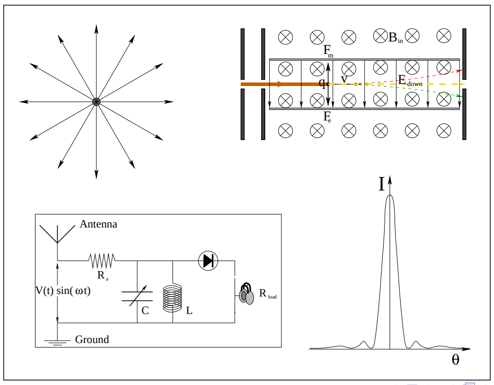
Introductory Physics 142/152/162

Self-Guided Learning Problems

Robert G. Brown

Fall, 2020



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About These Problems

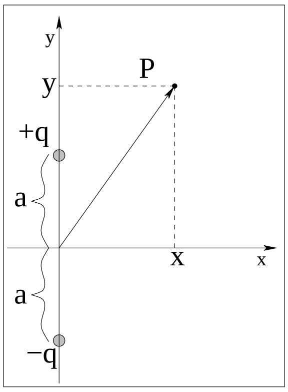
This is an experiment. Beamer allows me to make slides that will successively reveal lines of math-heavy text. This gives me a unique opportunity to build a collection of self-guided learning problems for physics that do what I've fantasized about doing for years now – present a problem, then provide a hint, then another hint, then another (or reveal a step, and then another step) until finally, the entire souution is presented, annotated.

Hopefully these problems will help students everywhere as they struggle to learn physics problems solving techniques and learn to "think like a physicist" as they do so.

To use this resource, pick a problem or topic from the table of contents and go directly to it, or work your way through all the problems systematically. Work on a separate sheet of paper, and when you get stuck, page down through the frames to see (hopefully) where you went wrong.

Remember, the point is to **master these problems**, not just to get through them. Make sure that before you are done, you can do every problem **without looking, without hints, and without remembering the exact solution** but rather, understanding *how* to find it!

E-Field of Electric Dipole at point **P**



- A simple electric dipole aligned with the y-axis is drawn to the left. We need to find the electric field at an arbitrary point in the x-y plane (a solution that, rotated around the y axis, would work everywhere in space).

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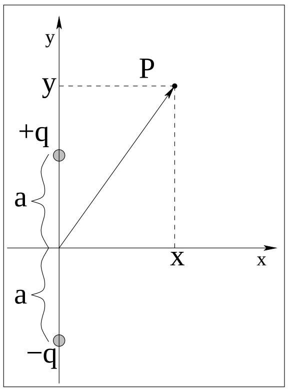

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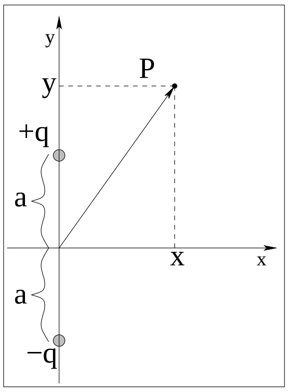
- Our solution strategy here is extremely straightforward. We start with the field of a point-charge in charge-centered coordinates and draw a vector arrow representing the fields of the specific charges in on the figure.

$\boldsymbol{E}$-Field of Electric Dipole at point $\boldsymbol{P}$



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- Next, we decompose the vectors in cartesian coordinates...

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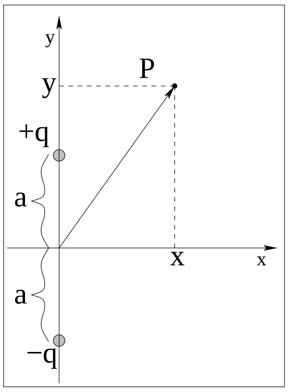

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- And add the vectors componentwise...

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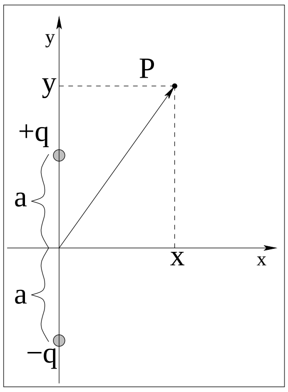

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- And admire the result!

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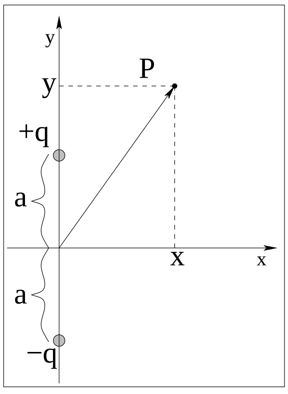

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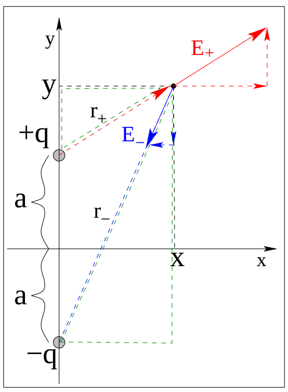
$\boldsymbol{E}$-Field of Electric Dipole at point $\boldsymbol{P}$-Solution



- I've decorated the figure to the left with:

$$r_+ = \left(x^2 + (y-a)^2\right)^{1/2} \qquad r_+ = \left(x^2 + (y+a)^2\right)^{1/2}$$

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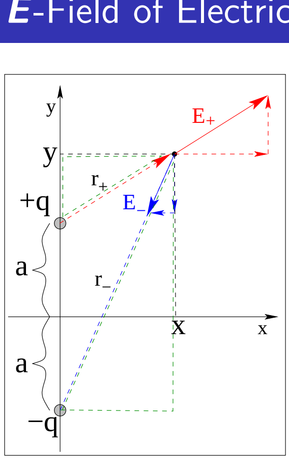

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- And:

$$E_+ = \frac{k_e q}{r_+^2} \qquad E_- = \frac{k_e q}{r_-^2}$$

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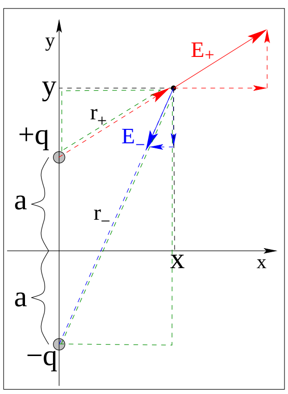

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- And:

$$E_{+} = \frac{k_e q}{r_{+}^2} \qquad E_{-} = \frac{k_e q}{r_{-}^2}$$

- So:

$$E_{+,x} = \frac{k_e q}{r_{+}^2}\frac{x}{r_{+}} \qquad E_{-,x} = -\frac{k_e q}{r_{-}^2}\frac{x}{r_{-}}$$

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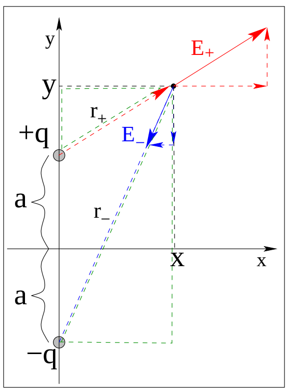

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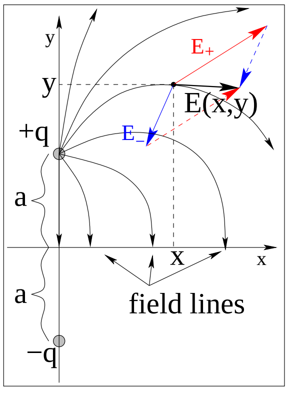
- And:

$$E_{+,y} = \frac{k_e q}{r_+^2}\frac{y-a}{r_+} \quad E_{-,x} = -\frac{k_e q}{r_-^2}\frac{y+a}{r_-}$$

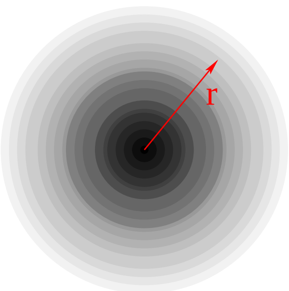
$\vec{E}$-Field of Electric Dipole at point P-Solution (Cont)



- Or (summing the terms and assembling all of the pieces, see below):
- Note that the solution is *tangent to the dipolar field lines* that run from $+q$ to $-q$.
- It might not be the *best* way to represent the field – spherical polar coordinates are better – but we definitely have the skill to find the $\vec{E}$-field at arbitrary points in space from arbitrary numbers of discrete charges in cartesian coordinates!

$$\vec{E}(x,y) = k_e q \left\{ \frac{x}{(x^2+(y-a)^2)^{3/2}} - \frac{x}{(x^2+(y+a)^2)^{3/2}} \right\} \hat{x} + k_e q \left\{ \frac{y-a}{(x^2+(y-a)^2)^{3/2}} - \frac{y+a}{(x^2+(y+a)^2)^{3/2}} \right\} \hat{y}$$

Finding the Field of a Hydrogen Atom

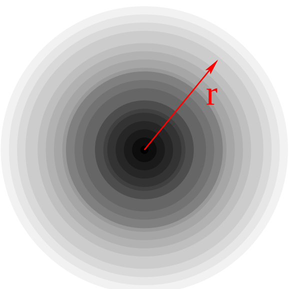


- Find the electric field at all points in space of a spherical charge distribution with radial charge density:

$$\rho(r) = \rho_0 \frac{e^{-r/2a}}{r^2}$$

and determine ρ_0 such that the total charge Q in the distribution is $-e$. This is the charge distribution of the electron cloud about a hydrogen atom in the ground state.

Finding the Field of a Hydrogen Atom



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- The key part of this solution will be to evaluate the total charge inside a Gaussian surface of radius r by integration.

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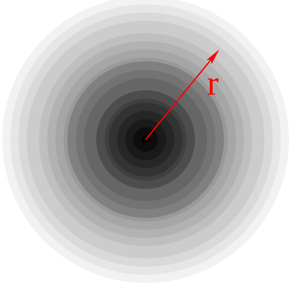

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- Then we can use this result plus Gauss's Law to find E_r.

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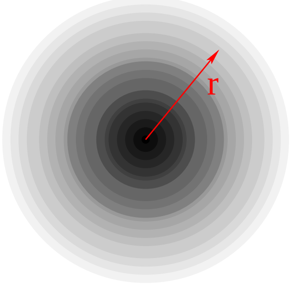

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- If we evaluate this same integral for $r \to \infty$ and set the result equal to $-e$, we can solve for ρ_0.

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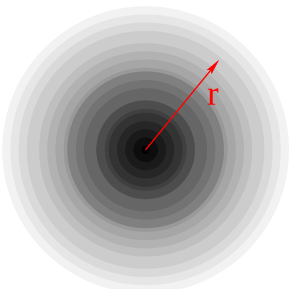

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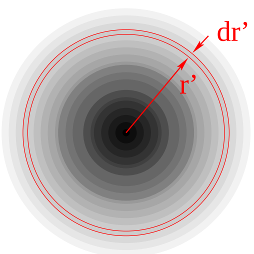

- We start by finding the total charge in a spherical shell of radius r' and thickness dr', using $dq = \rho dV$.

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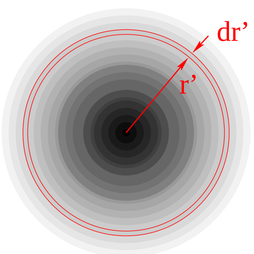

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$$dq = \rho(r')4\pi r'^2 dr' = 4\pi\rho_0 e^{-r'/2a} dr'$$

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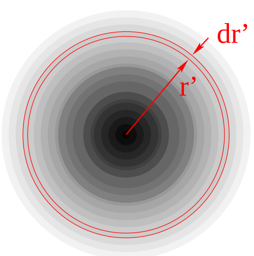

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- Next, we integrate this from 0 to r to find the total charge inside r:

$$q(r) = 4\pi\rho_0 \int_0^r e^{-r'/2a} dr'$$

$$u = -r'/2a \qquad \Rightarrow q(r) = -8\pi\rho_0 a \int_0^{-r/2a} e^u du = 8\pi\rho_0 a \left(1 - e^{-r/2a}\right)$$

- Now we use Gauss's Law to find the E-field:

$$\oint_S \vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{n}} dA = E_r 4\pi r^2 == \frac{8\pi\rho_0 a}{\epsilon_0}\left(1 - e^{-r/2a}\right) \Rightarrow \boxed{E_r(r) = \frac{2\rho_0 a}{\epsilon_0 r^2}\left(1 - e^{-r/2a}\right)}$$

Finding the Field of a Hydrogen Atom-Solution (Cont)

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- Hence:

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- **Note:** This solution already continues to $r \to \infty$. Quantum mechanically, $\rho(r)$ never *quite* reaches zero, even though (of course) the exponential term cuts off rapidly once $r \gg 2a$ (where a is the characteristic "size" of the hydrogen atom at ~ 0.5 Å).

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$$q(r \to \infty) = -e = 8\pi\rho_0 a \quad \Rightarrow \quad \boxed{\rho_0 = \frac{-e}{8\pi a}}$$

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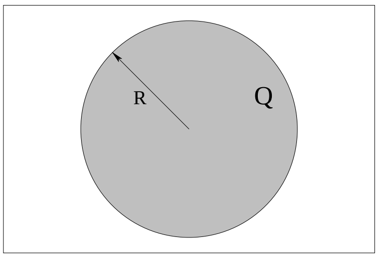
- If you want a further challenge, see if you can integrate by parts to find $V(r)$!

Potential Energy of Solid Sphere of Charge



- A solid sphere of charge with radius R and total charge Q, uniformly distributed, is shown to the left. We'd like to compute its total electrostatic potential energy. This formula can serve as an estimate or reference form for many problems of interest, such as the approximate electrostatic potential energy of quarks bound into a proton.

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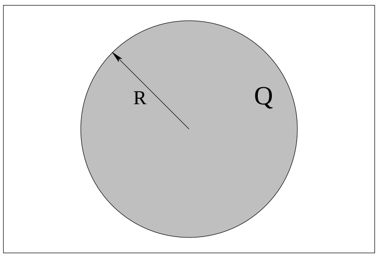

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- Our solution stratey here is to "build a sphere". We can imagine we have a sphere of radius $r' < R$ with the appropriate fractional charge, compute the electrostatic potentiual at its surface, and use this to compute the work dW required to bring in the next chunk of charge dQ of thickness dr' to increase its radius.

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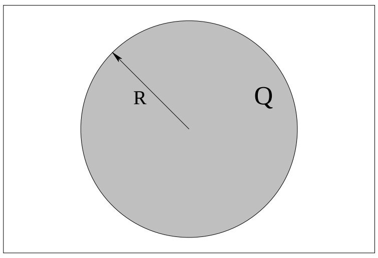

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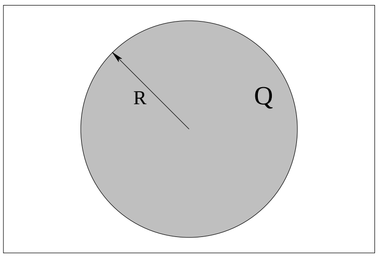

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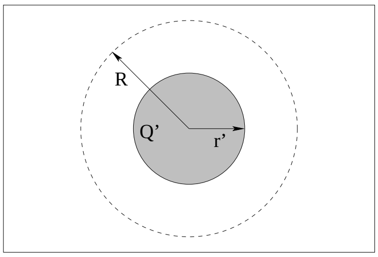
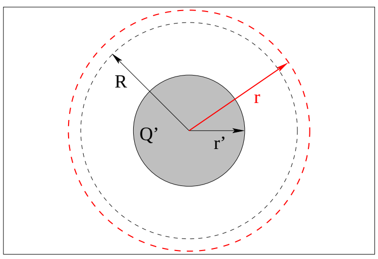
Potential Energy of Solid Sphere of Charge-Solution



- The charge Q' inside r' is easy:

$$Q' = Q \frac{\frac{4\pi}{3} r'^3}{\frac{4\pi}{3} R^3} = Q \frac{r'^3}{R^3}$$

Potential Energy of Solid Sphere of Charge-Solution



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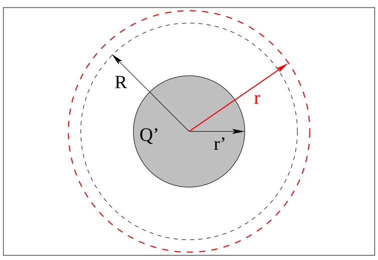
- Gauss's Law at $r > r'$ (careful with r vs r'!) is:

$$E_r 4\pi r^2 = \frac{1}{\epsilon_0}Q' \Rightarrow E_r(r) = \frac{k_e Q r'^3}{R^3}\frac{1}{r^2} \ (r > r')$$

Potential Energy of Solid Sphere of Charge-Solution



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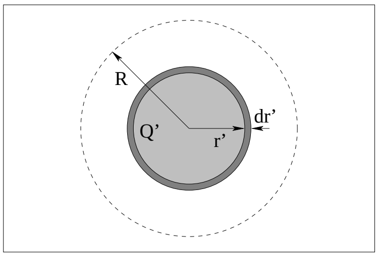
- Gauss's Law at $r > r'$ (careful with r vs r'!) is:

$$E_r 4\pi r^2 = \frac{1}{\epsilon_0}Q' \Rightarrow E_r(r) = \frac{k_e Q r'^3}{R^3}\frac{1}{r^2} \; (r > r')$$

- Next we integrate this to find the potential at $r = r'$:

$$V(r = r') = -\int_{\infty}^{r'} E_r(r)dr = \frac{k_e Q r'^2}{R^3}$$

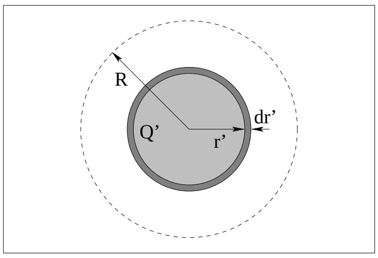
Potential Energy of Solid Sphere of Charge-Solution (Cont)



- Now we're ready to find the work required to bring in a charge dQ' with thickness dr' at radius r'. We start by finding dQ':

$$dQ' = \rho dV = \frac{3Q}{4\pi R^3} 4\pi r'^2 dr' = \frac{3Q}{R^3} r'^2 dr'$$

Potential Energy of Solid Sphere of Charge-Solution (Cont)



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$$dU = dW = V(r')dQ' = \frac{k_e Q r'^2}{R^3} \times \frac{3Q}{R^3} r'^2 dr' = \frac{3 k_e Q^2}{R^6} r'^4 dr'$$

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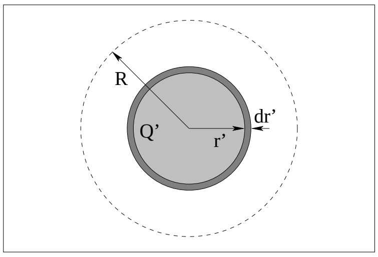

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- And finally, we integrate this from $r' = 0$ to R:

$$U = \int dU = \int_0^R \frac{3k_e Q^2}{R^6} r'^4 dr' \quad \text{or} \quad \boxed{U = \frac{3}{5} \frac{k_e Q^2}{R}}$$

Potential Energy of Solid Sphere of Charge-Solution Summary

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- This answer "makes sense". It has the right units, for one thing.
- For another, if we assemble a *shell* of charge Q and radius R, we expect

$$U_{\text{shell}} = \frac{1}{2}\frac{k_e Q}{R}$$

and this is a bit larger than this as we expect because we've pushed the charge a bit closer together throughout the interior.

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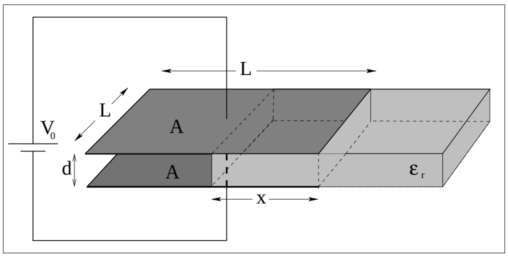
- Be sure to come back and verify that the two approaches give you the same answer!

Force on a Dielectric Slab



- Here's a tricky one, one that fooled me back even as an instructor when I first started teaching physics. In the figure to the left, a square capacitor ($L \times L \times d$) with a **constant potential V_0** maintained across it by a batter has a dielectric slab with relative permittivity ϵ_r inserted a distance x into it.

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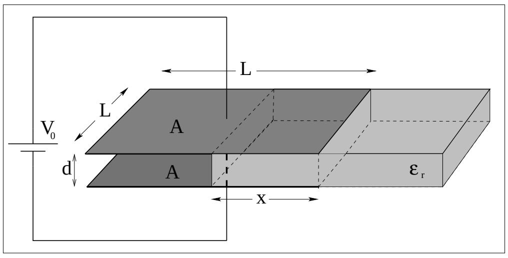

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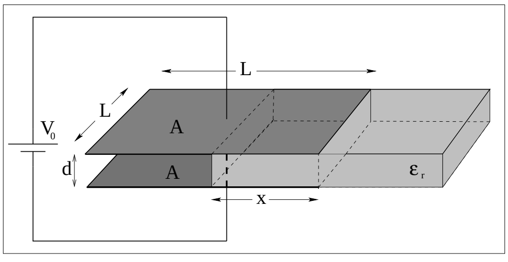

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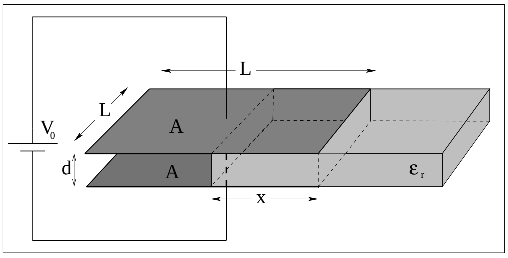

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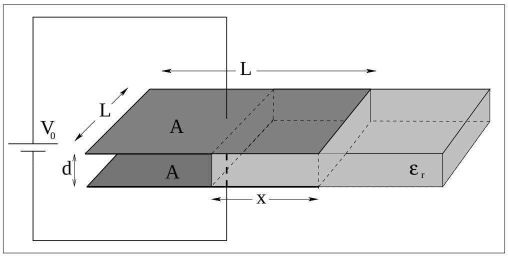

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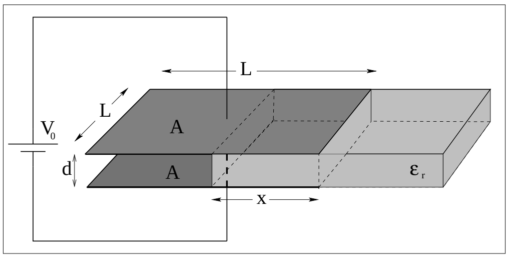

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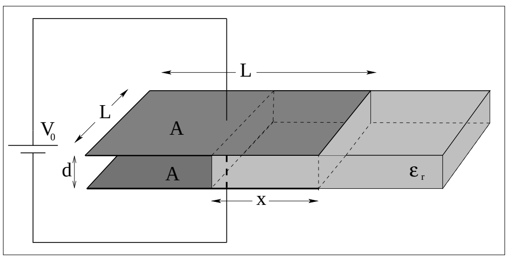
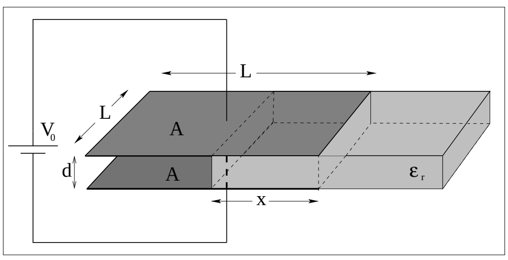
Force on a Dielectric Slab-Solution



- We'll start by expressing the potential energy of the x-filled capacitor as a function of x. Note well that in this case $V_C = V_0$ no matter what, so $U = \frac{1}{2}CV_0^2$ is likely to be the easiest if we find C as a function of x.

Force on a Dielectric Slab-Solution



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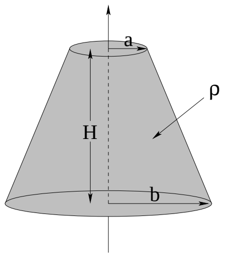
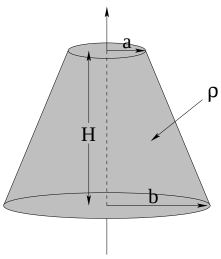
Resistance of a Truncated Cone



- We'd like to find the total vertical resistance of a truncated right circular cone made from material of a uniform resistivity ρ with height H and radii a and b at the top and bottom respectively. Give this a try on your own with looking up anything before uncovering the hints below.

Resistance of a Truncated Cone



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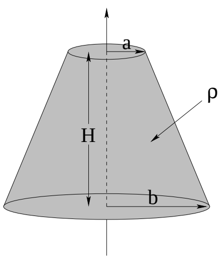

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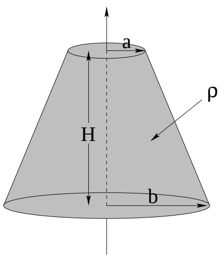

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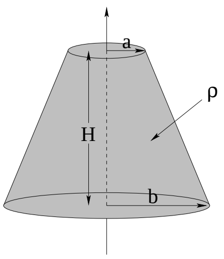

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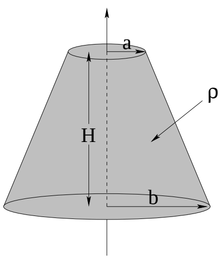
* - The solution is on the next page. Don't advance until you are ready!

Resistance of a Truncated Cone-Solution



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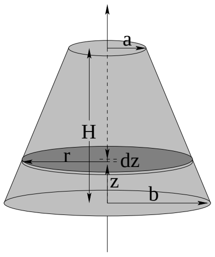

- First, we know that (for a short cylindrical resistor) $R = \dfrac{\rho\ell}{A}$.
- Second, we should choose **cylindrical** coordinates to treat the cone as a stack of short cylindrical resistors *in series*

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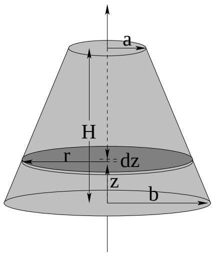

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- In order to compute A for each thin slice of height dz and radius r and do the sum/integral, we need (e.g. point-slope formula):

$$r(z) = b - \frac{b - a}{H} z$$

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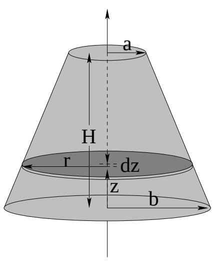

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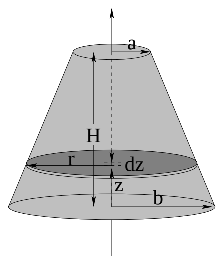
- Now we can find the resistance dR of the thin dark shaded slice:

$$dR = \frac{\rho\, dz}{\pi r^2} = \frac{\rho\, dz}{\pi \left(b - \frac{b-a}{H} z\right)^2}$$

Resistance of a Truncated Cone-Solution (Cont)



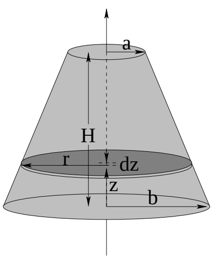
Our job is now reduced to a simple math problem – integrating

$$dR = \frac{\rho\, dz}{\pi r^2} = \frac{\rho\, dz}{\pi \left(b - \frac{b-a}{H} z\right)^2}$$

from $z = 0$ to $z = H$. This is easy if we use **u-substitution** with:

$$u = b - (b-a)z/H \qquad \Rightarrow \qquad du = \left(-\frac{(b-a)}{H}\right) dz$$

Resistance of a Truncated Cone-Solution (Cont)



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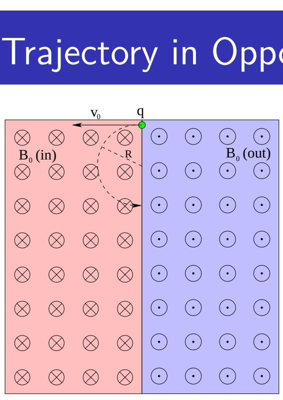
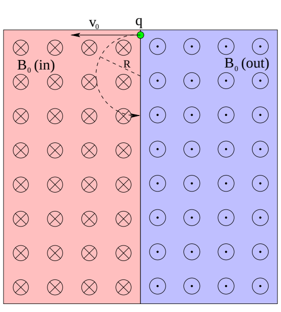
$$= \frac{\rho H}{\pi (b-a)}\left(\frac{1}{a} - \frac{1}{b}\right) \qquad \Rightarrow \qquad \boxed{R = \frac{\rho H}{\pi a b}}$$

Trajectory in Opposing $\boldsymbol{B}$-fields



- A region is split with two equal and opposite magnetic fields sitting side by side. A charged particle q and mass m is initially travelling to the left at speed v_0 as shown, and is bent into a semicircular trajectory of radius R by the magnetic field B_0 into the page.

Trajectory in Opposing $\boldsymbol{B}$-fields



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- Find: The sign of q.

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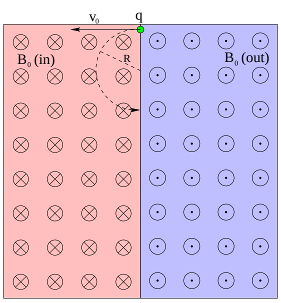

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- Find: The sign of q. The mass m in terms of q, R, B_0, and v_0.

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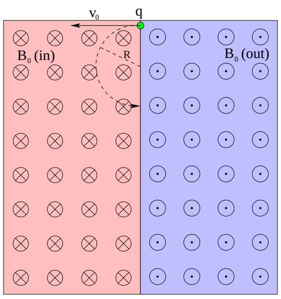

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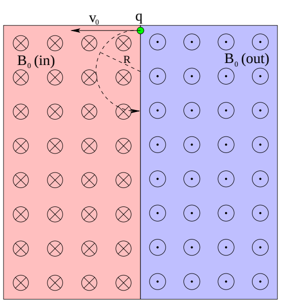

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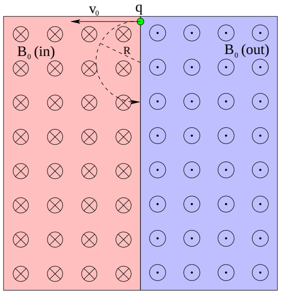
Hints:

Trajectory in Opposing $\vec{B}$-fields



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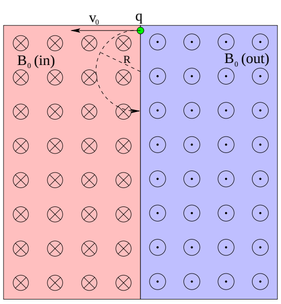
Hints:

- Use the right hand rule and $\vec{F} = q(\vec{v} \times \vec{B})$ to see what the sign of the charge must be.

Trajectory in Opposing $\boldsymbol{B}$-fields

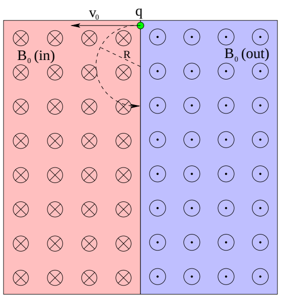


- A region is split with two equal and opposite magnetic fields sitting side by side. A charged particle q and mass m is initially travelling to the left at speed v_0 as shown, and is bent into a semicircular trajectory of radius R by the magnetic field B_0 into the page.
- Find: The sign of q. The mass m in terms of q, R, B_0, and v_0. The time it will take to complete the semicircle and enter the field B_0 out of the page. Finally, draw the subsequent trajectory of the particle until it exits the fields.

Hints:

- Use the right hand rule and $\vec{\boldsymbol{F}} = q(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}})$ to see what the sign of the charge must be.
- Use Newton's Second Law and centripetal acceleration to come up with an algebraic formula for m.

Trajectory in Opposing $\vec{B}$-fields



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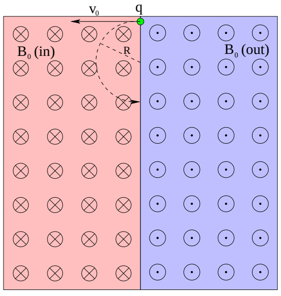

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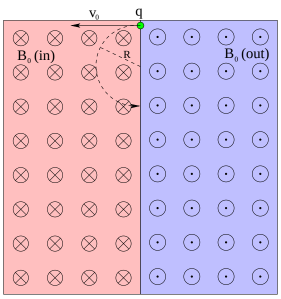
Trajectory in Opposing $\vec{B}$-fields



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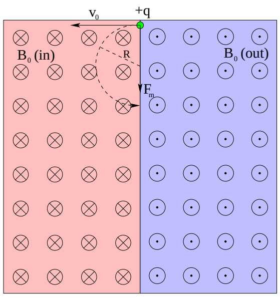
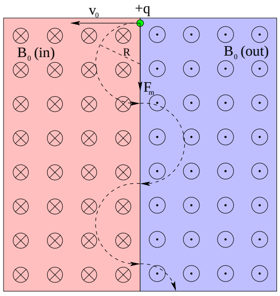
Trajectory in Opposing $\vec{\boldsymbol{B}}$-fields-Solution



- The magnetic force initially points down to make the charge curve in towards the center of curvature of the displayed trajectory. $\vec{\boldsymbol{F}}_m = +q(\vec{\boldsymbol{v}}_0 \times \vec{\boldsymbol{B}}_0)$ is down, so the charge must be $\boxed{+q}$ positive.

Trajectory in Opposing $\vec{B}$-fields-Solution



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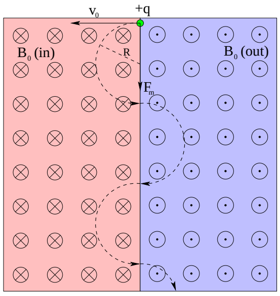
- $\vec{B}$ does no work; so v_0 does not change, only its direction. When it hits the (blue shaded) $\vec{B}$ field on the right hand side, again the magnetic force must point down (with the same magnitude) so the charge again curves back to the (red shaded) $\vec{B}$ field.

Trajectory in Opposing $\vec{B}$-fields-Solution



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- To solve for m, use the magnitude of the force in N2 plus centripetal acceleration:

$$qv_0B_0 = m\frac{v_0^2}{R} \qquad \Rightarrow \qquad \boxed{m = \frac{qB_0R}{v_0}}$$

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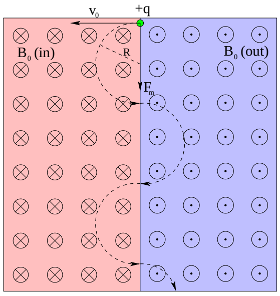

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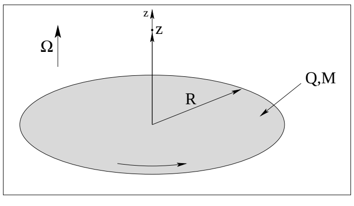
- In terms of the givens:

$$v_0t_c = \pi R \qquad \Rightarrow \qquad \boxed{t_c = \frac{\pi R}{v_0}}$$

which doesn't depend on the magnetic field, mass, or charge, because v_0 and R were given!

$\vec{B}$-field of a Spinning Disk on its Axis



- A disk of radius R and uniformly distributed mass M and charge Q is rotating around the z-axis at a constant angular velocity $\vec{\Omega} = \Omega\hat{z}$. Find the $\vec{B}$-field at an arbitrary point on the z-axis, and show that for $z \gg R$ it can be written:

$$\vec{B} = \frac{k_m Q \vec{L}}{2Mz^3}$$

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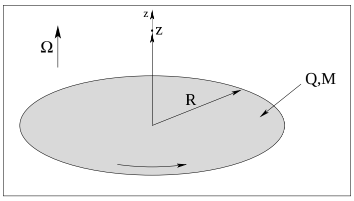

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- **Solution:** In words: We'll find the charge of a tiny chunk of the disk in coordinates we can integrate over to cover the disk. Using the Biot-Savart Law (the form appropriate for a point charge moving with a speed $v \ll c$) we'll find its differential $\vec{B}$-field at a point on the z-axis and integrate.

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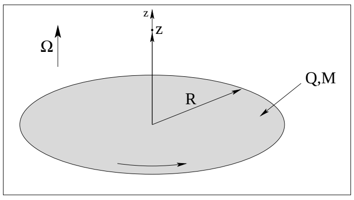

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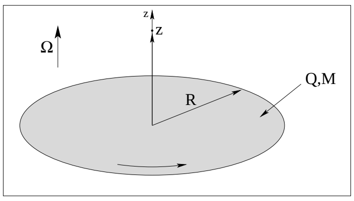

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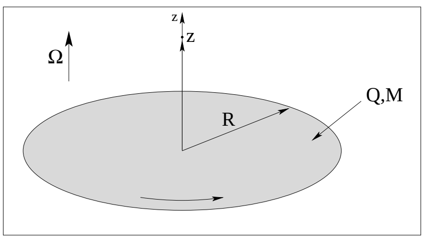
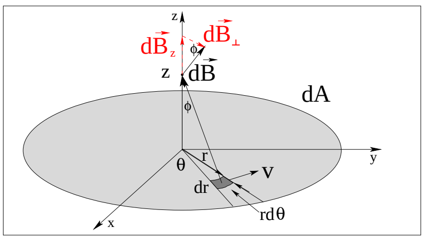


- Our first chore is to add x and y axes, define θ and r as integration variables, and decorate the figure with $d\vec{\boldsymbol{B}}$ and its components (and an angle ϕ we'll need as we proceed).

$\vec{B}$-field of a Spinning Disk on its Axis-Solution



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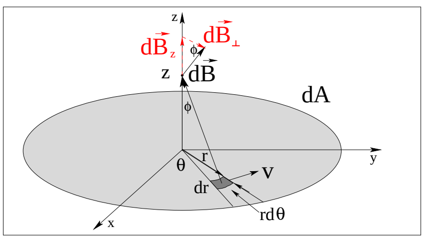

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- Then:

$$d\vec{\boldsymbol{B}} = k_m \frac{dq\vec{\boldsymbol{v}} \times (\vec{\boldsymbol{z}} - \vec{\boldsymbol{r}})}{|\vec{\boldsymbol{z}} - \vec{\boldsymbol{r}}|^3} \Rightarrow dB = k_m \frac{v\,dq}{(r^2 + z^2)} = k_m \frac{vQrdr\,d\theta}{\pi R^2(r^2 + z^2)}$$

and

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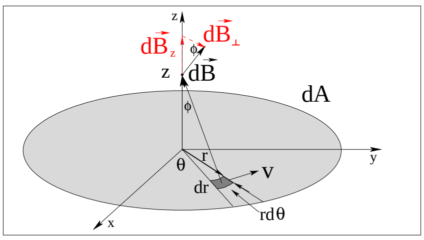

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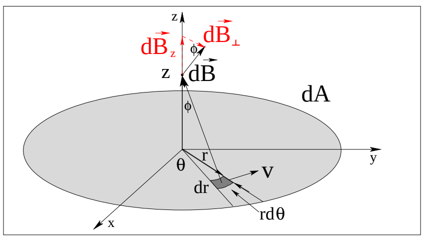
and

$$dB_z = dB \sin\phi \qquad dB_x = dB \cos\phi \cos\theta \qquad dB_y = dB \cos\phi \sin\theta$$

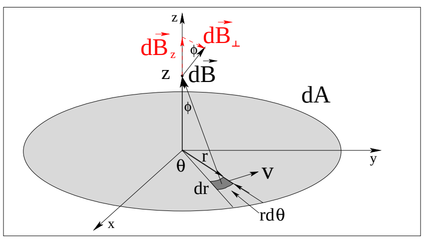
$\boldsymbol{B}$-field of a Spinning Disk on its Axis-Solution (Cont)



- Note that:

$$\sin\phi = \frac{r}{(r^2 + z^2)^{1/2}} \qquad \cos\phi = \frac{z}{(r^2 + z^2)^{1/2}}$$

$\boldsymbol{B}$-field of a Spinning Disk on its Axis-Solution (Cont)



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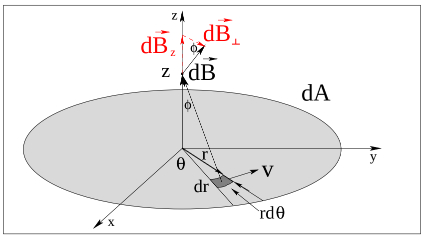
$$\sin\phi = \frac{r}{(r^2+z^2)^{1/2}} \qquad \cos\phi = \frac{z}{(r^2+z^2)^{1/2}}$$

- Hence, with $v = r\Omega$:

$$dB_z = \frac{k_m Q \Omega r^3 dr\, d\theta}{\pi R^2 (r^2+z^2)^{3/2}}$$

$\boldsymbol{B}$-field of a Spinning Disk on its Axis-Solution (Cont)



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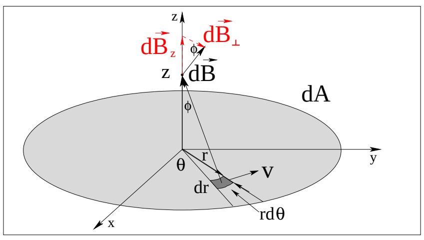
- Then (integrating):
$$B_z = \int_0^R \int_0^{2\pi} \frac{k_m Q \Omega r^3 dr\, d\theta}{\pi R^2 (r^2+z^2)^{3/2}} = \frac{2 k_m Q \Omega}{R^2} \int_0^R \frac{r^3 dr}{(r^2+z^2)^{3/2}}$$

$\boldsymbol{B}$-field of a Spinning Disk on its Axis-Solution (Cont)



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$$B_z = \int_0^R \int_0^{2\pi} \frac{k_m Q\Omega r^3 dr\, d\theta}{\pi R^2(r^2+z^2)^{3/2}} = \frac{2k_m Q\Omega}{R^2}\int_0^R \frac{r^3 dr}{(r^2+z^2)^{3/2}}$$

- This integral is tricky – it requires (careful!) integration by parts. Choose $u = r^2$, $dv = \frac{2r\,dr}{(r^2+z^2)^{3/2}}$ and get:

$$B_z = \frac{k_m Q\Omega}{R^2}\left\{ -\frac{2R^2}{(R^2+z^2)^{1/2}} + 4(R^2+z^2)^{1/2} - 4z \right\}$$

$\boldsymbol{B}$-field of a Spinning Disk on its Axis-Solution (Cont)

- This must be expanded for $z \gg R$:

$\boldsymbol{B}$-field of a Spinning Disk on its Axis-Solution (Cont)

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$$
\begin{aligned}
B_z &= \frac{k_m Q\Omega}{R^2}\left\{-\frac{2R^2}{z}\left(1+\frac{R^2}{z^2}\right)^{-1/2} + 4z(1+\frac{R^2}{z^2})^{1/2} - 4z\right\} \\
&\approx \frac{k_m Q\Omega}{R^2}\left\{-\frac{2R^2}{z}\left(\cancel{1} - \frac{1}{2}\frac{R^2}{z^2} + \ldots\right) + 4z\left(\cancel{1} + \cancel{\frac{R^2}{2z^2}} - \frac{R^4}{8z^4} + \ldots\right) \cancel{-4z}\right\} \\
&\approx \frac{k_m Q\Omega}{R^2}\left\{\left(\frac{R^4}{z^3} - \frac{R^4}{2z^3}\right)\right\} = \frac{k_m Q\Omega R^2}{2z^3} = 2k_m \frac{Q}{2M}\frac{\frac{1}{2}MR^2\Omega}{z^3}
\end{aligned}
$$

$\boldsymbol{B}$-field of a Spinning Disk on its Axis-Solution (Cont)

- This must be expanded for $z \gg R$:

$$B_z = \frac{k_m Q\Omega}{R^2}\left\{-\frac{2R^2}{z}\left(1+\frac{R^2}{z^2}\right)^{-1/2} + 4z(1+\frac{R^2}{z^2})^{1/2} - 4z\right\}$$

$$\approx \frac{k_m Q\Omega}{R^2}\left\{-\frac{2R^2}{z}\left(\cancel{1}-\frac{1}{2}\frac{R^2}{z^2}+\ldots\right)+4z\left(\cancel{1}+\cancel{\frac{R^2}{2z^2}}-\frac{R^4}{8z^4}+\ldots\right)\cancel{-4z}\right\}$$

$$\approx \frac{k_m Q\Omega}{R^2}\left\{\left(\frac{R^4}{z^3}-\frac{R^4}{2z^3}\right)\right\} = \frac{k_m Q\Omega R^2}{2z^3} = 2k_m\frac{Q}{2M}\frac{\frac{1}{2}MR^2\Omega}{z^3}$$

- or:

$$\boxed{B_z = 2k_m\left(\frac{Q}{2M}L_z\right)\frac{1}{z^3} = \frac{2k_m m_z}{z^3}}$$

as expected.

$\boldsymbol{B}$-field of a Spinning Disk on its Axis-Solution (Cont)

- Note that:

$$dB_x = dB\cos\phi\cos\theta = k_m \frac{vQzrdr\ \cos\theta d\theta}{\pi R^2(r^2+z^2)^{3/2}} \quad \Rightarrow$$

$$B_x = \frac{k_m Q\Omega z r^2 dr}{\pi R^2(r^2+z^2)^{3/2}} \times \int_0^{2\pi} \cos\theta d\theta = 0$$

$$dB_y = dB\cos\phi\sin\theta = k_m \frac{vQzrdr\ \sin\theta d\theta}{\pi R^2(r^2+z^2)^{3/2}} \quad \Rightarrow$$

$$B_y = \frac{k_m Q\Omega z r^2 dr}{\pi R^2(r^2+z^2)^{3/2}} \times \int_0^{2\pi} \sin\theta d\theta = 0$$

$\boldsymbol{B}$-field of a Spinning Disk on its Axis-Solution (Cont)

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$$B_x = \frac{k_m Q\Omega z r^2 dr}{\pi R^2 (r^2+z^2)^{3/2}} \times \int_0^{2\pi} \cos\theta d\theta = 0$$
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$$B_y = \frac{k_m Q\Omega z r^2 dr}{\pi R^2 (r^2+z^2)^{3/2}} \times \int_0^{2\pi} \sin\theta d\theta = 0$$

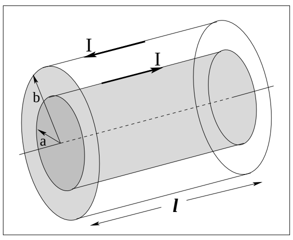
- We can then conclude via explicit (if painful) integration that:

$$\boxed{\vec{\boldsymbol{B}}(z) = \frac{2k_m \vec{\boldsymbol{m}}}{z^3}}$$

where

$$\vec{\boldsymbol{m}} = \frac{Q}{2M}\vec{\boldsymbol{L}} = \frac{Q}{2M} \times \frac{1}{2} M R^2 \Omega \hat{\boldsymbol{z}}$$

Self-Induction of a Coaxial Cable



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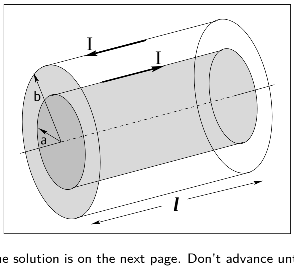
Self-Induction of a Coaxial Cable



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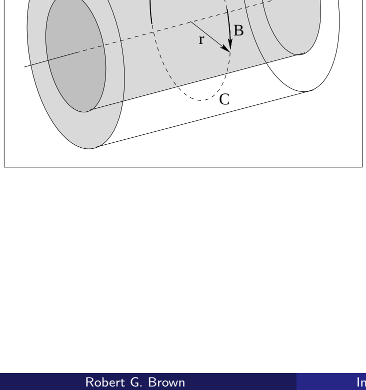
Self-Induction of a Coaxial Cable-Solution



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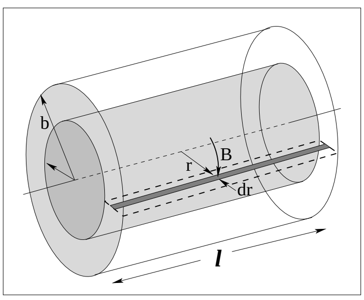

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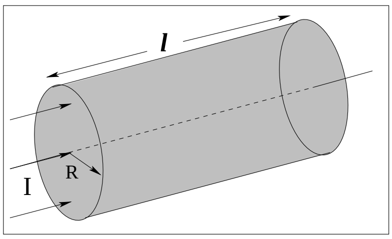
The Self-Induction of a Thick Wire



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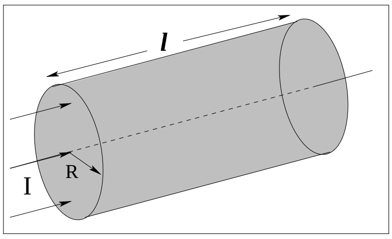

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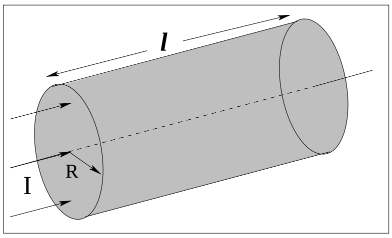

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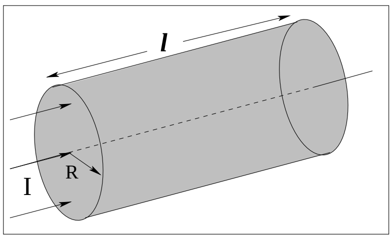

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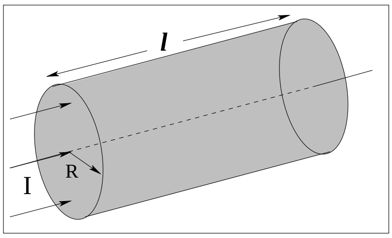

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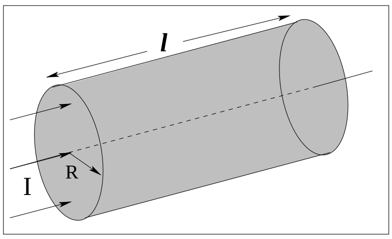

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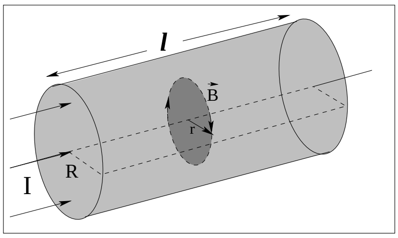
The Self-Induction of a Thick Wire-Solution



- To find the $\vec{\boldsymbol{B}}$-field, we find the current through the shaded disk at radius r:

$$I(r) = \int_S \vec{\boldsymbol{J}} \cdot \hat{\boldsymbol{n}} dA = I \frac{r^2}{R^2}$$

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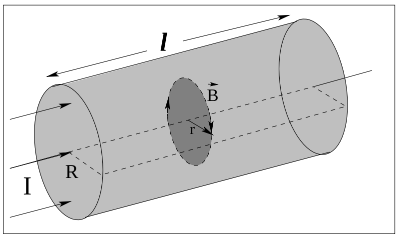

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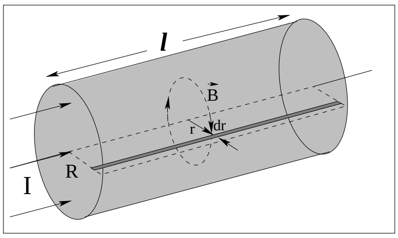

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- Then we find the magnetic flux through the shaded strip of length ℓ and thickness dr at radius r. However, we can't use the entire current I in this expression as it doesn't pass inside r and is not "linked" to the flux, so we multiply by an extra r^2/R^2 to correct for this:

$$d\phi_m = B(r)\frac{r^2}{R^2}\ell\, dr = \frac{\mu I \ell r^3 dr}{2\pi R^4} \quad \Rightarrow \quad \phi_m = \frac{\mu I \ell}{2\pi R^4}\int_0^R r^3 dr = \frac{\mu I \ell}{8\pi}$$

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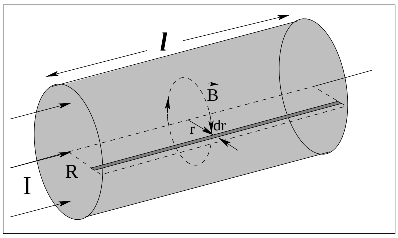

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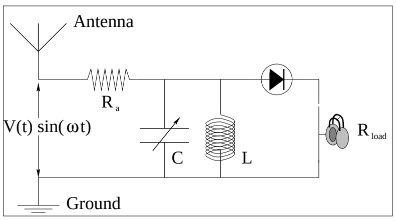
- Hence:

$$\boxed{\frac{\phi_m}{I\ell} = \frac{L}{\ell} = \frac{\mu}{8\pi}}$$

A Crystal Radio Circuit



- We would like to find the total current $I(t) = I_0 \sin(\omega t + \phi)$ in the circuit to the left, a common one used in ***crystal radios***, and use this current to discuss the power delivered to the headphones as a function of ω. Note that it is a ***parallel LRC circuit*** with an extra resistor in series.

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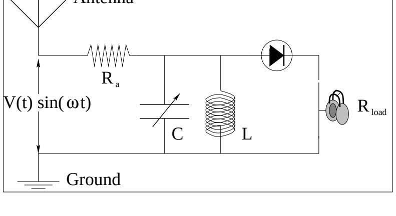

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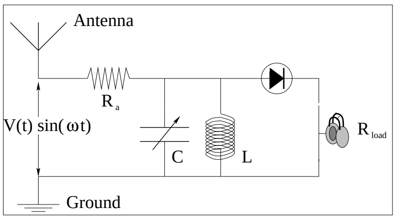

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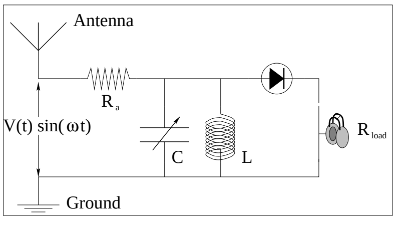

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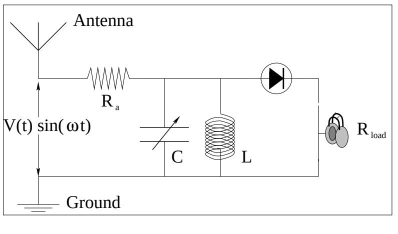

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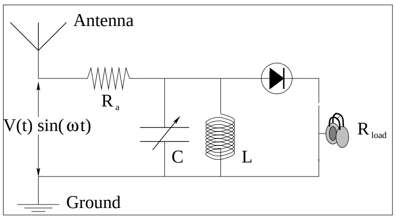

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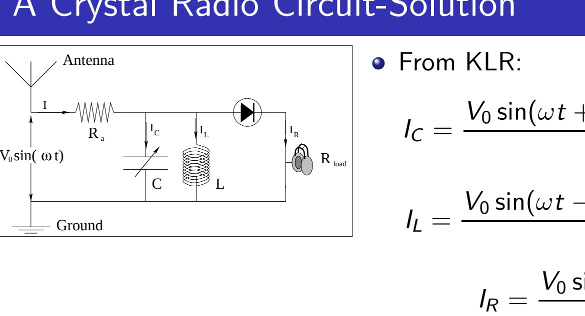

- From KLR:

$$I_C = \frac{V_0 \sin(\omega t + \pi/2) - I_0 \sin(\omega t + \phi + \pi/2) r}{\chi_C}$$

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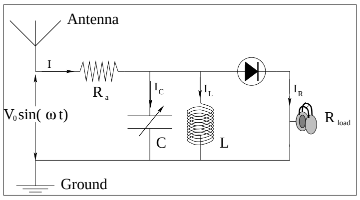

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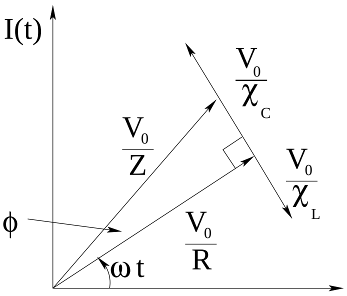

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A Crystal Radio Circuit-Solution (Cont)

- We can rearrange this to put all of the I_0 terms on one side. Note well that ϕ is the phase angle of the current through r only – it is no longer the phase angle of the "unperturbed" phasor on the right below.

$$I_0 \left\{ \left(\frac{R+r}{R} \right) \sin(\omega t + \phi) + \frac{r}{\chi_C} \sin(\omega t + \phi + \pi/2) + \frac{r}{\chi_L} \sin(\omega t + \phi - \pi/2) \right\}$$

$$= V_0 \left\{ \frac{1}{\chi_C} \sin(\omega t + \pi/2) + \frac{1}{\chi_L} \sin(\omega t - \pi/2) + \frac{1}{R} \sin(\omega t) \right\}$$

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- We rewrite this in terms of the phase angle ϕ_0 of the total voltage across R, C, and L:

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$$= \frac{V_0}{Z_0} \sin(\omega t + \phi_0) \quad \text{where:}$$

$$\boxed{\frac{1}{Z_0} = \left(\frac{1}{R^2} + \left\{ \frac{1}{\chi_C} - \frac{1}{\chi_L} \right\}^2 \right)^{1/2}} \qquad \boxed{\phi_0 = \tan^{-1} \left(\frac{R(\chi_L - \chi_C)}{\chi_L \chi_C} \right)}$$

A Crystal Radio Circuit-Solution (Cont)

- The phase angle of the ***dimensionless*** term in $\{\}$ brackets on the left has to match that of the term on the right. It's effect will be to lower I_0 relative to its value when $r = 0$ (where $\phi = \phi_0$ and $I_0 = V_0/Z_0$). We'll call this the ***attenuation factor***, A, so that:

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- To conclude:

$$\boxed{I(t) = \frac{V_0}{A Z_0} \sin(\omega t + \phi)}$$

with A, Z_0, and ϕ defined above. When $r \to 0$, $A \to 1$, $\phi \to \phi_0$ as expected.

A Crystal Radio Circuit-Solution (Cont)

- Consider:

$$\boxed{I(t) = \frac{V_0}{AZ_0}\sin(\omega t + \phi)}$$

with

$$\boxed{A = \left\{\left(\frac{R+r}{R}\right)^2 + \left(\frac{r}{\chi_C} - \frac{r}{\chi_L}\right)^2\right\}^{1/2}} \qquad \boxed{\phi = \tan^{-1}\left(\frac{(\chi_L - \chi_C)rR}{(\chi_C\chi_L)(r+R)}\right)}$$

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- If $\chi_L = \chi_C \Rightarrow \omega = \omega_0 = 1/\sqrt{LC}$, $Z_0 = R$, $A = (r+R)/R$, $\phi = \phi_0 = 0$.

$$I_0 = \frac{V_0}{r+R}$$

as expected, and we've already shown that in this case we'll have the *maximum possible* power dissipated in R if and only if $r = R$ (the resistances of antenna and load match). This is ***resonance*** in an ***well-matched circuit***.

A Crystal Radio Circuit-Solution (Cont)

- If $\chi_C \ll r$ (high frequency) or $\chi_L \ll r$ (low frequency) *and* $r = R$,

$$A \approx \frac{r}{\chi_{C,L}} \gg 1$$

and

$$I(t) \to 0$$

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  as the remaining reactance gets much smaller than r (or R).
- This is a kind of ***band pass filter***, but one that modulates the *fraction* of the power delivered to the load resistor by essentially shorting it out through the capacitor or inductor for frequencies much larger or smaller (respectively) than the resonant frequency.

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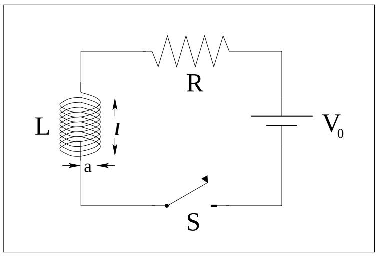
- Consequently, the power delivered to the circuit is strongly suppressed as AZ_0 gets large.
- Also:
$$\phi \to \tan^{-1}\left(-\frac{R}{2\chi_{L,C}}\right) \to -\frac{\pi}{2}$$
as the remaining reactance gets much smaller than r (or R).
- This is a kind of ***band pass filter***, but one that modulates the *fraction* of the power delivered to the load resistor by essentially shorting it out through the capacitor or inductor for frequencies much larger or smaller (respectively) than the resonant frequency.
- (In case you forgot, $\chi_L = \omega L$, $\chi_C = 1/\omega C$).

Energy Flow into an Inductance



- At time $t = 0$ the switch is closed, and the current through the solenoidal inductance (initially zero) ***increases***. Find the flux of the Poynting vector through the sides of the inductor, which has N turns, length ℓ, and radius a as shown. Show that it equals the rate that power flows into the inductor into its total energy store, $U = \frac{1}{2}LI^2$.

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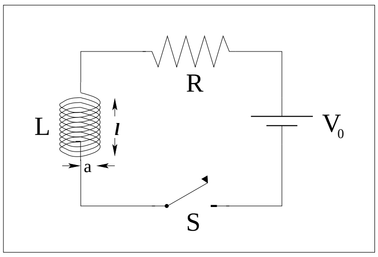

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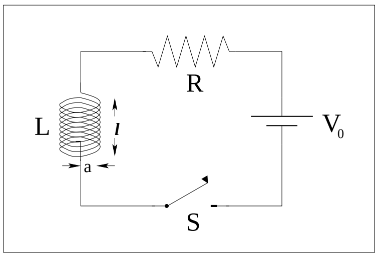

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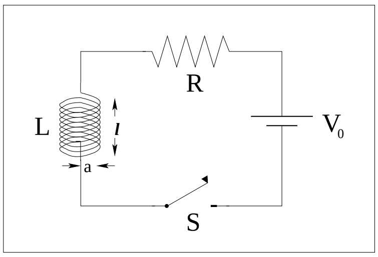

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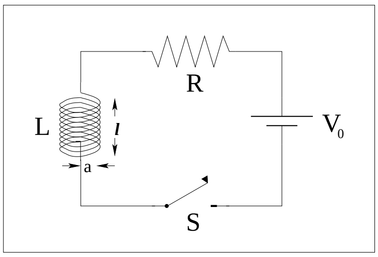

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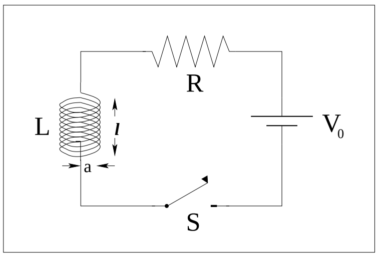

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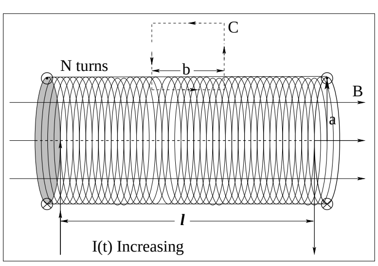
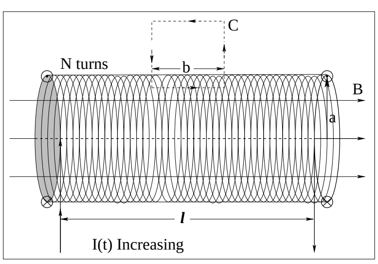
Energy Flow into an Inductance-Solution



- Ampere's Law:

$$\phi_m = \oint_C \vec{B} \cdot d\vec{\ell} = Bb = \mu_0 \frac{N}{\ell} I b$$

Energy Flow into an Inductance-Solution



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Or:

$$\boxed{B_{\rm in} = \mu_0 \frac{N}{\ell} I \quad \text{to the right}}$$

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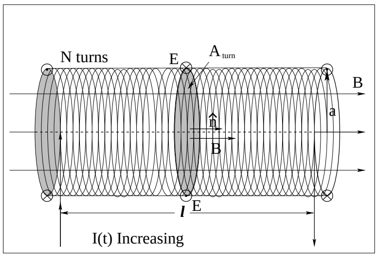

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- The total flux is $\phi_m = NB_{\rm in}A = \frac{\mu_0 N^2 I \pi a^2}{\ell}$, so:

$$L = \frac{\phi_m}{I} = \frac{\mu_0 N^2 \pi a^2}{\ell}$$

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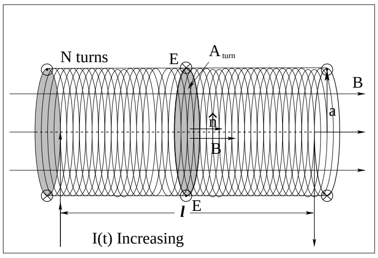

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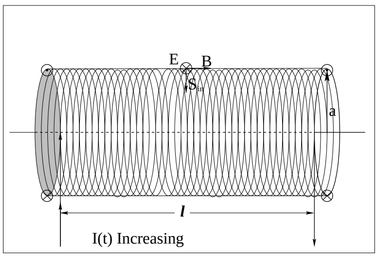
- This let's us find:

$$|V_{\text{ind}}| = NE2\pi a = L\frac{dI}{dt} \quad \Rightarrow \quad \boxed{E = \frac{\mu_0 Na}{2\ell} \frac{dI}{dt}}$$

Note that as I increases, $\vec{\boldsymbol{E}}$ points in the ***opposite*** direction to I in the wire!

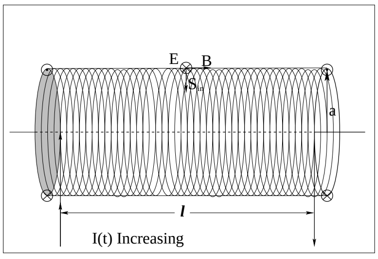
Energy Flow into an Inductance-Solution (Cont)



- Now we find the Poynting vector:

$$\vec{\boldsymbol{S}} = \frac{1}{\mu_0}\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} = \frac{1}{\mu_0} \times \frac{\mu_0 N I}{\ell} \times \frac{\mu_0 N a}{2\ell}\frac{dI}{dt} \ \text{(in)}$$

Energy Flow into an Inductance-Solution (Cont)



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- with its magnitude:

$$S_{\rm in} = \frac{\mu_0 N^2 a}{2\ell^2} I \frac{dI}{dt}$$

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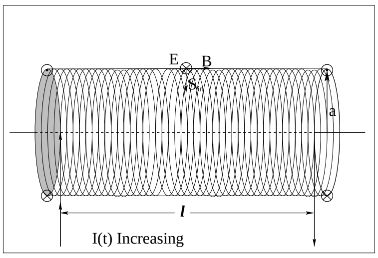

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$$\phi_S = \int_{\rm side} \vec{\boldsymbol{S}} \cdot \hat{\boldsymbol{n}} dA = S_{\rm in} 2\pi a \ell = \frac{\mu_0 N^2 \pi a^2}{\ell} I \frac{dI}{dt}$$

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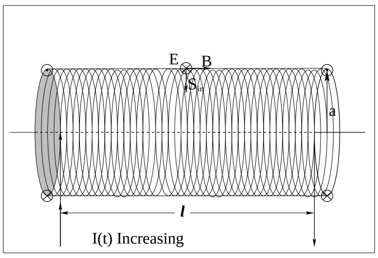

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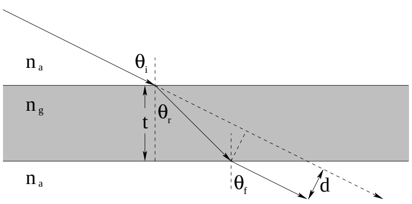
- Or (identifying L from the previous page):

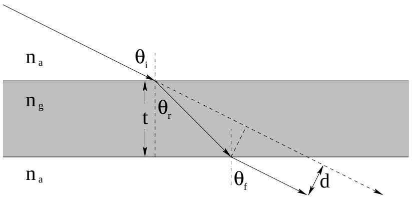
$$\boxed{P_{\rm in} = LI\frac{dI}{dt} = \frac{1}{2}L\frac{dI^2}{dt}} \quad \text{(Q.E.D.)}$$

Lateral Deflection of a Light Beam



- In the figure to the left, a beam of light is incident on a slab of glass of thickness t. It is surrounded by air. The beam emerges displaced from its original direction by a lateral distance d as shown.

Lateral Deflection of a Light Beam



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- We need to show that $\theta_i = \theta_f$ (the incident and emergent beams are parallel) and we also need to show/prove that:

$$d = \frac{sin(\theta_i - \theta_r)}{\cos(\theta_r)} t$$

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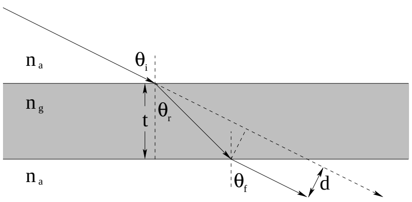

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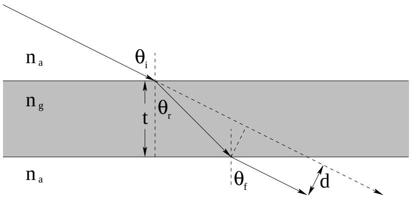

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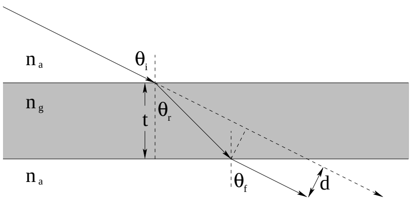

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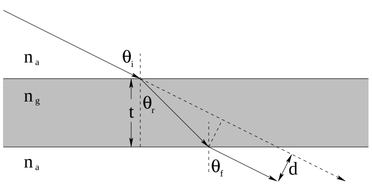

- Snell's Law:

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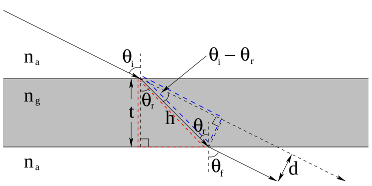

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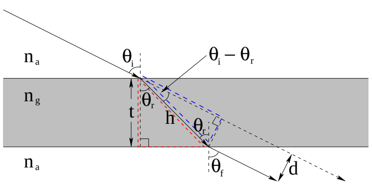

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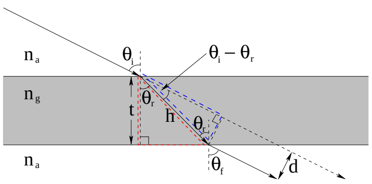

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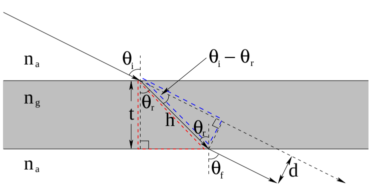

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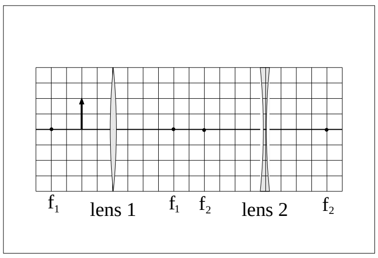
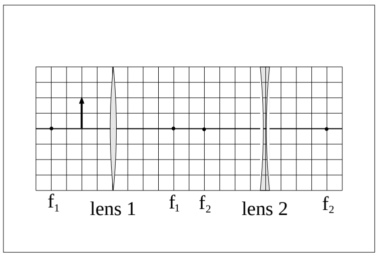
- So, not so difficult as all of that in terms of the number of steps, but it requires some insight to "see" the correct triangles and rules to use to get the answer!
- If you want a challenge, try to evaluate this for $\theta_i = 60^\circ$ and $t = 1$ cm. You should get $d \approx 5$ mm...

Solving a Two Lens Optical System



- In the figure, two lenses and an object are drawn ***to scale*** – each box represents "1 cm". $f_1 = +4$ cm, $f_2 = -4$ cm, and the 2 cm high object is located at $s_1 = 2$ cm to the left of the first lens. The lenses are separated by 10 cm.

Solving a Two Lens Optical System



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- We need to find "everything": s_1', s_2, s_2', m_1, m_2, m_{tot}, and to characterize each image, especially the final one as seen looking through lens 2 on the right.

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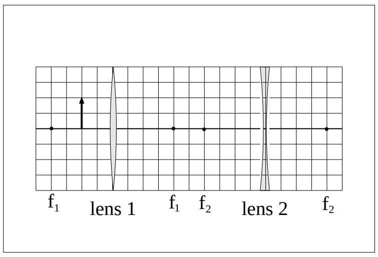

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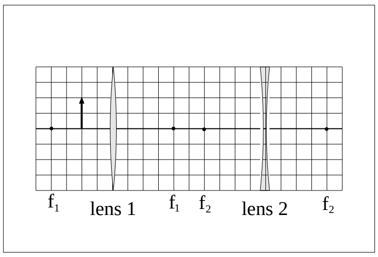

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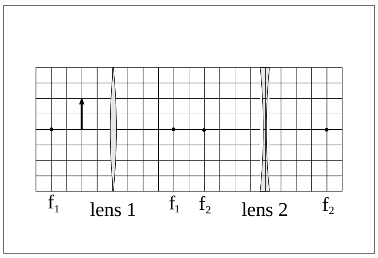

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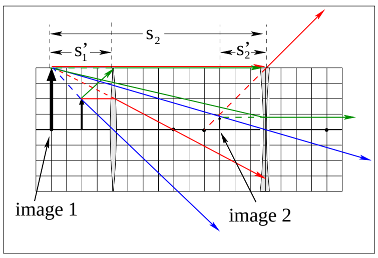
Solving a Two Lens Optical System-Solution



- The order of doing these things isn't very important. I did the ray diagram to the left first. Note that red = parallel ray, blue = central ray, green = focal ray (for both lenses). The focal ray for lens 1 becomes the parallel ray for lens 2 (so the rays overlap). Note that we can treat the image of the first lens *exactly* as if it is a (virtual) object for the second lens.

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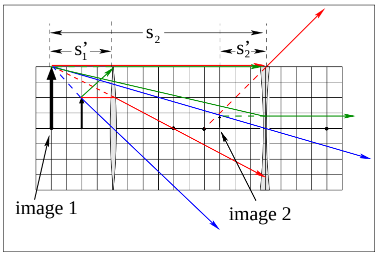

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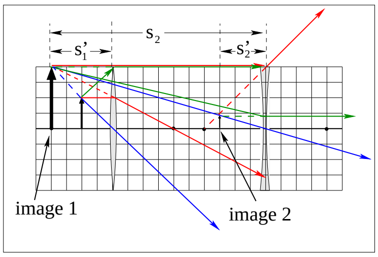

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- From the diagram alone I'd guess $s_1' = -4$ cm, $s_2 = 14$ cm, $s_2' \approx -3$ cm, $m_1 = +2$, $m_2 \approx 0.2$, $m_{\rm tot} \approx 0.4$. Let's see how this guess works out.

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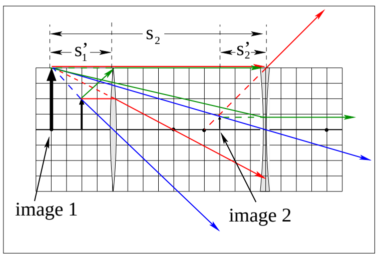

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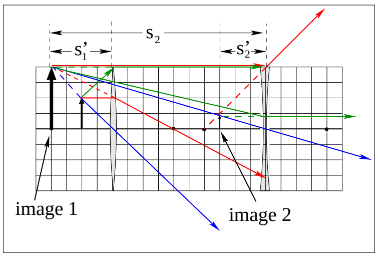
- First:

$$\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f_1} \Rightarrow \frac{1}{s_1'} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \Rightarrow \boxed{s_1' = -4 \text{ cm}} \Rightarrow \boxed{m_1 = -\frac{s_1'}{s_1} = +2}$$

Solving a Two Lens Optical System-Solution



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So far, so good.

Solving a Two Lens Optical System-Solution (Cont)

- Next, the lenses are 10 cm apart. We add $-s_1'$ to get $s_2 = +14$ cm, and then:

$$\frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f_2} \Rightarrow \frac{1}{s_2'} = -\frac{1}{4} - \frac{1}{14} = -\frac{18}{52} \Rightarrow \boxed{s_2' = -\frac{52}{18} \approx -3 \text{ cm}}$$

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Solving a Two Lens Optical System-Solution (Cont)

- Next, the lenses are 10 cm apart. We add $-s'_1$ to get $s_2 = +14$ cm, and then:

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Still right on the money, within a few percent.

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- Finally:

$$\boxed{m_{\rm tot} = m_1 m_2 = 2 \times 0.21 \approx 0.42}$$

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  Super! The numbers all correspond with the ray diagram within the acceptable limits imposed by line thickness, etc!

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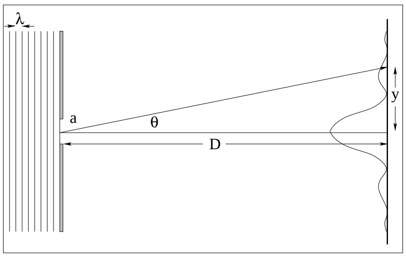
- We can categorize the images. Image 1 is erect and virtual. Image two is erect and virtual. The first is larger than the object, the second smaller than both object and first image. We are good to go!

Diffraction of Two Wavelengths Through a Slit



- A slit of width $a = 2800$ nm is illuminated by white light. We'd like to determine whether or not we expect the white light to be resolved into "rainbow colors" in any of the secondary maxima.

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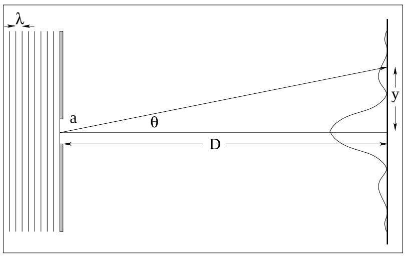

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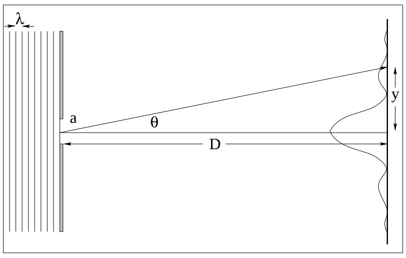

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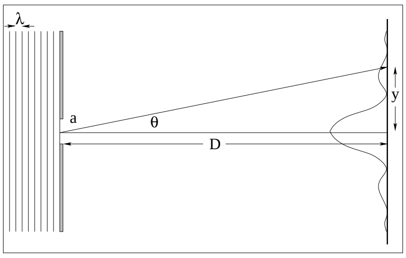

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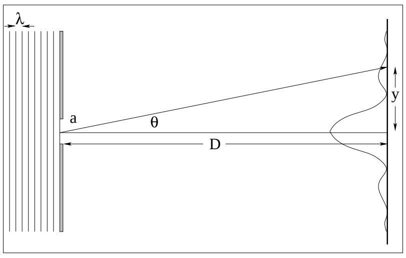

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Diffraction of Two Wavelengths Through a Slit-Solution

- Recall that minima occur when $a \sin\theta = m\lambda$ for $m = 1, 2, 3...$, or:

m	θ_r (red)	θ_v (violet)
1	$\sin^{-1}\frac{1}{4} = 14.5^\circ$	$\sin^{-1}\frac{1}{7} = 8.2^\circ$
2	$\sin^{-1}\frac{2}{4} = 30.0^\circ$	$\sin^{-1}\frac{2}{7} = 16.6^\circ$
3	$\sin^{-1}\frac{3}{4} = 48.6^\circ$	$\sin^{-1}\frac{3}{7} = 25.4^\circ$
4	$\sin^{-1}\frac{4}{4} = 90.0^\circ$	$\sin^{-1}\frac{4}{7} = 34.9^\circ$
5	NA	$\sin^{-1}\frac{5}{7} = 45.6^\circ$
6	NA	$\sin^{-1}\frac{6}{7} = 59.0^\circ$
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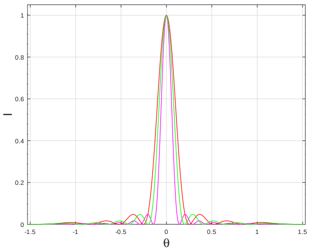
- Our estimates for the maxima are (from $a \sin\theta \approx \left(m + \frac{1}{2}\right)\lambda$):

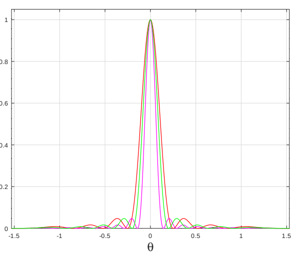
m	θ_r (red)	θ_v (violet)
1	NA	NA
2	$\sin^{-1}\frac{3}{8} = 22.0^\circ$	$\sin^{-1}\frac{3}{14} = 12.4^\circ$
3	$\sin^{-1}\frac{5}{8} = 38.7^\circ$	$\sin^{-1}\frac{5}{14} = 20.9^\circ$
4	$\sin^{-1}\frac{7}{8} = 61.0^\circ$	$\sin^{-1}\frac{7}{14} = 30.0^\circ$
5	NA	$\sin^{-1}\frac{9}{14} = 40.0^\circ$
6	NA	$\sin^{-1}\frac{11}{14} = 51.8^\circ$
7	NA	$\sin^{-1}\frac{13}{14} = 68.2^\circ$

Diffraction of Two Wavelengths Through a Slit-Solution (Cont)



- From this table we might well conclude that the spectrum *would* be resolved into colored bars in the higher order maxima and minima. However, by chance, the second violet maxima occurs right at the first red maximum, so those to particular colors aren't separated in the first bright sidebar.

Diffraction of Two Wavelengths Through a Slit-Solution (Cont)



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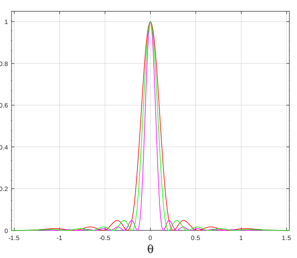
- However, other colors in between are! I plotted the actual diffraction intensity curves for red, green, and violet light; this fiuure makes it clear that green's first maximum happens at the second minimum of violet and close to the first minimum for red!

Diffraction of Two Wavelengths Through a Slit-Solution (Cont)



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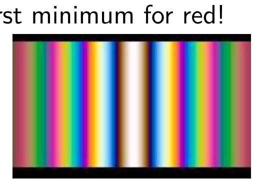

We therefore might reasonably expect the sidebars to be mixes of colors instead of white, and as you can see, for slit widths with many secondary maxima, they are!

The End

Feedback Welcome

Send Comments To: rgb at duke dot edu