Physics 315: Problem Set 3

1. Desai 27.6

2. Find the probability (to order ϵ^2) that a state $|j, m = m_{max} = j\rangle$ is rotated by an infinitesimal angle ϵ about the y–axis and is found unchanged.

3. Use Clebsch-Gordan coefficients to block diagonalize (decompose) the 4-dimensional $2 \otimes 2$ individual-basis operators $\vec{J}^2$, J_x, J_y, and J_z to their 4-dimensional block diagonal composite-basis form. Find the matrix which performs this block-diagonalization. What does this do to the group elements generated by these operators? Hints:

   - You may have to reorder your states in order to see the block-diagonal form.
   - $J_+ = S_{1+} + S_{2+}$, etc., where S are the spin-1/2 operators.
   - Find the operators by taking matrix elements, using for example, $\vec{J}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$.
   - $\langle m_1 m_2 | J_i | m_1' m_2' \rangle = \sum_{lm} \sum_{l'm'} \langle m_1 m_2 | lm \rangle \langle lm | J_i | l'm' \rangle \langle l'm' | m_1' m_2' \rangle$.
   - You are to find the matrix S such that $\mathcal{O}_{\text{(individual basis)}} = S\mathcal{O}_{\text{(combined basis)}}S^\dagger$, where $\mathcal{O}$ is any of the four operators above, and the same S works for all of them.

4. (from Merzbacher Exercise 17.20) As we discussed in 212, the Clebsch-Gordan coefficients take us from the individual product basis to the block-diagonal irreducible basis for the rotation matrices $D^{(j)}$:

$$D^{(j_1)}_{\mu_1 m_1}(R) D^{(j_2)}_{\mu_2 m_2}(R) = \sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_M \sum_\mu \langle j_1 j_2; m_1 m_2 | JM \rangle \langle j_1 j_2; \mu_1 \mu_2 | J\mu \rangle D^{(J)}_{\mu M}(R) \quad .$$

   Using this, the unitarity of the D matrices, and the orthogonality of the C-G, show that

$$\sum_{\mu_2} \langle j_1 j_2; \mu_1 \mu_2 | JM' \rangle D^{(j_2)}_{\mu_2 m_2}(R) = \sum_{m_1} \sum_M D^{(J)}_{M'M}(R) \langle j_1 j_2; m_1 m_2 | JM \rangle D^{(j_1)*}_{\mu_1 m_1}(R) \quad .$$

5. Gauges

   (a) Show that the Hamiltonian for a charged particle subject to a scalar potential U and a vector potential $\vec{A}$ is a different operator in different gauges. Under what circumstance would the Hamiltonian be a physical observable?

   (b) Show that the Schrodinger equation remains invariant under a gauge transformation.

   (c) Show that the probability density and current are gauge invariant.

6. Construct a non-trivial (that is, not manifestly block-diagonal) six dimensional representation of the permutation group S_3 we discussed in class. Decompose it into a sum of irreducible representations.

7. Which of the following are groups?

   (a) All real numbers, with group multiplication being ordinary multiplication.

   (b) All real numbers, with group multiplication being addition

   (c) All complex numbers except zero with group multiplication being ordinary multiplication

   (d) All positive rational numbers with group multiplication between two group elements a and b being a/b

   (e) The three Pauli matrices σ_x, σ_y, σ_z, along with the unit matrix I.

8. Consider the group of two-dimensional linear coordinate transformations:

$$\tilde{x} = ax + by + c$$
$$\tilde{y} = dx + ey + f$$

(a) What is the identity element of this group?

(b) Find the six generators of this group using the limit definition given in class.

9. Consider the group consisting of the elements a and e, where group combination yields the following:

$$ee = e \quad aa = e \quad ea = ae = a$$

(a) Verify that a and e do indeed form a group under the combination rule given.

(b) Find two one-dimensional representations of this group and one two-dimensional representation of the group. Demonstrate that the group properties are obeyed.

(c) Form the character table and find the number of character classes for the three representations you have found. Which are irreducible representations?

10. Show that the group $SU(n)$ has $n^2 - 1$ independent real parameters. Show that the group of special real orthogonal matrices $SO(n, \mathcal{R})$ has $n(n-1)/2$ independent real parameters. ($SO(n)$ matrices have their inverses equal to their transposes and determinant equal to one.)