Physics 315 Problem Set 5

1. Combine two irreducible SU(2) tensors, S_{ab} and w_c, and decompose the product into irreducible representations of SU(2). It will help to proceed as follows:

   (a) Recognize that S_{ab} must be symmetric under interchange of its indices and therefore that it is a triplet. Recognize that w_c is a doublet.

   (b) Note what you expect your result to be: you are decomposing $S_{ab}w_c$. This has 6 components. If you combined a triplet (spin 1) with a doublet (spin 1/2) you expect to get a four component object (spin 3/2) and a two component object (spin 1/2).

   (c) It might help to remind yourself of what the C-G coefficients are which relate this "individual basis" $|l = 1, m_l; s = 1/2, m_s\rangle$ (where $m_l = -1, 0, +1$ and $m_s = -1/2, +1/2$) to the "combined basis" $|j = 3/2, m_j\rangle$ (where $m_j = 3/2, 1/2, -1/2. - 3/2$) and $|j = 1/2, m_j\rangle$ (where $m_j = -1/2, +1/2$).

   (d) You know that you need to find a tensor containing the available three indices which is completely symmetric under the interchange of these indices. We will call this object T_{abc}. Convince yourself that it has four independent components. This, as you suspect from looking at the C-G coefficients above, will be the j=3/2 irreducible representation.

   (e) There are only three combinations available if you use the symmetry of S_{ab}. So write

   $$S_{ab}w_c = A\left(S_{ab}w_c + S_{ac}w_b + S_{bc}w_a\right) + BS_{ab}w_c + CS_{ac}w_b + DS_{bc}w_a \quad .$$

   The first (symmetric) term will become your T_{abc} once you have found the coefficient A. The coefficients B, C, and D will be found by imposing consistency.

   (f) You certainly know that what will become your second term, which will be your expression for the j=1/2 irreducible representation, is *not* completely symmetric under interchange of indices. Therefore, you expect that if $a = b = c$ this term should vanish.

   (g) With the above conditions imposed, along with matching up the left hand side with the right hand side, find the coefficients A, B, C, and D.

2. Find expressions for M_k and J_k, $k = 1, 2, 3$ so that $i\omega_{ok}M_k + \frac{i}{2}\epsilon_{klm}\omega_{lm}J_k$ is the most general matrix ω consistent with $\Lambda^T g\Lambda = g$, where $\Lambda = 1 + \omega + \cdots$.

3. Verify by calculating commutations relations that the new basis $J_i^{(+)} = \frac{1}{2}(J_i + iM_i)$ and $J_i^{(-)} = \frac{1}{2}(J_i - iM_i)$, $i = 1, 2, 3$ form two commuting SU(2)s.

4. In class we argued that $D^{(1/2,1/2)} \otimes D^{(1/2,1/2)} = D^{(0,0)} \oplus D^{(0,1)} \oplus D^{(1,0)} \oplus D^{(1,1)}$. Use tensor methods to show that the decomposition $a_i b_j c_k d_l$, $(i, j, k, l = 1, 2)$ gives the same result.

5. These problems are out of W.K. Tung, which is on reserve in the library. You don't need to refer to that text, but you might find it helpful to read sections 10.1-10.4. Note that Tung's metric is different than the one we have been using in class.

   (a) Tung, problem 10.5. Show that an anti-symmetric second-rank tensor in Minkowski space transforms as the $(1, 0) \oplus (0, 1)$ representation of the Lorentz group.

   (b) Tung, problem 10.6. (i) Show that the trace of a second-rank tensor t^μ_μ is invariant under all Lorentz transformations, so that it transforms as the (0,0) representation. (ii) Show that the traceless symmetric tensor $t^{\{\mu,\nu\}} - g^{\mu\nu}t^\lambda_\lambda/4$ transforms irreducibly under Lorentz transformations as the $(1, 1)$ representation.

6. CP transformation. It is a fundamental property of field theory that the product of charge conjugation C, parity P, and time reversal T transformations is a symmetry of nature. (It is still possible that this will be experimentally refuted and we will have to come up with another theory to describe physics. We will ignore that possibility for this problem.) Therefore, if T is violated, so must CP, the product of charge conjugation and parity operations. Neutral kaons exhibit this symmetry violation. The valence quantum numbers are as follows: $|K^0\rangle \sim |\overline{s}d\rangle$ and $|\overline{K}^0\rangle \sim |s\overline{d}\rangle$, where $s(\overline{s})$ is the strange (anti-strange) quark and $d(\overline{d})$ is the down (anti-down) quark. Charge conjugation, C, takes particles to antipariticles, with possibly some phase factor. $|K^0\rangle$ and $|\overline{K}^0\rangle$ are called antiparticles of each other because C takes one into the other.

(a) You are given that

$$CP|K^0\rangle = -|\overline{K}^0\rangle$$
$$CP|\overline{K}^0\rangle = -|K^0\rangle$$

Neutral kaons are subject to strong interactions through H_{QCD} and weak interactions through H_{weak}. CPT invarience requires that the following "masses" be equal:

$$m_{K^0} = \langle K^0|H_{QCD}|K^0\rangle$$
$$m_{\overline{K}^0} = \langle \overline{K}^0|H_{QCD}|\overline{K}^0\rangle$$

Conservation of the strange quantum number by QCD interactions yields $\langle K^0|H_{QCD}|\overline{K}^0\rangle$ =0. The (much smaller) weak interaction allows both decay of the neutral kaons and for the antiparticles to "oscillate" into each other. Because of this, the Hamiltonian involving the neutral kaons alone is not hermitian. We write (ignoring the EM interaction at this time),

$$H = H_{QCD} + H_{weak} = M - i\Gamma \quad .$$

Conservation of probability yields

$$\frac{d}{dt}|\psi(t)|^2 + 2\pi \sum_i \rho(\phi_i)|\langle\phi_i|H_{weak}|\psi(t)\rangle|^2 = 0 \quad ,$$

where ϕ_i are the possible final states into which neutral kaons may decay, and $\rho(\phi_i)$ is the density of final states. Using the above, CPT invariance, and the fact that H_{weak} matrix elements are small corrections, diagonalize H to find the physical states in terms of the relevant matrix elements. Note that these states will not be orthogonal. Show that the physical state may be written as

$$|K_L^0\rangle = \frac{|K_{-1}\rangle + \epsilon|K_1\rangle}{(1+|\epsilon|^2)^{1/2}}$$
$$|K_s^0\rangle = \frac{|K_1\rangle + \epsilon|K_{-1}\rangle}{(1+|\epsilon|^2)^{1/2}} \quad ,$$

where ϵ is small (you don't have to show that), measured to be around 10^{-3}. The L and S designations on the kaon states stand for "long-lived" and "short-lived." $K_{\pm 1}$ are the eigenvectors of the CP operator.

(b) The kaons decay into pions (in addition to oscillating into each other via $|K^0\rangle \leftrightarrow |\overline{K}^0\rangle$). You have analyzed the pions, their quark structure, and the possible electrically neutral two-pion states in problem set 4. Pions are spin-0, isospin-1 objects. Therefore, they obey Bose statistics and any legal wavefunction involving pions must be appropriately symmetrized. If CP were conserved, then the CP=1 kaon eigenstate, $|K_1^0\rangle$, would decay only to the CP=1 eigenstate of pions: $|\pi^+\pi^-\rangle$, which is really $\frac{1}{\sqrt{2}}(|\pi_1^+\pi_2^-\rangle + |\pi_1^-\pi_2^+\rangle)$, where we have given subscripts to the pions for purposes of symmetrizing them; and $|\pi^0\pi^0\rangle$. The CP=–1 eigenstate, $|K_{-1}^0\rangle$, would decay only to a three pion state (and other CP=–1 final states we won't talk about) were CP conserved. Experimentally, however, it is observed that the kaon particles violate CP. You found above that the actual physical particles are linear combinations of the CP eigenstates. Experimentally, one observes that the long-lived kaons, which should decay mostly to three-pion final states, actually sometimes decay to two-pion final states; this provides a measure for the amount of CP violation (or $|K_1\rangle$ admixture) in $|K_L^0\rangle$. Final states are characterized by their total isospin. If the $I = 0$ channel dominates (it does – this is called the "$\Delta I = 1/2$ rule" (why is it called this?) and nobody knows why it is true), find the experimental observable

$$\frac{\langle\pi^0\pi^0|H_{weak}|K_L^0\rangle}{\langle\pi^0\pi^0|H_{weak}|K_S^0\rangle}$$

to first order in ϵ. You will need to invoke CPT invariance again to simplify the result. (We are neglecting the small effect of final state interaction phase shifts here.)