Physics 315: Problem Set 2

1. Fun with Four Vectors

   Recall that our metric tensor is defined by $g_{\mu\nu} = g^{\mu\nu} = diag(1,-1,-1,-,1)$; the space-time variables are combined into $x^\mu = (x^0, \vec{x})$; $\partial_\mu = \frac{\partial}{\partial x^\mu}$; and the totally antisymmetric tensor is $\epsilon^{\alpha\beta\gamma\delta}$ with the convention that $\epsilon^{0123} = +1$.

   (a) Find x_μ.

   (b) Use the four vector $q^\mu = (q^0, \vec{q})$ to form a Lorentz scalar.

   (c) Show that the equation $p^\mu = i\partial^\mu$, where p^μ is the four momentum, recaptures the expected quantum mechanical energy and momentum operators.

   (d) Evaluate $i\partial_\mu e^{-ik\cdot x}$, where k and x are both four-vectors.

   (e) In a scattering process, two particles of four momentum q and l, respectively, combine to form a third particle of four momentum p. Express conservation of energy and momentum in four vector notation.

   (f) The Lagrange density for a free (no potential) spinless field theory is

   $$\mathcal{L} = \frac{1}{2}\left(\partial^\mu\phi(x)\partial_\mu\phi(x) - m^2\phi^2\right) \quad .$$

   Think of ϕ as being a real valued field dependent upon the real space-time four vector x. Using the Euler-Lagrange formula

   $$\partial^\nu \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad , \tag{1}$$

   show that the equation of motion for a free spinless field theory is $(\partial^\mu\partial_\mu + m^2)\phi = 0$. This is the Klein-Gordon Equation.

   (g) Under what condition is $e^{ik\cdot x}$ a solution of the Klein-Gordon equation?

   (h) Extra Credit: Derive the Euler-Lagrange equation of motion, Eqn. (1), above.

2. A Lorentz transformation takes a four vector x^μ into another four vector x'^μ,

   $$x'^\mu = \Lambda^\mu_\nu x^\nu \quad .$$

   Find an explicit four by four matrix Λ which represents a rotation in the xy plane. Find another explicit four by four matrix Λ which represents a boost in the x direction.

3. Which of the following are groups?

   (a) All real numbers, with group multiplication being ordinary multiplication.

   (b) All real numbers, with group multiplication being addition

   (c) All complex numbers except zero with group multiplication being ordinary multiplication

   (d) All positive rational numbers with group multiplication between two group elements a and b being a/b

   (e) The three Pauli matrices σ_x, σ_y, σ_z, along with the unit matrix I.

4. Consider the group of two-dimensional linear coordinate transformations:

   $$\tilde{x} = ax + by + c$$
   $$\tilde{y} = dx + ey + f$$

   (a) What is the identity element of this group?

   (b) Find the six generators of this group using the limit definition given in class.

5. Construct a six dimensional representation of the permutation group S_3 we discussed in class. Decompose it into a sum of irreducible representations.