1. Show that the relationship we obtained from imposing the condition that our atoms are in a steady state of thermal equilibrium at temperature $T$ and the applied radiation field is also in thermal equilibrium at temperature $T$:

$$\frac{B_{ij} \frac{\hbar \omega^3}{\pi c^3} \frac{1}{e^{\hbar \omega/kT} - 1}}{A_{ji} + B_{ji} \frac{\hbar \omega^3}{\pi c^3} \frac{1}{e^{\hbar \omega/kT} - 1}} = \frac{g_j e^{-\hbar \omega/kT}}{g_i}$$

requires that

$$g_i B_{ij} = g_j B_{ji} \text{ and } A_{ji} = \frac{\hbar \omega^3}{\pi c^3} B_{ji}$$

if the Einstein coefficients are independent of temperature.

2. Show that the number of photon states per unit volume per unit frequency range for radiation confined to a box is $\frac{\omega^2}{\pi c^2}$.

3. Obtain the lifetime of the $n=2$, $l=1$ level of hydrogen against spontaneous decay by dipole radiation.

4. Show (I already sketched this in class; you need to justify the steps) that the ratio of the typical size of a magnetic dipole matrix element to the typical size of an electric dipole matrix element is $Z\alpha$, for a hydrogenic atom with $Z$ protons. Find the ratio of a typical quadrupole matrix element to a typical electric dipole matrix element. What does that tell you about the relative size of the transition rates?

5. Rate of energy loss

In class we found an expression for the rate of energy loss by spontaneous emission of unpolarized dipole radiation of frequency $\omega_{ji}$ in terms of $A_{ji}$, the Einstein coefficient for spontaneous decay from level $j$ to level $i$. Now compare this to the case of a classical electron oscillator:

(a) Consider the one dimensional equation of motion

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0 ,$$

where $\omega_0$ is the oscillation frequency and choose

$$\gamma = \frac{1}{4\pi\epsilon_0} \frac{2 e^2 \omega_0^2}{3 mc^3} .$$

Show that $\gamma$ is small compared to $\omega_0$ for atomic energies.

(b) Find a solution which is valid in this limit.

(c) Calculate the approximate rate of energy loss.

(d) Identify the classical analog of the dipole matrix element; that is, comparing your result to our expression from class, what plays the role of the dipole matrix element?

6. The possibility that spontaneous emission may occur broadens the energy of an excited state. The state has a lifetime $\tau$ and through the HUP, $\Delta E \Delta t \sim \hbar$, its energy is no longer exactly known. At $t=0$ $N(0)$ atoms are placed in an excited state with energy $E_X$.

(a) Given the Einstein coefficient, $A$, for spontaneous emission, write the differential equation satisfied by $N(t)$, the number of particles existing in the excited state at time $t$. Solve this equation. If the probability of finding the atom in the excited state, with wavefunction $\psi_X$, is unity at $t=0$, what is the probability of finding the atom in the state $\psi_X$ at some later time $t$?

(b) If $\psi_X$ is a stationary state in the absence of spontaneous emission, then in the presence of spontaneous emission we may write $\Psi(\vec{r},t) = c_X(t) \psi_X(\vec{r}) + \text{other states}$. Using your result above, what is $c_X(t)$?
(c) Now find the width of the excited state by taking the Fourier transform of your result above:

\[ g(\omega) \sim \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \]

Do not forget the initial condition that the population is zero for times less than \( t = 0 \). Sketch \( |g(\omega)|^2 \) and label the width of the resonance in terms of \( A \). What is the physical interpretation of this sketch?

(d) Calculate the typical size for this width by considering a hydrogenic atom in the excited state \( X \) spontaneously decaying to the ground state, using

\[ \langle 100|\vec{r}|X \rangle \sim \frac{a_0}{Z} \]

and \( \omega_{X1} = \omega_X - \omega_1 \sim \text{ionization energy from the ground state} \). What is the fractional frequency width (the ratio of the frequency width to \( \omega_X \))? \( (E_i = \hbar \omega_i) \).

7. Create your own problem involving an application of Fermi’s Golden Rule. Then solve it.