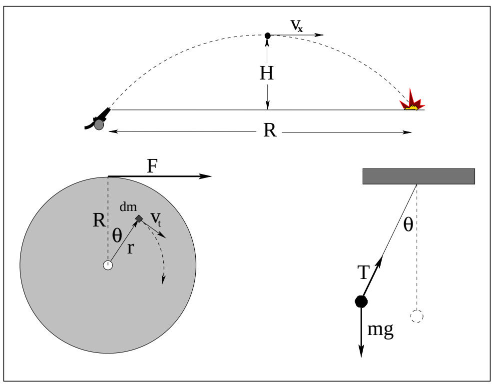
Introductory Physics 141/151/161

Self-Guided Learning Problems

Robert G. Brown

Summer, 2020



Contents

About These Problems

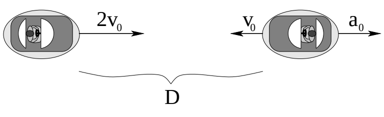
This is an experiment. Beamer allows me to make slides that will successively reveal lines of math-heavy text. This gives me a unique opportunity to build a collection of self-guided learning problems for physics that do what I've fantasized about doing for years now – present a problem, then provide a hint, then another hint, then another (or reveal a step, and then another step) until finally, the entire souution is presented, annotated.

Hopefully these problems will help students everywhere as they struggle to learn physics problems solving techniques and learn to "think like a physicist" as they do so.

To use this resource, pick a problem or topic from the table of contents and go directly to it, or work your way through all the problems systematically. Work on a separate sheet of paper, and when you get stuck, page down through the frames to see (hopefully) where you went wrong.

Remember, the point is to **master these problems**, not just to get through them. Make sure that before you are done, you can do every problem **without looking, without hints, and without remembering the exact solution** but rather, understanding *how* to find it!

Kinematics: Two Bumper Cars



- Two bumper cars are headed straight at one another, one travelling at $2v_0$ to the right, the other at speed v_0 to the left. When they are separated by a distance D, the car on the right slows down with a constant acceleration a_0. Does the right hand car manage to stop before being hit by the left hand car?

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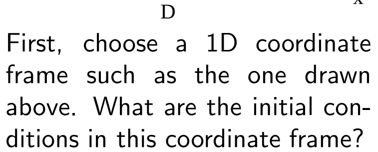

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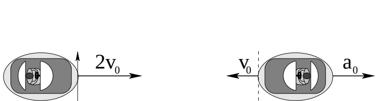

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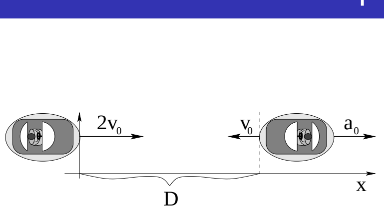

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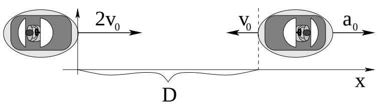

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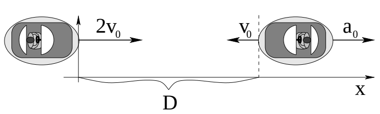

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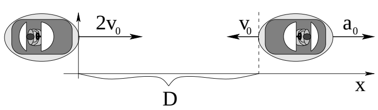

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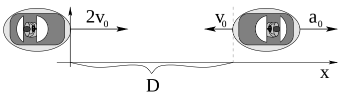

Solution: $x_l(t) = 2v_0 t, \quad v_l(t) = 2v_0$

$$x_r(t) = D - v_0 t + \frac{1}{2} a_0 t^2, \quad v_r(t) = -v_0 + a_0 t$$

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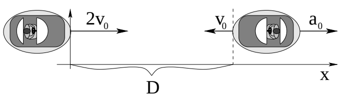

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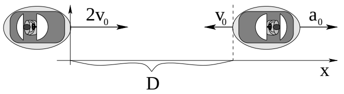

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Find the time: $t_r = \frac{v_0}{a_0}$. The left hand car is then at $x_l(t_r) = \frac{2v_0^2}{a_0}$ and the right hand car is at $x_r(t_r) = D - \frac{v_0^2}{a_0} + \frac{v_0^2}{2a_0} = D - \frac{v_0^2}{2a_0}$.

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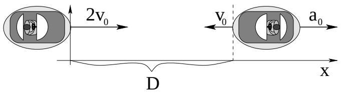

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$$D < \frac{5v_0^2}{2a_0}$$

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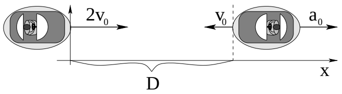

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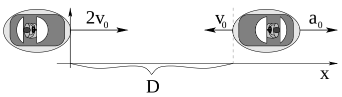

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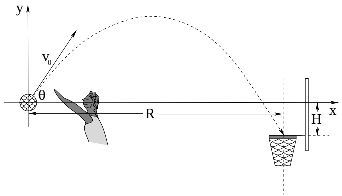
It makes sense – larger D makes it *less* likely to collide, larger a_0 makes it *less* likely they will collide, larger v_0 makes it *more* likely. Knowing they collide, if we write:

$$x_l(t_c) = 2v_0 t_c = D - v_0 t_c + \frac{1}{2} a_0 t_c^2 = x_r(t_c)$$

would let us find the time of collision and answer other questions about e.g. their relative velocity at that time. This is a simple quadratic equation for t_c.

Kinematics: 2D Basketball Trajectory



- A basketball player shoots a jump hook at a (horizontal) distance R from the basket, releasing the ball at a height H above the rim as shown. To shoot over his opponent's outstretched arm, he releases the basketball at an angle θ with respect to the horizontal.

Find v_0, the **speed** he must release the basketball with (in terms of H, R, g and θ) for the ball to go through the hoop "perfectly" as shown. Assume that his release is on line and undeflected, at initial speed v_0 and that the acceleration of the basketball is $\vec{a} = -g\hat{j}$, ignoring drag.

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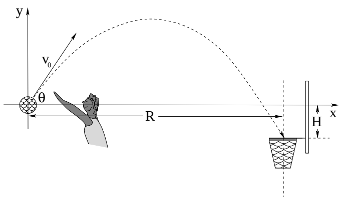

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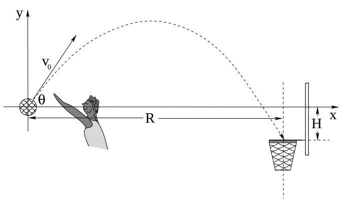

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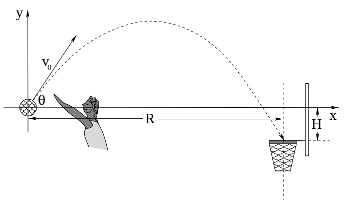

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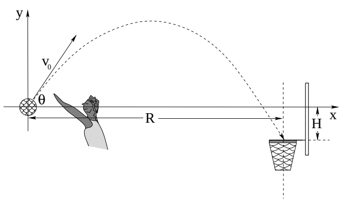

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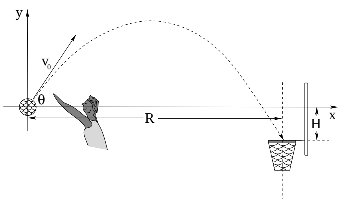

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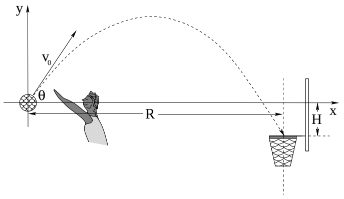

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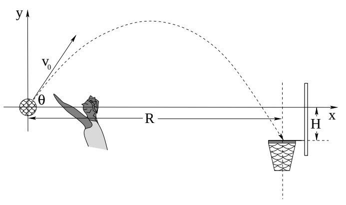

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The solution is on the next page. Don't advance until you are ready!

Kinematics: 2D Basketball Trajectory-Solution

Initial Conditions:

$$a_x = 0, v_{0x} = v_0 \cos\theta, x_0 = 0 \text{ and } a_y = -g, v_{0y} = v_0 \sin\theta, y_0 = 0$$

Integrate:

$$x(t) = v_0 \cos\theta t \quad v_x(t) = v_0 \cos\theta \quad y(t) = -\frac{1}{2}gt^2 + v_0 \sin\theta t \quad v_y(t) = v_0 \sin\theta - gt$$

Find the **time** t_b that the basketball reaches the horizontal position of the hoop:

$$R = v_0 \cos\theta t_b \Rightarrow t_b = R/(v_0 \cos\theta)$$

This must also be the time that the ball has exactly the height of the hoop:

$$-H = -\frac{1}{2}gt_b^2 + v_0 \sin\theta t_b \Rightarrow \frac{gR^2}{2v_0^2 \cos^2\theta} = R\tan\theta + H$$

And finally, we solve for v_0:

$$\boxed{v_0 = \sqrt{\frac{gR^2}{2(R\sin\theta\cos\theta + H\cos^2\theta)}}}$$

After doing the algebra, check the dimensions. Are they OK?

Firing a Speargun

Hints:

- An underwater fisherman fires her speargun at a distant fish. The neutral-buoyancy spear leaves the gun at initial speed v_0 and experiences a *linear* drag force $F_d = -bv$ opposite to its velocity. Find $v(t)$ and R, the maximum range of the spear.

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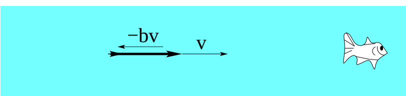

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Hints:
First, draw a picture of the *general* motion and drag force.

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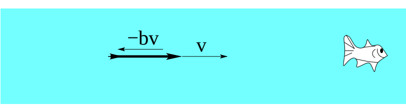

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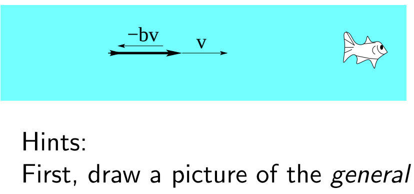
Hints:
First, draw a picture of the *general* motion and drag force. Then, write *Newton's Second Law*, expressing a in terms of v, not x.

Firing a Speargun



- An underwater fisherman fires her speargun at a distant fish. The neutral-buoyancy spear leaves the gun at initial speed v_0 and experiences a *linear* drag force $F_d = -bv$ opposite to its velocity. Find $v(t)$ and R, the maximum range of the spear.

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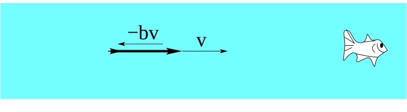

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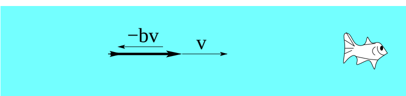

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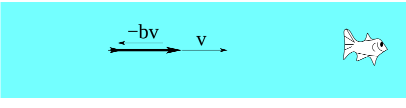

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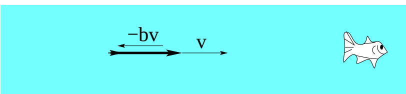

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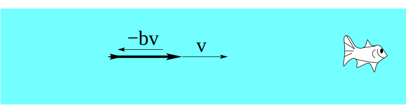

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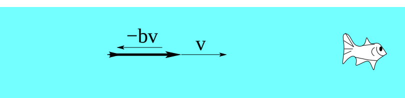

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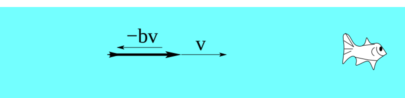

$$F = -bv = m\frac{dv}{dt} = ma \Rightarrow \frac{dv}{v} = -\frac{b}{m}dt$$

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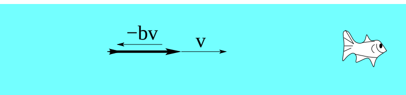

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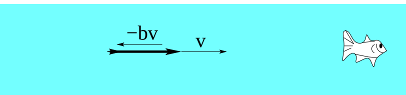

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$$\Rightarrow \ln\left(\frac{v}{v_0}\right) = -\frac{b}{m}t \Rightarrow e^{\left\{\ln\left(\frac{v}{v_0}\right) = -\frac{b}{m}t\right\}} \Rightarrow \frac{v}{v_0} = e^{-\frac{b}{m}t} \Rightarrow \boxed{v(t) = v_0 e^{-\frac{b}{m}t}}$$

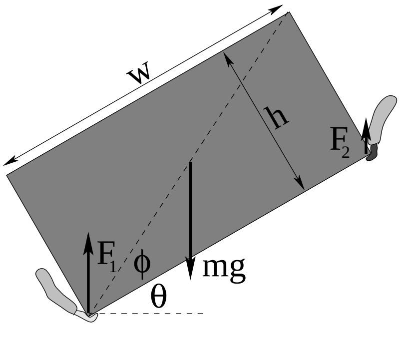
$$\frac{dx}{dt} = v_0 e^{-\frac{b}{m}t} \Rightarrow x(t) = \int_0^x dx = \int_0^t v_0 e^{-\frac{b}{m}t} dt = -\frac{mv_0}{b}\int_0^t e^{-\frac{b}{m}t}\left(-\frac{b}{m}\right)dt$$

$$x(t) = \frac{mv_0}{b}\int_{-bt/m}^{0} e^u du = \frac{mv_0}{b}\left(1 - e^{-\frac{b}{m}t}\right)$$

so the range R is:

$$\boxed{R = x(t \to \infty) = \frac{mv_0}{b}}$$

Static Equilibrium: Carrying A Box Up the Stairs



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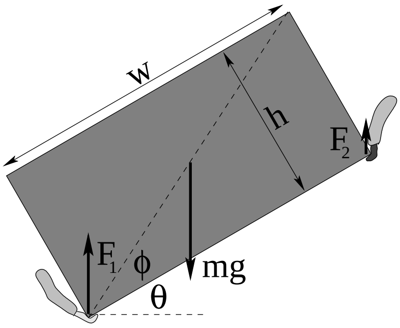

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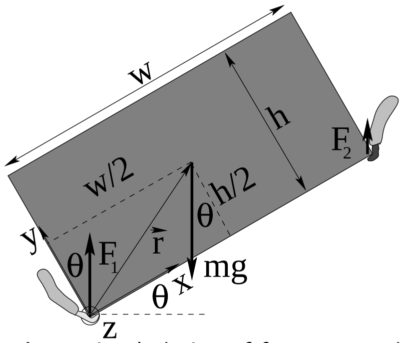

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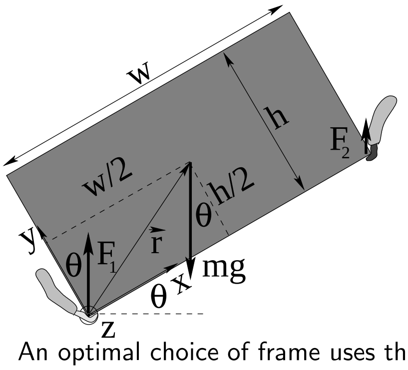

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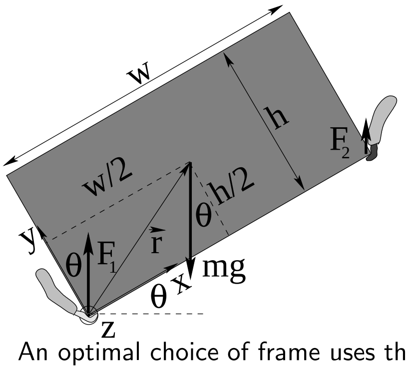

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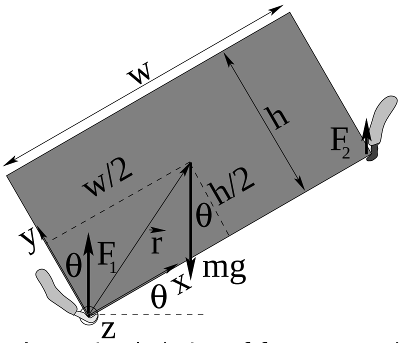

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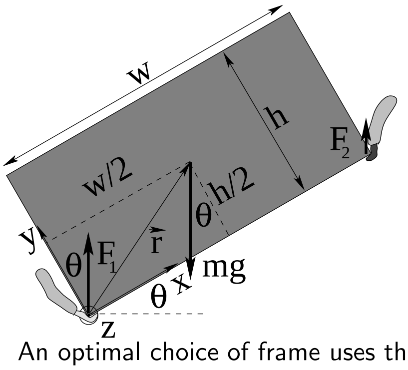

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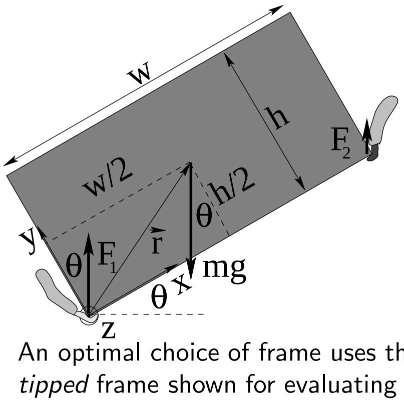

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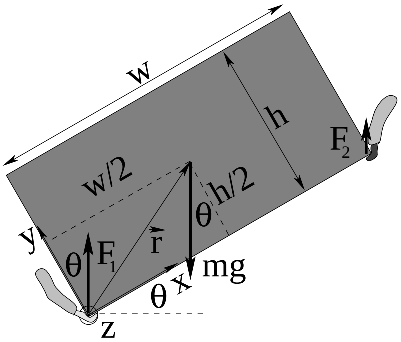

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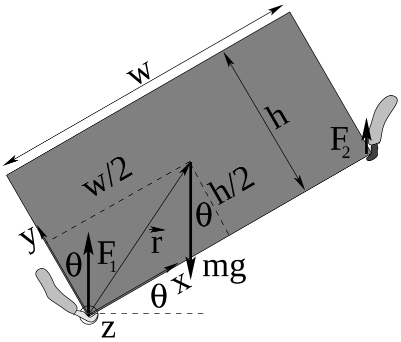

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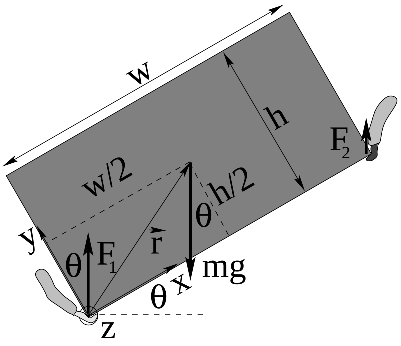

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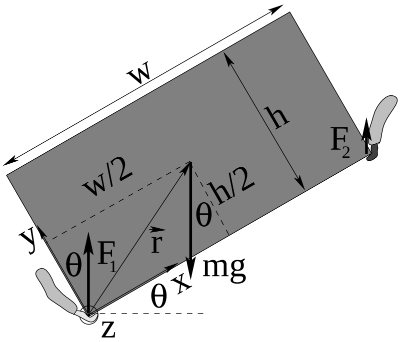

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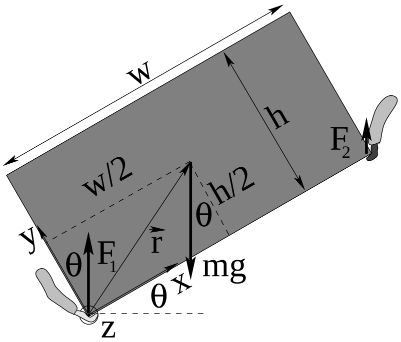

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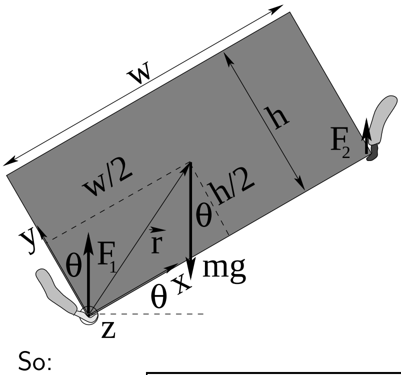

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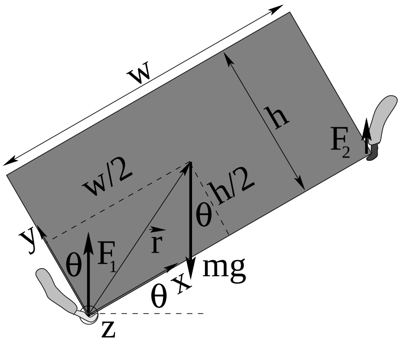

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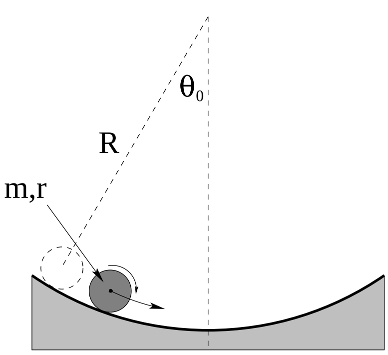
Note: $F_1 > F_2$ for all $\theta \in (0, \pi/2)$. It's better to be the student at the top! Note: $F_1 = mg$ when $\theta = \tan^{-1}\frac{w}{h}$. At this point $F_2 = 0$ and the student at the top is just helping balance the load!

Oscillations: A Rolling Pendulum



- A disk of mass m and radius r is gently set on a rough circular floor so that it makes an angle θ_0 relative to a vertical through the center of curvature of the floor, with its center of mass a distance R from the center of curvature as shown, and is released from rest at $t = 0$ so that it **rolls without slipping and oscillates**.
- Find: a) ω of the oscillation; b) $\theta(t)$; c) $f_s(t)$ (the force of static friction *as a function of time!*).

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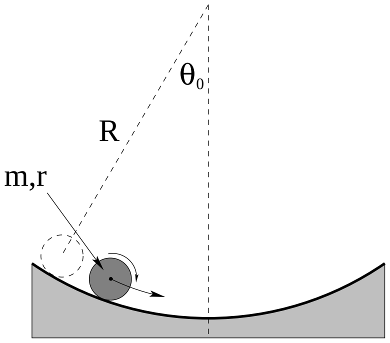

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Draw a force diagram on the rolling mass at a general angle θ.

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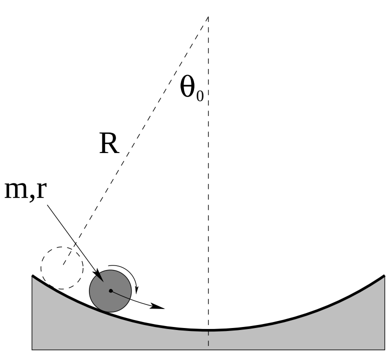

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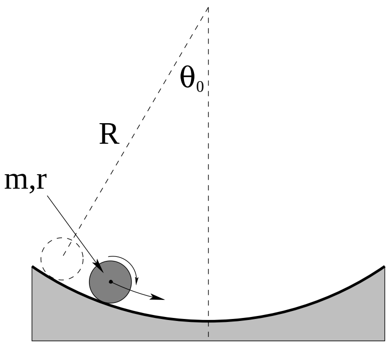
Draw a force diagram on the rolling mass at a general angle θ. Choose a positive direction for θ and (I recommend) choose to use $s = R\theta$ as a linear coordinate and choose a pivot either at the CM of the disk or the point where it touches the floor.

Oscillations: A Rolling Pendulum



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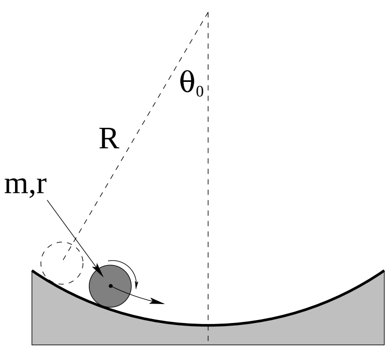
Oscillations: A Rolling Pendulum



- A disk of mass m and radius r is gently set on a rough circular floor so that it makes an angle θ_0 relative to a vertical through the center of curvature of the floor, with its center of mass a distance R from the center of curvature as shown, and is released from rest at $t = 0$ so that it **rolls without slipping and oscillates**.
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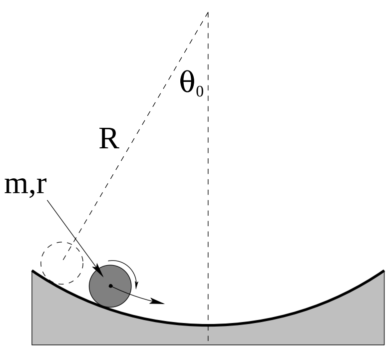

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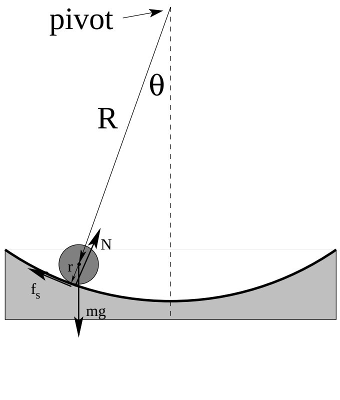
* - The solution is on the next page. Don't advance until you are ready!

Oscillations: A Rolling Pendulum-Solution



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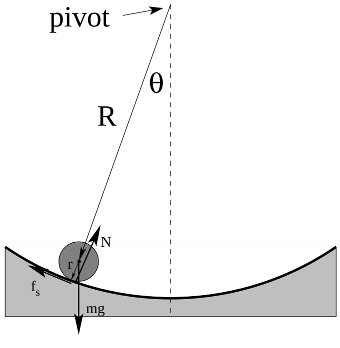

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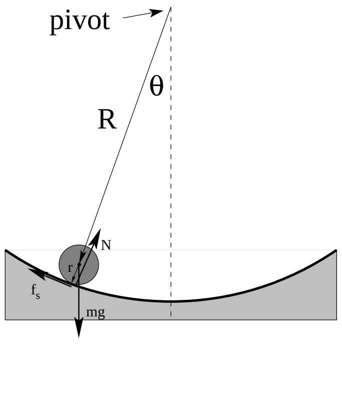

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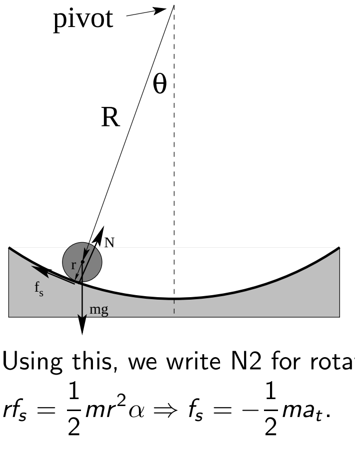

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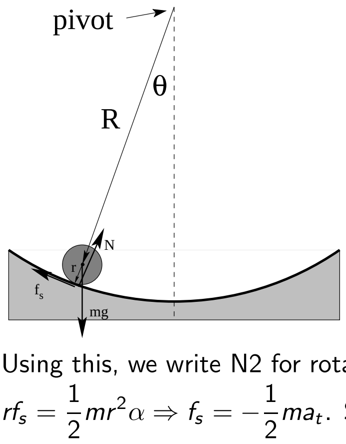

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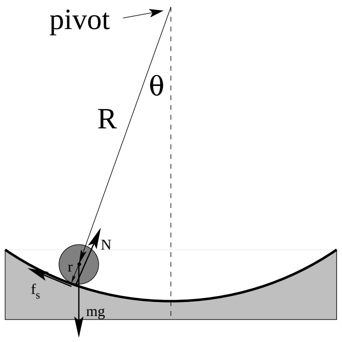

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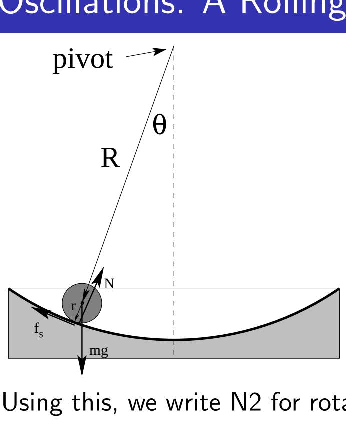

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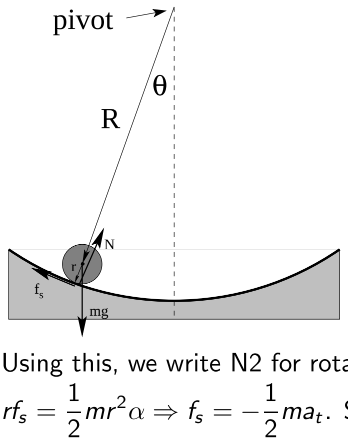

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$$\boxed{\omega = \sqrt{\frac{2g}{3R}}} \qquad \boxed{\theta(t) = \theta_0\cos(\sqrt{\frac{2g}{3R}}t)} \qquad \boxed{f_s = -\frac{1}{2}ma_t = \frac{mg}{3}\theta_0\cos(\sqrt{\frac{2g}{3R}}t)}$$

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- Note f_s always has the same sign as $\theta(t)$ – when it is positive (on the left half of the curved floor) f_s points up the incline, when it is negative (or the right half of the curved floor) f_s points *up the incline, to the right!* It is *symmetric*, as it must be as we could be viewing the solution from the other side of the page!

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  This too makes sense! At $t = 0$, the disk starts to roll *down to the right*, so $\Omega_{\rm disk}$ is into the page, positive. You should be able to trace each quarter cycle of its oscillation and see that everything is consistent and correct.

The End

Feedback Welcome

Send Comments To: rgb at duke dot edu