” so that you can count the multiplicity of the result. Label each irreducible representation using the (p_1, p_2, p_3) language and verify the multiplicity found using your standard arrangement with the formula

$$\left(\frac{p_1+1}{1}\right) \left(\frac{p_2+1}{1}\right) \left(\frac{p_3+1}{1}\right) \cdot \left(\frac{p_1+p_2+2}{2}\right) \left(\frac{p_2+p_3+2}{2}\right) \cdot \left(\frac{p_1+p_2+p_3+3}{3}\right) \quad .$$

(Note that its generalization to $SU(N)$ is clear. The reason there is no $(p_1 + p_3 + 2)$ term is because of a rule to stick to consecutive p labels in the formula.)

6. $SU(3)$: Decompose $\bar{3} \otimes \bar{3} \otimes \bar{3}$ using Young diagrams. Verify that this would have been easier to obtain by finding $3 \otimes 3 \otimes 3$ and then conjugating everything.