Useful Identities for Electrodynamics

Vector Identities

Triple Products

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

First Order Differential Product Rules

$$\vec{\nabla}(fg) = f(\vec{\nabla}g) + g(\vec{\nabla}f)$$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$$

Note: $(\vec{A}\cdot\vec{\nabla}) = A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}$

$$\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$$

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$$

Second Order Differential Product Rules

$$\vec{\nabla}\cdot(\vec{\nabla}f) = \nabla^2 f \text{ (definition)}$$

$$\vec{\nabla}\times(\vec{\nabla}f) = 0$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

Fundamental Theorems of Calculus

Gradient: $\int_{\vec{a}}^{\vec{b}}(\vec{\nabla}f)\cdot d\vec{\ell} = f(\vec{b}) - f(\vec{a})$

Divergence Theorem: $\int_{V/S}(\vec{\nabla}\cdot\vec{A})\;dV = \oint_S \vec{A}\cdot d\vec{a}$

Curl/Stokes Theorem: $\int_{S/C}(\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint_C \vec{A}\cdot d\vec{\ell}$

Cartesian Coordinates

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

Line: $d\vec{\ell} = dx\hat{x} + dy\hat{y} + dz\hat{z}$

Area: $dA = dx\;dy$ (e.g.)

Volume: $dV = dx\;dy\;dz$

Gradient: $\vec{\nabla}f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$

Divergence: $\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

Curl: $\vec{\nabla}\times\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$

Laplacian: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Spherical Coordinates

$$\vec{r} = r\hat{r} + r\theta\hat{\theta} + r\sin(\theta)\phi\hat{\phi}$$

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

$$\hat{r} = \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z}$$

$$\hat{\theta} = \cos(\theta)\cos(\phi)\hat{x} + \cos(\theta)\sin(\phi)\hat{y} - \sin(\theta)\hat{z}$$

$$\hat{\phi} = -\sin(\phi)\hat{x} + \cos(\phi)\hat{y}$$

$$x = r\sin(\theta)\cos(\phi)$$

$$y = r\sin(\theta)\sin(\phi)$$

$$z = r\cos(\theta)$$

$$r = +(x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

Line: $d\vec{\ell} = dr\;\hat{r} + rd\theta\;\hat{\theta} + r\sin(\theta)d\phi\;\hat{\phi}$

Area: $dA = r^2\sin(\theta)d\theta\;d\phi$

Volume: $dV = r^2\sin(\theta)dr\;d\theta\;d\phi$

Gradient: $\vec{\nabla}f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial\phi}\hat{\phi}$

Divergence:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(\sin(\theta)A_\theta) + \frac{1}{r\sin(\theta)}\frac{\partial A_\phi}{\partial\phi}$$

Curl:

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial\theta}(\sin(\theta)A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\boldsymbol{r}}$$

$$+ \quad \frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\boldsymbol{\theta}}$$

$$+ \quad \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial f}{\partial r}) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}(\sin(\theta)\frac{\partial f}{\partial\theta}) + \frac{1}{r^2\sin^(\theta)}\frac{\partial^2 f}{\partial\phi^2}$$

Integration by Parts (examples)

$$\vec{\boldsymbol{\nabla}} \cdot (f\vec{\boldsymbol{A}}) = f(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}) + (\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{\nabla}})f$$

$$\int_{V/S} \vec{\boldsymbol{\nabla}} \cdot (f\vec{\boldsymbol{A}}) dV = \int_{V/S} f(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}) + \int_{V/S} (\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{\nabla}})f$$

$$\oint_S (f\vec{\boldsymbol{A}}) \cdot d\vec{\boldsymbol{a}} = \int_{V/S} f(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}) + \int_{V/S} (\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{\nabla}})f \quad \text{(divergence theorem)}$$

$$\int_{V/S} f(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}) = \oint_S (f\vec{\boldsymbol{A}}) \cdot d\vec{\boldsymbol{a}} - \int_{V/S} (\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{\nabla}})f \quad \text{(integration by parts, or:)}$$

$$\int_{V/S} (\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{\nabla}})f = \oint_S (f\vec{\boldsymbol{A}}) \cdot d\vec{\boldsymbol{a}} - \int_{V/S} f(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}) \quad \text{(integration by parts)}$$

$$\vec{\boldsymbol{\nabla}} \times (f\vec{\boldsymbol{A}}) = f(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) - (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{\nabla}})f$$

$$\int_{S/C} \vec{\boldsymbol{\nabla}} \times (f\vec{\boldsymbol{A}}) \cdot d\vec{\boldsymbol{a}} = \int_{S/C} f(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) \cdot d\vec{\boldsymbol{a}} - \int_{S/C} (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{\nabla}})f \cdot d\vec{\boldsymbol{a}}$$

$$\oint_C f\vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{\ell}} = \int_{S/C} f(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) \cdot d\vec{\boldsymbol{a}} - \int_{S/C} (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{\nabla}})f \cdot d\vec{\boldsymbol{a}} \quad \text{(Stokes' theorem)}$$

$$\int_{S/C} f(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) \cdot d\vec{\boldsymbol{a}} = \oint_C f\vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{\ell}} + \int_{S/C} (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{\nabla}})f \cdot d\vec{\boldsymbol{a}} \quad \text{(integration by parts)}$$