Physics 315: Problem Set 2

1. Fun with Four Vectors

   Recall that our metric tensor is defined by $g_{\mu\nu} = g^{\mu\nu} = diag(1, -1, -1, -, 1)$; the space-time variables are combined into $x^\mu = (x^0, \vec{x})$; $\partial_\mu = \frac{\partial}{\partial x^\mu}$; and the totally antisymmetric tensor is $\epsilon^{\alpha\beta\gamma\delta}$ with the convention that $\epsilon^{0123} = +1$.

   (a) Find x_μ.

   (b) Use the four vector $q^\mu = (q^0, \vec{q})$ to form a Lorentz scalar.

   (c) Show that the equation $p^\mu = i\partial^\mu$, where p^μ is the four momentum, recaptures the expected quantum mechanical energy and momentum operators.

   (d) Evaluate $i\partial_\mu e^{-ik\cdot x}$, where k and x are both four-vectors.

   (e) In a scattering process, two particles of four momentum q and l, respectively, combine to form a third particle of four momentum p. Express conservation of energy and momentum in four vector notation.

   (f) A Lorentz transformation takes a four vector x^μ into another four vector x'^μ,

   $$x'^\mu = \Lambda^\mu_\nu x^\nu \quad .$$

   Find an explicit four by four matrix Λ which represents a rotation in the xy plane. Find another explicit four by four matrix Λ which represents a boost in the x direction.

2. A free (i.e., noninteracting) scalar (i.e., spin=0) field theory has the Lagrange density

   $$L = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) \quad .$$

   (a) Show, using the Euler-Lagrange equations we derived in class, that this leads to the Klein-Gordon equation,

   $$(\partial_\mu \partial^\mu + m^2)\phi = 0 \quad .$$

   (b) Under what condition is $\phi \sim e^{ik\cdot x}$ a solution to the Klein-Gordon equation? Interpret the condition physically. (Here, k and x are both four-vectors.)

3. Now look at the massless free scalar field theory (i.e., set $m = 0$.) The Lagrange density is

   $$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi \quad .$$

   The classical action of this theory is invariant under the "scale transformation"

   $$\phi'(x) = e^\alpha \phi(e^\alpha x) \quad ,$$

   where x is a four-vector and α is a positive real number.

   (a) Show that the scale transformation is indeed a symmetry of the action

   (b) Find the conserved current and the conserved charge associated with this symmetry.