Physics 315: Problem Set 4

1. Merzbacher Exercise 17.27

   Show that the tensor operators $S_k^q = (-1)^q T_k^{-q\dagger}$ and T_k^q transform the same way under rotation. Prove that $\langle \alpha' j || S_k || \alpha j \rangle = \langle \alpha j || T_k || \alpha' j \rangle^*$ and deduce the identity $\langle jkmq | jkjm' \rangle = (-1)^q \langle jkm' \, -q | jkjm \rangle$. [Note the double line notation to indicate a reduced matrix element.]

2. adapted from Sakurai Ch. 3 Problem 25:

   Construct a spherical tensor of rank 1 [k=1] and a spherical tensor of rank 2 [k=2] out of two different vectors $\vec{U} = (U_x, U_y, U_z)$ and $\vec{V} = (V_x, V_y, V_z)$. Explicitly write $T^{(1)}_{\pm 1,0}$ and $T^{(2)}_{\pm 2, \pm 1, 0}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$. What else is needed to completely decompose the $U_i V_j$ product?

3. Sakurai Ch. 3 Problem 28:

   (a) Write xy, xz, and $(x^2 - y^2)$ as components of a spherical (irreducible) tensor of rank 2.

   (b) The expectation value

   $$Q = e\langle n, j, m = j | (3z^2 - r^2) | n, j, m = j \rangle \quad .$$

   is known as the *quadrupole moment*. Evaluate

   $$e\langle n, j, m' | (x^2 - y^2) | n, j, m = j \rangle \quad ,$$

   where $m' = j, j-1, j-2, ...$, in terms of Q and appropriate Clebsch-Gordan coefficients.

4. We argued that a single-indexed object u_i, where i=1,2, generates the 2x2 representation of SU(2), which we wrote in its explicit most general form in class in terms of the α, β, and γ parameters. Call this most general group element Σ. u_i transforms via Σ while u^i transforms as its conjugate, Σ^*. In what follows, the summation convention is implied.

   (a) Review (we did this in class) by explicit calculation using the Σ transformation matrices that the Levi-Civita tensor remains invariant:

   $$\Sigma_{ac}\Sigma_{bd}\varepsilon_{ab} = \varepsilon_{cd} \quad .$$

   (b) Show by explicit calculation using the Σ matrices that the naive Kronecker-delta does *not* remain invariant under SU(2) tranformations:

   $$\Sigma_{ac}\Sigma_{bd}\delta_{ab} \neq \delta_{cd} \quad .$$

   (c) Now show that the Kronecker delta δ^i_j is invariant:

   $$\Sigma_{ac}(\Sigma^*)^{bd}\delta^b_a = \delta^d_c \quad .$$

5. Pions are spin-0 (but isospin-1) objects. There are three of them: π^+, π^0, and π^-. The superscript indicates the electric charge of the particle. Their valence quantum numbers are $|\pi^+\rangle \sim |u\overline{d}\rangle$, $|\pi^0\rangle \sim \frac{1}{\sqrt{2}}(|u\overline{u}\rangle - |d\overline{d}\rangle)$, and $|\pi^-\rangle \sim |d\overline{u}\rangle$. They are obtained by combining two isospin-1/2 objects to form a triplet of isospin-1 objects (the pions) and an isospin-0 object (the η).

   (a) First, confirm the isospin triplet of pions assignment given above. That is, begin with the doublet

   $$\begin{pmatrix} u \\ d \end{pmatrix} \quad ,$$

   where the up quark u is categorized by its isospin state $|I, I_3\rangle = |1/2, 1/2\rangle$ and the down quark d is categorized by its isospin state $|I, I_3\rangle = |1/2, -1/2\rangle$. Recall that the antiquarks $\overline{u}$ and $\overline{d}$ have opposite electric charge and opposite I_3 quantum number from their quark counterparts. Therefore, $\overline{u}$ has the isospin quantum numbers $|I, I_3\rangle = |1/2, -1/2\rangle$ and $\overline{d}$ has the isospin quantum numbers $|I, I_3\rangle = |1/2, 1/2\rangle$. The doublet $\begin{pmatrix} u \\ d \end{pmatrix}$ transforms as a 2 under SU(2) (multiplicity=$2 \cdot (I = 1/2) + 1$). $\begin{pmatrix} \overline{u} & \overline{d} \end{pmatrix}$ on the other hand, transforms like a $\overline{2}$ under SU(2). However, you just showed that the Levi-Civita tensor is invariant under an SU(2) rotation, so it can be used to change the $\overline{2}$ objects to 2 objects: If a_i $(i = 1, 2)$ indicates the components of $\begin{pmatrix} \overline{u} & \overline{d} \end{pmatrix}$, find $b^j = \epsilon^{jk} a_k$, summation implied. So you have just found how to express the antiquarks as a 2. (The epsilon tensor behaves as a raising or lowering operator in SU(2) the way the metric tensor $g_{\mu\nu}$ is used to raise or lower Lorentz indices.) You know how to combine $2 \otimes 2$ from your spin work, so do that in order to write the triplet of pions, $|I, I_3\rangle = |1, 1\rangle, |1, 0\rangle$, and $|1, -1\rangle$ in terms of the quark basis. How do the expressions compare to the triplet you obtain by combining two (regular) spin-1/2 objects?

(b) Show that all possible electrically neutral two-pion final states have total isospin $I = 0$ or 2 and $I_3 = 0$. Express these states in terms of the single pion basis.