Physics 315: Problem Set 4

1. Merzbacher Exercise 17.27
Show that the tensor operators $S^q_k = (-1)^q T^q_{-q} - T^q_{q}$ and $T^q_k$ transform the same way under rotation. Prove that $\langle \alpha' j | S_k | \alpha j \rangle = \langle \alpha j | T_k | \alpha' j \rangle^*$ and deduce the identity $\langle jkm|jkjm' \rangle = (-1)^q \langle jkm' - q|jkjm \rangle$. [Note the double line notation to indicate a reduced matrix element.]

2. adapted from Sakurai Ch. 3 Problem 25:
Construct a spherical tensor of rank 1 [$k=1$] and a spherical tensor of rank 2 [$k=2$] out of two different vectors $\vec{U} = (U_x, U_y, U_z)$ and $\vec{V} = (V_x, V_y, V_z)$. Explicitly write $T^{(1)}_{1} \pm 1, 0$ and $T^{(2)}_{2} \pm 2, \pm 1, 0$ in terms of $U_x, U_y, U_z$ and $V_x, V_y, V_z$. What else is needed to completely decompose the $U_i V_j$ product?

3. Sakurai Ch. 3 Problem 28:
(a) Write $xy$, $xz$, and $(x^2 - y^2)$ as components of a spherical (irreducible) tensor of rank 2.
(b) The expectation value $Q = e \langle n, j, m = j | (3z^2 - r^2) | n, j, m = j \rangle$ is known as the quadrupole moment. Evaluate $e \langle n, j, m' = j, j - 1, j - 2, \ldots, m = j | (x^2 - y^2) | n, j, m = j \rangle$, where $m' = j, j - 1, j - 2, \ldots$, in terms of $Q$ and appropriate Clebsch-Gordan coefficients.

4. We argued that a single-indexed object $u_i$, where $i=1,2$, generates the 2x2 representation of SU(2), which we wrote in its explicit most general form in class in terms of the $\alpha, \beta$, and $\gamma$ parameters. Call this most general group element $\Sigma$. $u_i$ transforms via $\Sigma$ while $u'$ transforms as its conjugate, $\Sigma^*$. In what follows, the summation convention is implied.

(a) Review (we did this in class) by explicit calculation using the $\Sigma$ transformation matrices that the Levi-Civita tensor remains invariant:
$$\Sigma_{ac} \Sigma_{bd} \epsilon_{ab} = \epsilon_{cd}. $$
(b) Show by explicit calculation using the $\Sigma$ matrices that the naive Kronecker-delta does not remain invariant under SU(2) transformations:
$$\Sigma_{ac} \Sigma_{bd} \delta_{ab} \neq \delta_{cd}. $$
(c) Now show that the Kronecker delta $\delta^i_j$ is invariant:
$$\Sigma_{ac} (\Sigma^*)^{bd} \delta^j_b = \delta^i_c.$$ 

5. Pions are spin-0 (but isospin-1) objects. There are three of them: $\pi^+, \pi^0$, and $\pi^-$. The superscript indicates the electric charge of the particle. Their valence quantum numbers are $|\pi^+\rangle \sim |ud\rangle$, $|\pi^0\rangle \sim \frac{1}{\sqrt{2}}(|u\bar{d}\rangle - |d\bar{u}\rangle)$, and $|\pi^-\rangle \sim |d\bar{u}\rangle$. They are obtained by combining two isospin-1/2 objects to form a triplet of isospin-1 objects (the pions) and an isospin-0 object (the $\eta$).

(a) First, confirm the isospin triplet of pions assignment given above. That is, begin with the doublet
$$\begin{pmatrix} u \\ d \end{pmatrix},$$
where the up quark $u$ is categorized by its isospin state $|I, I_3\rangle = |1/2, 1/2\rangle$ and the down quark $d$ is categorized by its isospin state $|I, I_3\rangle = |1/2, -1/2\rangle$. Recall that the antiquarks $\bar{\pi}$ and $\bar{d}$ have opposite
electric charge and opposite $I_3$ quantum number from their quark counterparts. Therefore, $\bar{u}$ has the isospin quantum numbers $|I, I_3⟩ = |1/2, -1/2⟩$ and $\bar{d}$ has the isospin quantum numbers $|I, I_3⟩ = |1/2, 1/2⟩$. The doublet $\begin{pmatrix} u \\ d \end{pmatrix}$ transforms as a 2 under SU(2) (multiplicity=2 · $(I = 1/2) + 1$). $(\bar{u} \ d)$ on the other hand, transforms like a $\bar{2}$ under SU(2). However, you just showed that the Levi-Civita tensor is invariant under an SU(2) rotation, so it can be used to change the $\bar{2}$ objects to 2 objects: If $a_i$ ($i = 1, 2$) indicates the components of $(\bar{u} \ d)$, find $b^j = \epsilon^{jk} a_k$, summation implied. So you have just found how to express the antiquarks as a 2. (The epsilon tensor behaves as a raising or lowering operator in SU(2) the way the metric tensor $g_{\mu\nu}$ is used to raise or lower Lorentz indices.) You know how to combine $2 \otimes 2$ from your spin work, so do that in order to write the triplet of pions, $|I, I_3⟩ = |1, 1⟩, |1, 0⟩$, and $|1, -1⟩$ in terms of the quark basis. How do the expressions compare to the triplet you obtain by combining two (regular) spin-1/2 objects?

(b) Show that all possible electrically neutral two-pion final states have total isospin $I = 0$ or 2 and $I_3 = 0$. Express these states in terms of the single pion basis.