Equations Du Jour

for Physics 52

Robert G. Brown

Physics Department, Duke University

Note: These notes are provided for the convenience and edification of students in my classes (or even other classes). They are not to be removed from the library! Photocopy them if you like, but bring them back so others may benefit from them as well. If you are so poor as to be unable to afford to photocopy them, seek financial support from appropriate sources on campus or come ask me! Copying them by hand on a weekly basis would only be good for you as well in your quest to learn physics. BUT:

IF YOU STEAL THESE NOTES GOD WILL SEE YOU!

What more can I say.

R. G. Brown

Equations Du Jour

A leisurely journey through physics 52 with nothing but the essential facts and derivations on a lecture by lecture basis.

EDJ 1. The Electric Field

True Facts:

1) Charge is quantized. Charge is conserved. Charge has (microscopic) units called electrostatic units = esu which is the charge on a single proton or electron. Charge has macroscopic units (SI) called Coulombs.

$$1 \text{ esu} = 1.6 \times 10^{-19} \text{Coulombs}$$

2) Pairs of charged objects are observed in Nature to exert forces on one another. This force is observed to be proportional to both charges and inversely proportional to the square of the distance separating them (if they are "small" i.e.--point like objects) and acts along the straight line joining the objects. We summarize this as Coulomb's Law:

$$\vec{F} = \frac{kq_1q_2}{|r_{12}|^2}\hat{r}_{12} \qquad \text{with } k = 9 \times 10^9 \frac{N - m^2}{C}.$$

3) Forces are what we measure (with spring balances and the like). However, our minds like to think that things (including forces) have causes. With the electric force, there are "no strings" attached to the two charges, and it is difficult to imagine what causes the force. To alleviate this difficulty, we make up something that "connects" the two charges. It is something we imagine to be at all points in space surrounding a charge, and the effect of this something is to cause another charge, placed at one of those points, to feel a certain force. We call this something "The Electric Field". Its definition is:

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}.$$

Practically speaking, this definition is equivalent to

$$\vec{E} = \frac{kq}{r^2}\hat{r} \qquad \text{for point charges.}$$

EDJ 2. The Electric Field (continued).

Recall from last time: Charge is quantized. Charge is conserved.

Coulomb's Law (for point charges)

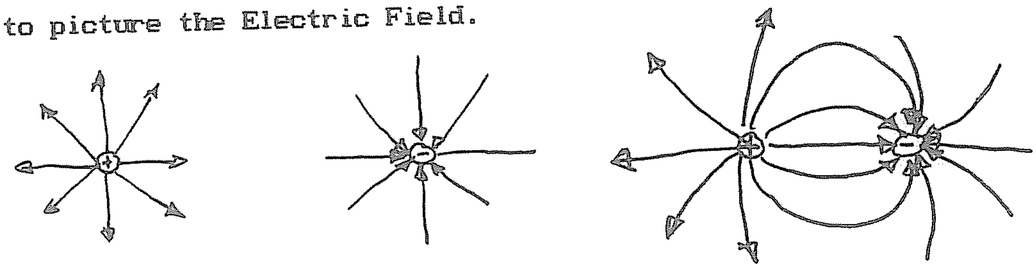
$$\vec{F} = \frac{kq_1q_2}{|r_{12}|^2}\hat{r}_{12} \qquad \text{with } k = 9 \text{ X } 10^9 \ \frac{N - m^2}{C^2}.$$

The Electric Field (for point charges)

$$\vec{E} = \frac{kq}{r^2}\hat{r}.$$

Today we will cover:

1) How to draw Lines of Force. These are just a convenient way to picture the Electric Field.



2) How to evaluate the Electric Field (or force...) at a point in space resulting from one, two, or more point charges.

This is done by using the <u>superposition principle</u> for the electric field. The electric field at a given point in space is the sum of the fields due to all the charges in space:

$$\vec{E}(\vec{r}) = \sum_i \frac{kq_i}{r_i^2}\hat{r}_i$$

where $\hat{r}_i$ is a unit vector from charge q_i to the point of observation, $\vec{r}$. We will apply this to one and two point charges located on the y-axis.

If the charge is continuously distributed throughout some region of space, we use the integral version of this equation:

$$\vec{E}(\vec{r}) = \int_{\mathbb{R}^3} k\rho(\vec{r}_0)\frac{(\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^3}d^3r_0.$$

Note that $\rho(\vec{r}_0)$ is the continuous charge density distribution discussed below; the $\vec{r}/|\vec{r}|^3$ term is just a unit vector over r^2 in the direction of $\vec{r} - \vec{r}_0$. We will apply this to a ring of charge, a disk of charge, and a line of charge.

Lecture Notes

These notes accompany the EDJ to describe physics 52. They contain the derivations omitted from the EDJ and an outline.

Lecture 1. The Electric Field

1) Introduction and greetings. Syllabus, Study guide, Survey. (No Lab this week).

2) Drop/add in Physics Bldg today.

3) Motivate study pattern/describe course, grading, etc.

4) Motivate study of E&M. ← refer students to chap. 1.

a) 4 forces

i) Nuclear forces important only on nuclear scale (10^{-15}m). Only importance to chemistry is that massive, positive, point-like nuclei exist.

ii) Gravity only important on cosmological scale. (10^{8+}m). Can't even see it unless objects of planetary mass available.

Electric Forces dominate our experience. They are important from 10^{-10}m - 10^{+10} and are responsible from properties of atoms, chemistry, biology (life) and light (what we see). They are ubiquitous! We are (for all practical purposes) electromagnetic phenomena!

So, we study Electromagnetism very carefully. In doing so we come to understand the idea of "field" that frees us from the "action at a distance" dilemma.

Lecture actually begins here.

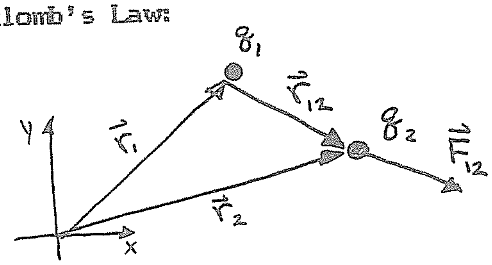
5) So, what is Electromagnetism, where does it come from? Read text for history. It is forces between charges.

True Facts:

1) Charge is quantized. Charge is conserved. Charge has (microscopic) units called electrostatic units = esu which is the charge on a single proton or electron. Charge has macroscopic units (SI) called Coulombs.

$$1 \text{ esu} = 1.6 \times 10^{-19} \text{Coulombs}$$

2) Pairs of charged objects are observed in Nature to exert forces on one another. This force is observed to be proportional to both charges and inversely proportional to the square of the distance separating them (if they are "small" i.e.--point like objects) and acts along the straight line joining the objects. We summarize this as Coulomb's Law:

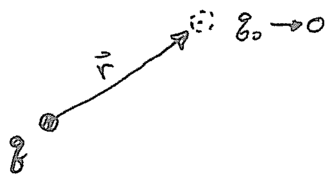


$$\vec{F} = \frac{kq_1q_2}{|r_{12}|^2}\hat{r}_{12} \qquad \text{with } k = 9 \times 10^9 \frac{N - m^2}{C}.$$

3) Forces are what we measure (with spring balances and the like). However, our <u>minds</u> like to think that things (including forces) have causes. With the electric force, there are "no strings" attached to the two charges, and it is difficult to imagine what causes the force. To alleviate this difficulty, we <u>make up</u> something that "connects" the two charges. It is something we imagine to be at all points in space surrounding a charge, and the effect of this something is to cause another charge, placed at one of those points, to feel a certain force. We call this something "The Electric Field". Its <u>definition</u> is:

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}. \quad = \quad \frac{k\, q\, q_0\, \vec{r}_0}{r_0^2\, q_0}$$

Practically speaking, this definition is equivalent to



$$\vec{E} = \frac{kq}{r^2}\hat{r} \qquad \text{for point charges.}$$

3) To do this, we need the following definitions:

Charge per Unit Volume/Area/Length

Volume charge density

i) $\rho = \frac{\Delta Q}{\Delta V} \approx \frac{dQ}{dV}$

Area charge density

ii) $\sigma = \frac{\Delta Q}{\Delta A} \approx \frac{dQ}{dA}$

Length charge density

iii) $\lambda = \frac{\Delta Q}{\Delta L} \approx \frac{dQ}{dL}$

So, $\Delta Q = \int_{\Delta V} \rho dV$ or $\Delta Q = \int_{\Delta A} \sigma dA$ or $\Delta Q = \int_{\Delta L} \lambda d\ell$ is the total charge contained in some volume/area/length of material $\Delta V/\Delta A/\Delta L$.

4) In order to understand our results we frequently want to discover what our fields look like when $\vec{r}$ is much greater than or less than the dimensions of the charge distribution. To do that we need and should remember the Binomial Expansion:

If $|y|<1$ then we can write

$$(1 + y)^n = 1 + ny + \frac{n(n-1)y^2}{2!} + \frac{n(n-1)(n-3)y^3}{3!} + \ldots$$

Since $|y|<1$ the terms get smaller rapidly. We rarely keep more than the first surviving term (other than 1).

Please note that in the EDJ I will not usually put down explicitly the equations that result from, for example, the application of $\vec{E} = \frac{kq}{r^2}\hat{r}$ to several point charges. That is because these results are not general! I want you to learn equations on the EDJ and be able to apply them to any simple problem that comes along, not just the ones presented in lecture. In general the EDJ will contain laws of physics, useful equations and concepts, and occasionally archetypical examples.

Lecture 2. The Electric Field (continued).

Recall from last time: Charge is quantized. Charge is conserved.

Coulomb's Law (for point charges)

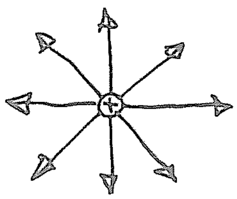
$$\vec{F} = \frac{kq_1q_2}{|\vec{r}_{12}|^2}\hat{r}_{12} \qquad \text{with } k = 9 \times 10^9 \ \frac{N - m^2}{C}.$$

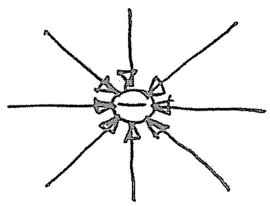
The Electric Field (for point charges)

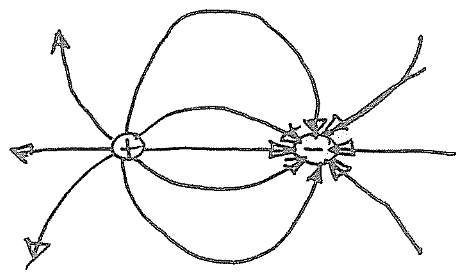
$$\vec{E} = \frac{kq}{|\vec{r}|^2}\hat{r}.$$

Today we will cover:

1) How to draw Lines of Force. These are just a convenient way to picture the Electric Field.







Rules: 1) Lines originate and end on charges.

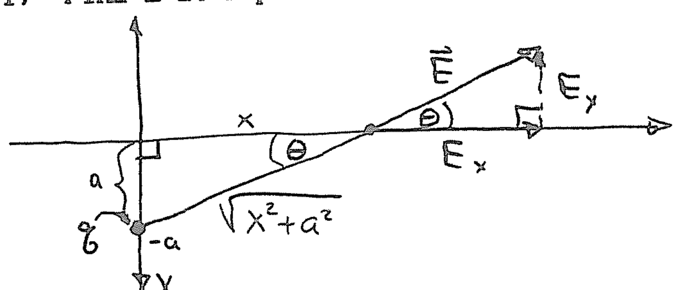
2) No two lines cross.

3) The number of lines entering or leaving a charge is proportional to the magnitude of the charge.

THE WHOLE IDEA is that the # of lines per unit area passing through a give surface is proportional to the magnitude of the field. Remember this idea; it is Gauss's Law and we will cover it in gory detail next week.

2) How to evaluate the Electric Field (or force...) at a point in space resulting from one, two, or more point charges.

i) Find $\vec{E}$ at a point on the x-axis in the diagram below.



Solution: $|\vec{E}| = \frac{kq}{r^2}$ where $r = (x^2 + a^2)^{1/2}$. From this we find the components $E_x = |E|\cos\theta$ and $E_y = |E|\sin\theta$. But we also need to find $\cos\theta = \frac{x}{r} = \frac{x}{(x^2 + a^2)^{1/2}}$

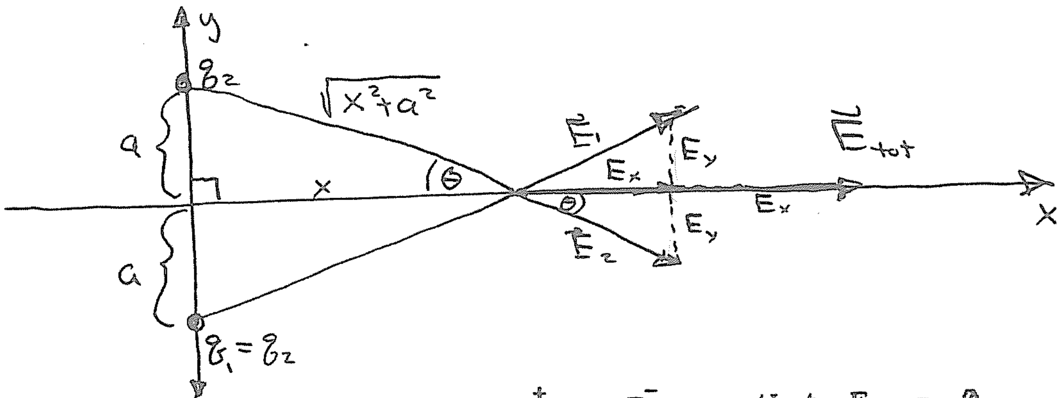
and $\sin\theta = \frac{a}{r} = \frac{a}{(x^2 + a^2)^{1/2}}$.

Putting the parts together yields

$$E_x = \frac{kqx}{(x^2 + a^2)^{3/2}} \quad \text{and} \quad E_y = \frac{kqa}{(x^2 + a^2)^{3/2}}.$$

Since two components specify a vector (in the plane) we are done.

ii) Find the $\vec{E}$-field on the x-axis from the diagram below. Note that this is a popular sort of quiz question.



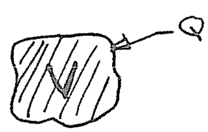
Solution: First of all, observe that $E_x^+ = -E_x^-$, so that $E_x = 0$ (if you don't see this immediately work it out on your own). Note further that $E_y^+ = E_y^-$, so that $E_y = 2E_y^+$. Following what we did in i), $|E^+| = |E^-| = \frac{kq}{r^2}$, $E_y^+ = |E^+|\sin\theta$, so putting it all together we get $\vec{E} = \frac{-2kqa}{(x^2 + a^2)^{3/2}}\hat{y}$.

3) How to use $\vec{E} = \frac{kq}{r^2}\hat{r}$ to evaluate $\vec{E}$ for (arbitrary?) charge distributions with applications. To do this we need to define a few things:

Charge per Unit Volume/Area/Length

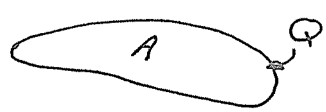
Volume charge density

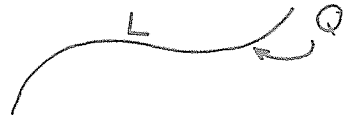
i) $\rho = \frac{\Delta Q}{\Delta V} \approx \frac{dQ}{dV}$



Area charge density

ii) $\sigma = \frac{\Delta Q}{\Delta A} \approx \frac{dQ}{dA}$



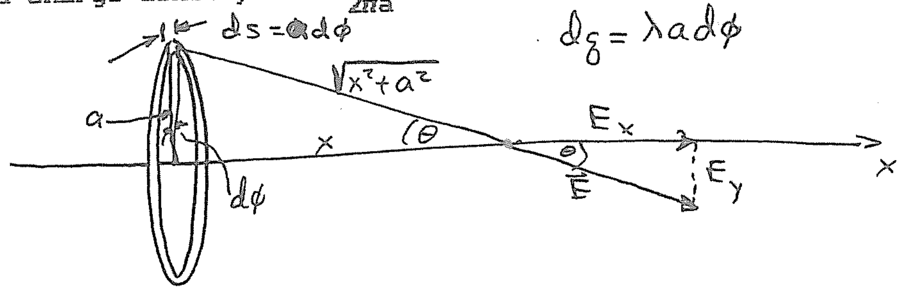


Length charge density

iii) $\lambda = \frac{\Delta Q}{\Delta L} \approx \frac{dQ}{dL}$

So, $\Delta Q = \int_{\Delta V} \rho dV$ or $\Delta Q = \int_{\Delta A} \sigma dA$ or $\Delta Q = \int_{\Delta L} \lambda d\ell$ is the total charge contained in some volume/area/length of material $\Delta V/\Delta A/\Delta L$. Got it? Then we

1) Find $\vec{E}$-field of ring of charge at an arbitrary point on (its) x-axis. The ring has total charge Q and radius a, so it has a charge density $\lambda = \frac{Q}{2\pi a}$.



Solution: We only know the electric field of a point charge. So we look at the electric field produced by the little "point" charge of length $d\ell$ at the top of the ring. Recall $d\ell = a d\phi$ is the length of an arc in terms of its subtended angle. Then the charge in the length $d\ell$ is $dQ = \lambda d\ell = \frac{Q d\phi}{2\pi}$. Then we have

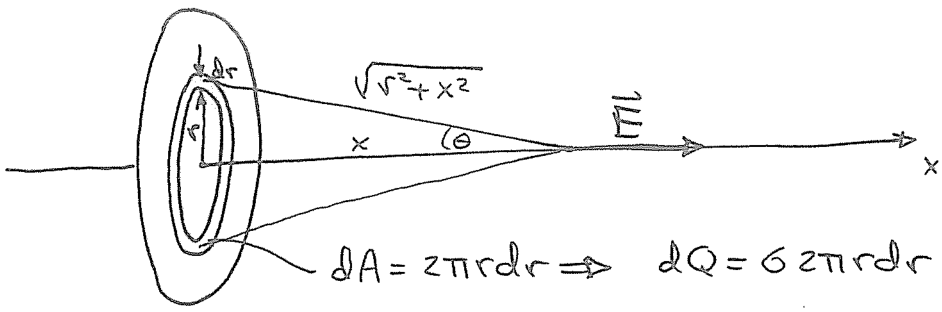
$|dE| = \frac{k\, dQ}{r^2}$ with $r^2 = (x^2 + a^2)$ as usual. Then

$dE_x = |dE| \cos\theta$ with $\cos\theta = \frac{x}{r}$ and

$dE_p = |dE| \sin\theta$. Note that $E_p = \int_0^{2\pi} dE_p = 0$ from symmetry. $dE_x = \frac{kQx\,d\phi}{2\pi\,(x^2 + a^2)^{3/2}}$, so

$$E_x = \int_0^{2\pi} dE_x = \frac{kQx}{(x^2 + a^2)^{3/2}}$$ and we are done.

2) Find the $\vec{E}$-field of a symmetric disk of total charge Q at an arbitrary point on the x-axis.



Solution: Consider a disk to be a lot of concentric rings. We know the field of a ring of radius r is (from above) $E_x = \frac{kQ_{ring}\,x}{(x^2 + r^2)^{3/2}}$, so we just add 'em up (by integrating, of course). To do this, we note that

$$Q_{ring} = dQ = \frac{Q}{\pi R^2}\,dA = \sigma dA = \sigma 2\pi r dr$$

(where Q is the total charge on the DISK and $dA = 2\pi r dr$ for the differential ring of radius r and width dr. That is,

$$dE_x = \frac{kQx(2\pi r dr)}{\pi R^2(x^2 + r^2)^{3/2}} = \frac{2kQxrdr}{R^2(x^2 + r^2)^{3/2}}$$

and

$$E_x = \frac{kQx}{R^2}\int_0^R \frac{2rdr}{(x^2 + r^2)^{3/2}} = \frac{2kQ}{R^2}\left\{1 - \frac{x}{(x^2 + R^2)^{1/2}}\right\}$$

$$= 2k\pi\,\sigma\left\{1 - \frac{x}{(x^2 + R^2)^{1/2}}\right\}$$

$E_p = 0$ (from symmetry).

We can now apply the BINOMIAL EXPANSION and obtain the limiting behavior of the field, i.e.--we can see what it looks like when $x \to \infty$ or $x \to 0$.

Recall (or learn for the first time) the Binomial Expansion. If $|y| < 1$ then we can write

$$(1 + y)^n = 1 + ny + \frac{n(n-1)y^2}{2!} + \frac{n(n-1)(n-3)y^3}{3!} + \ldots$$

Since $|y| < 1$ the terms get smaller rapidly. We rarely keep more than the first surviving term (other than 1).

Suppose, then, that we want to find E_x as $x \to \infty$. We look at the second term in the brackets $\{\}$ above:

$$\frac{x}{(x^2 + R^2)^{1/2}} = \frac{x}{x(1 + R^2/x^2)^{1/2}} = (1 + \frac{R^2}{x^2})^{-1/2}$$

$$= 1 - \frac{1}{2}\frac{R^2}{x^2} + O(\frac{1}{x^4}) + \ldots$$

(The last is pronounced "and terms of the order x^{-4} and higher" and just means they are so small we're not going to worry about them.) Substituting this into our result for E_x above, we get:

$$E_x = 2\pi k\sigma\left\{1 - \left[1 - \frac{1}{2}\frac{R^2}{x^2}\right]\right\} = \frac{kQ}{x^2} \;!$$

This just means that when we are far away from the disk ($x \gg R$) its electric field looks like that of a point charge. What does the field look like when $x \ll R$?

Next time: The electric field of an infinite line of charge--the hard way!

EDJ 3. The Electric Field (continued).

Recall from last time: the superposition principle for the electric field.

$$\vec{E}(\vec{r}) = \sum_i \frac{kq_i}{r_i^2}\hat{r}_i \quad \text{(recall point charges)}$$

or

$$\vec{E}(\vec{r}) = \int_{\mathbb{R}^3} k\rho(\vec{r}_0)\frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}d^3r_0. \quad \text{(recall ring/disk)}$$

Where

Volume charge density

i) $\rho = \frac{\Delta Q}{\Delta V} \approx \frac{dQ}{dV}$

Area charge density

ii) $\sigma = \frac{\Delta Q}{\Delta A} \approx \frac{dQ}{dA}$

Length charge density

iii) $\lambda = \frac{\Delta Q}{\Delta L} \approx \frac{dQ}{dL}$

So, $\Delta Q = \int_{\Delta V}\rho dV$ or $\Delta Q = \int_{\Delta A}\sigma dA$ or $\Delta Q = \int_{\Delta L}\lambda d\ell$ is the total charge contained in some volume/area/length of material $\Delta V/\Delta A/\Delta L$.

Also recall the Binomial Expansion:

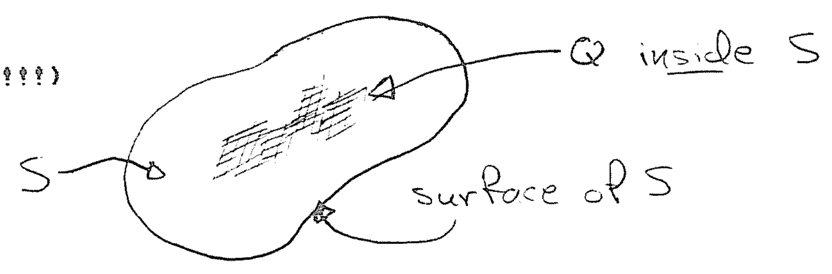
If $|y|<1$ then we can write

$$(1+y)^n = 1 + ny + \frac{n(n-1)y^2}{2!} + \frac{n(n-1)(n-3)y^3}{3!} + \ldots$$

Today we will apply the definition of the electric field to another continuous charge distribution,

1) the (infinite) line of charge. We will use the integral rule above, which is the "hard way". The easy way, as we shall see, is to use

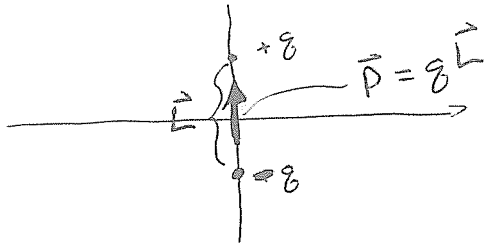
2) Gauss's Law (!!!)



$$\oint_S \vec{E}\cdot\hat{n}\, dA = 4\pi kQ_{\text{enclosed by } S}.$$

3) Before we discuss Gauss's Law (except to let you see it for the future, we do Dipoles. An electric dipole consists of two charges of equal and opposite charge separated by a (directed)

length $\vec{L}$.



Where $\vec{p}$ (the dipole moment) is given by

$$\vec{p} = q\vec{L}.$$

Then (true facts) the torque on the dipole in a uniform electric field $\vec{E}$ is

$$\vec{\tau} = \vec{p} \times \vec{E}.$$

The (potential) energy of the dipole is

$$U = -pE\cos\theta = -\vec{p}\cdot\vec{E}$$ $(dU = \tau d\theta = pE\sin\theta d\theta)$

and the net force on the dipole is

$$\vec{F} = 0.$$

In a non-uniform electric field the dipole may experience simulataneously a torque and a non-zero force. Atoms in electric fields tend to polarize and experience a dipole-dipole attraction (Van der Waals force).

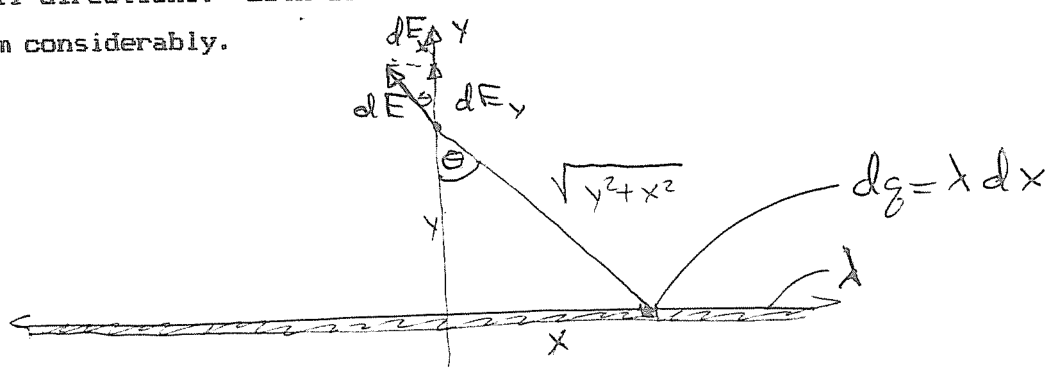
Then, if we have time, we will start Chapter 21 (Gauss' Law) by defining

4) Electric Flux: The "flow" of an electric field through a surface. It is the product of the surface area with the component the vector electric field that is normal, or perpindicular to the surface on a point by point basis. Generally, for $\vec{E}$ varying in both magnitude and angle over an arbitrary surface,

$$\phi = \int_S \vec{E}\cdot\hat{n}\, dA$$

Lecture 3. The Electric Field (continued).

Today we will apply the definition of the electric field to the infinite line of charge. We will integrate the expression containing (in this case) the charge distribution λ to obtain the electric field some distance from the line. First, note that the ∞ line has two useful symmetries. Any point can be the "middle", or every point on the line splits it in half. Also, if we are on a point some distance y away from the line, the line is <u>cylindrically symmetric</u>, that is, if we walk around the line it looks the same from all directions. Both of these observations will simplify the problem considerably.



Above is pictured the "infinite" line of charge with charge per unit length λ. We note from symmetry that the field must be perpindicular to the line. The geometry of the integral is indicated on the figure.

Solution: We know only that the field of a point charge is $\vec{E} = \frac{kq}{r^2}\hat{r}$. We therefore select a little chunk of the line of differential thickness (length dx). The charge in this (point like) chunk is $dQ = \lambda dx$. The magnitude of the electric field produced by this chunk is

$$|d\vec{E}| = \frac{kdq}{r^2} = \frac{k\lambda dx}{(x^2 + y^2)}$$

and its direction is as shown above. The component perpendicular to the line of charge is $dE_y = |dE|\cos\theta$ (with θ as drawn). since $\cos\theta = y/r$,

$$dE_y = \frac{(k\lambda y)dx}{(x^2 + y^2)^{3/2}}.$$

$dE_x = |dE| \sin\theta$ <u>cancels</u> since there is always an equal and opposite contribution from the other half of the line, so $dE_x = E_x = 0$. We want to do the integral

$$E_y = \int dE_y = \int_{-\infty}^{\infty} \frac{(k\lambda y)dx}{(x^2 + y^2)^{3/2}} . = \int \frac{k\lambda y dx}{r^3}$$

This is very difficult to do unless we change variables from x to θ, using the observation that

$$x = y \tan\theta$$

so

$$dx = y \sec^2\theta d\theta = y\left[\frac{r}{y}\right]^2 d\theta. \qquad \left(\sec\theta = \frac{1}{\cos\theta} = \frac{1}{y/r} = r/y\right)$$

Substituting these values into the equation above, we get

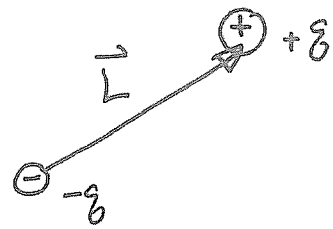
$$dE_y = \int_{-\theta_1}^{\theta_2} \frac{k\lambda}{y} \cos\theta \, d\theta \qquad \left(\frac{y}{r}\right)$$

where $\theta_2 = \theta_1 = \tan^{-1}\frac{L}{2y} = \tan^{-1}(\infty) = 90^\circ$. For the infinite line, then,

$$E_y = \frac{2k\lambda}{y}$$

We will see on Monday that this is very much the "hard way" to get this equation. The "easy way" is to use Gauss' Law.

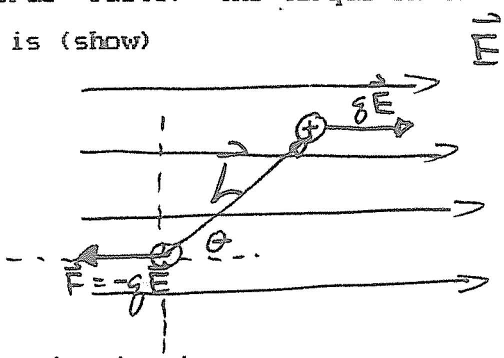
Dipoles. An electric dipole consists of two ~~charges of~~ equal and opposite charges separated by a (directed) length $\vec{L}$.



Where $\vec{p}$ (the <u>dipole moment</u>) is given by

$$\vec{p} = q\vec{L}.$$

Then (true facts) the torque on the dipole in a uniform electric field $\vec{E}$ is (show)



$$\vec{\tau} = \vec{p} \times \vec{E} = |p||E|\sin\theta$$

and the net force on the dipole is

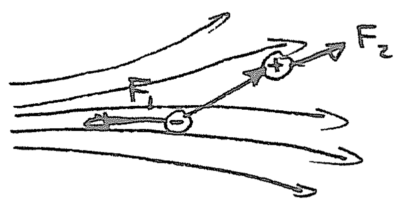
$$\vec{F} = 0.$$

The (potential) energy of the dipole is

$$U = -W(\theta) = +\int |\vec{\tau}|\,d\theta = -pE\cos\theta = -\vec{p}\cdot\vec{E}$$

done by us from 90° to θ.

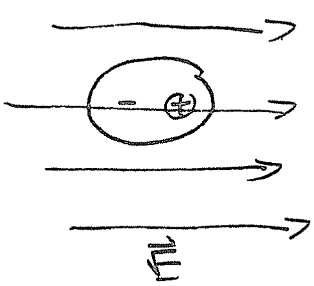
In a non-uniform electric field the dipole may experience simulataneously a torque and a non-zero force.

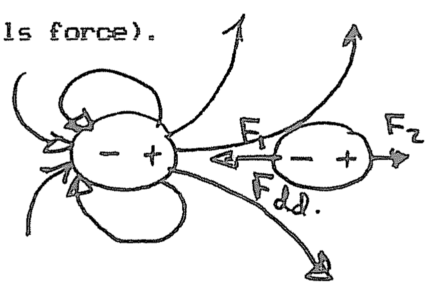


$|F_1| > |F_2|$

$F_1 \nparallel F_2$.

Atoms in electric fields tend to polarize and experience a dipole-dipole attraction (Van der Waals force).

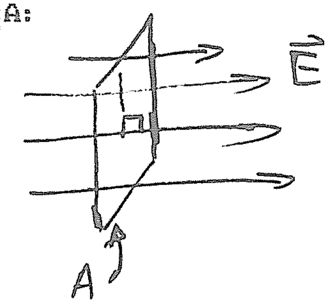




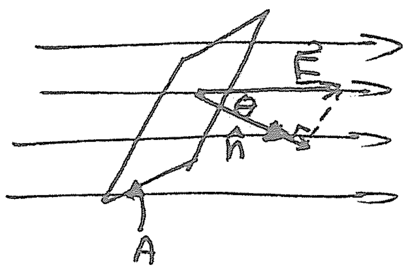
Then, if we have time, we will start Chapter 21 (Gauss' Law) by defining

4) <u>Electric Flux:</u> The "flow" of an electric field through a surface. It is the product of the surface area with the component the vector electric field that is normal, or perpendicular to the surface on a point by point basis.

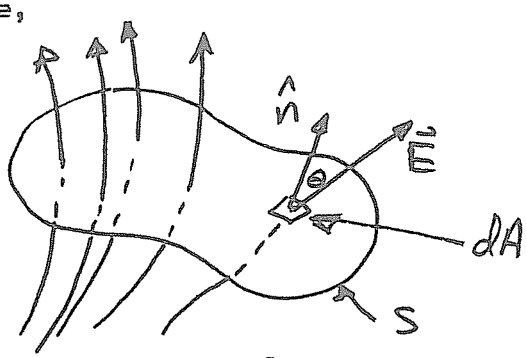
For $\vec{E} \perp A$, $\phi = |\vec{E}|A$:



For $\vec{E}$ at an angle θ to A, $\phi = \vec{E}\cdot\hat{n}A = |\vec{E}|A\cos\theta$



Generally, for $\vec{E}$ varying in both magnitude and angle over an arbitrary surface,



$$\phi_e = \int_S \vec{E}\cdot\hat{n}\, dA$$

EDJ 4. Gauss' Law.

Recall from last time: the superposition principle for the electric field.

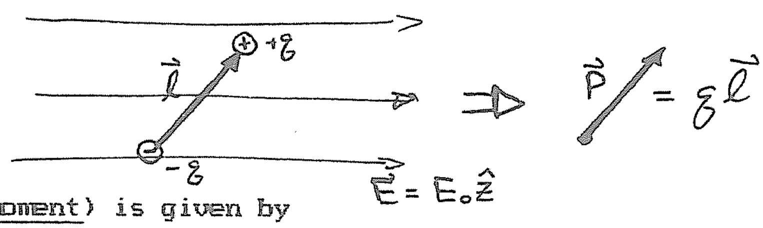
$$\vec{E}(\vec{r}) = \sum_i \frac{kq_i}{r_i^2}\hat{r}_i \quad \text{(recall point charges)}$$

or

$$\vec{E}(\vec{r}) = \int_{\mathbb{R}^3} k\rho(\vec{r}_0)\frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}d^3r_0. \quad \text{(recall ring/disk/line)}$$

Today:

1) Electric Dipoles: An electric dipole consists of two equal and opposite charges separated by a (directed) length $\vec{\ell}$.

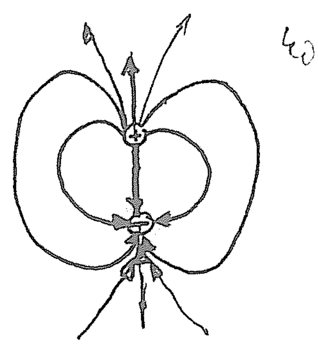


Where $\vec{p}$ (the dipole moment) is given by

$$\vec{p} = q\vec{\ell}.$$

Then (true facts) the torque on the dipole in a uniform electric field $\vec{E} = E_0\hat{z}$ is

$$\vec{\tau} = \vec{\ell} \times q\vec{E} = q\vec{\ell} \times \vec{E} = q|\vec{\ell}||\vec{E}|\sin\theta = \vec{p} \times \vec{E}.$$



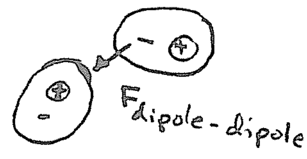
The (potential) energy of the dipole is

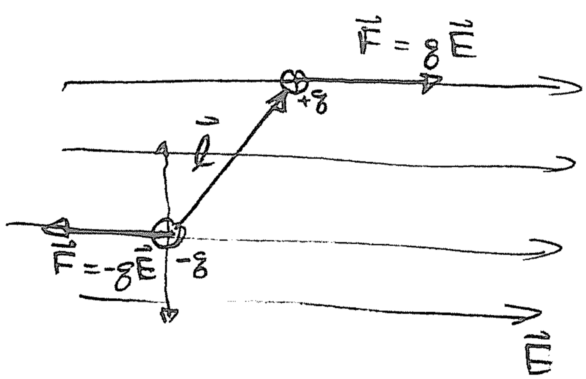
$$U = \int_{90^\circ}^{\theta} |\vec{\tau}|d\theta = \int_{90^\circ}^{\theta} |\vec{p}||\vec{E}|\sin\theta d\theta = -|\vec{p}||\vec{E}|\cos\theta = -\vec{p}\cdot\vec{E}$$

and the net force on the dipole is

$$\vec{F} = 0.$$

In a non-uniform electric field the dipole may experience simulataneously a torque and a non-zero force. Atoms in electric fields tend to polarize and experience a dipole-dipole attraction (Van der Waals force).



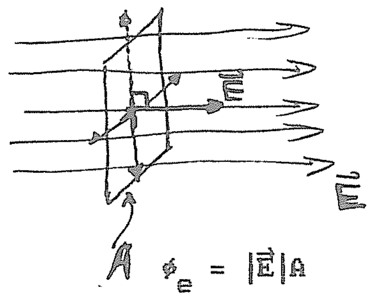


$$\vec{F}_{tot} = q\vec{E} - q\vec{E} = 0$$

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{\ell} \times q\vec{E} = (q\vec{\ell}) \times \vec{E} = \vec{p} \times \vec{E}$$

2) <u>Electric Flux</u>: The "flow" of an electric field through a surface. It is the product of the <u>surface area</u> with the <u>component the vector electric field</u> that is <u>normal</u>, or <u>perpendicular</u> to the surface on a point by point basis.

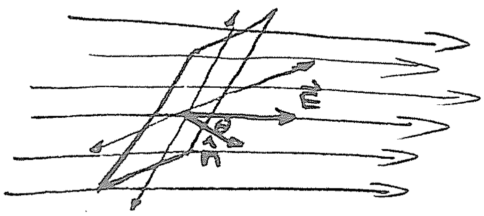
For a constant electric field $\vec{E} = E_0\hat{z}$, (with $A \perp \hat{z}$):

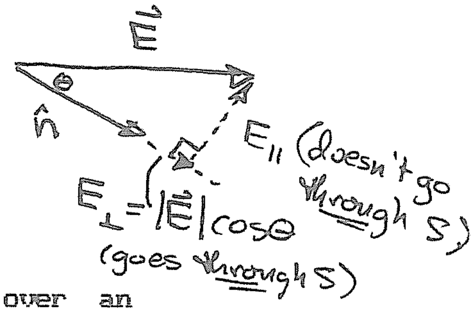


$$\phi_e = |\vec{E}|A$$

For constant $\vec{E} = E_0\hat{z}$ and A at some angle θ to $\hat{z}$:

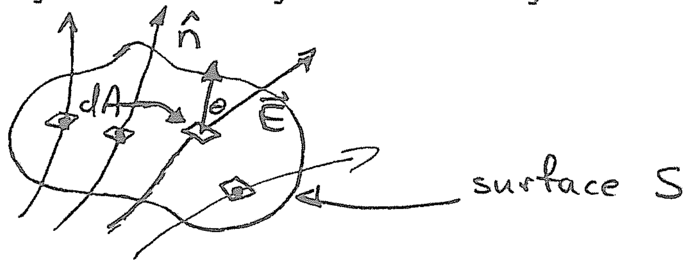
$|\hat{n}| = 1$ (unit vector $\perp$ A)





$$\phi_e = |\vec{E}|A\cos\theta = (\vec{E}\cdot\hat{n})A$$

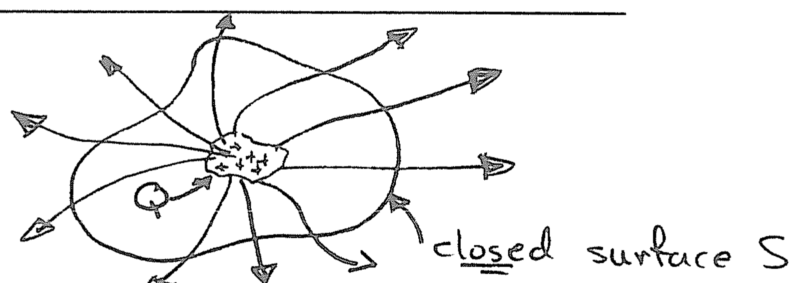
Generally, for $\vec{E}$ varying in both magnitude and angle over an arbitrary surface,



$$\phi_e = \int_S \vec{E}\cdot\hat{n}\, dA$$

---

3) Gauss's Law (!!!)

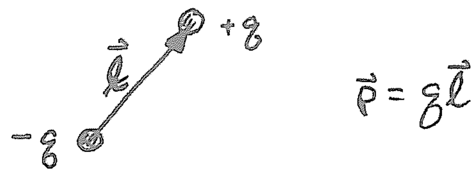


$$\phi_e^S = \oint_S \vec{E}\cdot\hat{n}\, dA = 4\pi k Q_{\text{enclosed by } S}$$

In words: The <u>electric flux</u> through a <u>closed surface S</u> is equal to a constant <u>(4πk)</u> times the <u>total charge enclosed by S</u>. Note that this is independent of the shape of S (as long as it is closed) and the distribution of charge inside S.

Lecture 4. Gauss' Law.

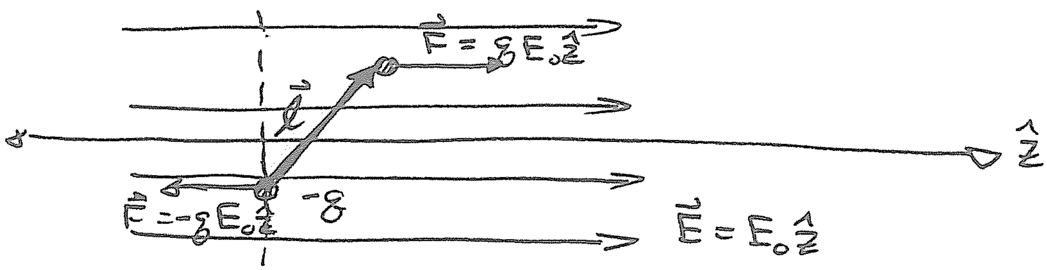
1) Electric Dipoles: An electric dipole consists of two equal and opposite charges separated by a (directed) length $\vec{\ell}$.



Where $\vec{p}$ (the dipole moment) is given by

$$\vec{p} = q\vec{\ell}.$$

In a uniform electric field $\vec{E} = E_0\hat{z}$:



$$\vec{\tau} = \vec{\ell} \times q\vec{E} = q\vec{\ell} \times \vec{E} = q|\vec{\ell}||\vec{E}|\sin\theta = \vec{p} \times \vec{E}. \qquad (\vec{\tau} = \vec{r} \times \vec{F})$$

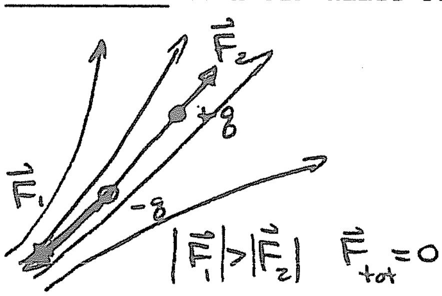
The net force on the dipole is

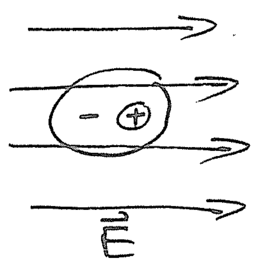
$$\vec{F} = 0.$$

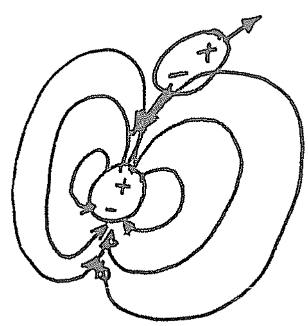
The (potential) energy of the dipole is

$$U = \int_{90^\circ}^{\theta} |\vec{\tau}|d\theta = \int_{90^\circ}^{\theta} |\vec{p}||\vec{E}|\sin\theta d\theta = -|\vec{p}||\vec{E}|\cos\theta = -\vec{p}\cdot\vec{E}.$$

In a non-uniform electric field the dipole may experience simulataneously a torque and a non-zero force. Atoms in electric fields tend to polarize. Free atoms also experience a dipole-dipole attraction (Van der Waals force).



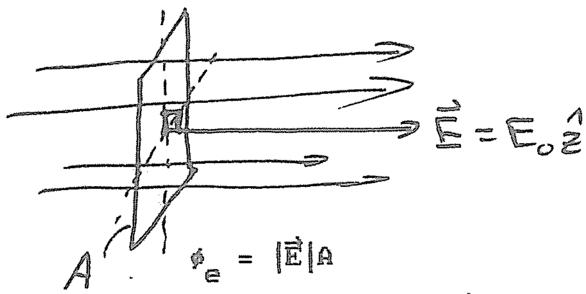




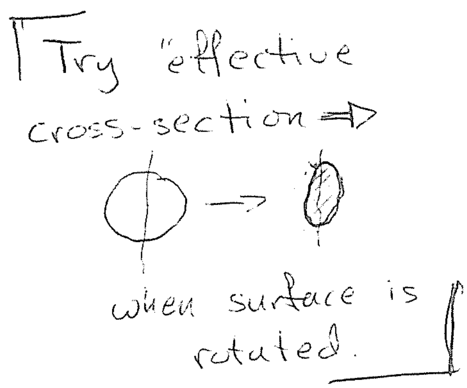
The electric field produced by a dipole drops off as $\frac{1}{r^3}$ (for $r \gg |\vec{\ell}|$). You should be able to show this using superposition and the Binomial Expansion.

2) Electric Flux: The "flow" of an electric field through a surface. It is the product of the surface area with the component the vector electric field that is normal, or perpendicular to the surface on a point by point basis.

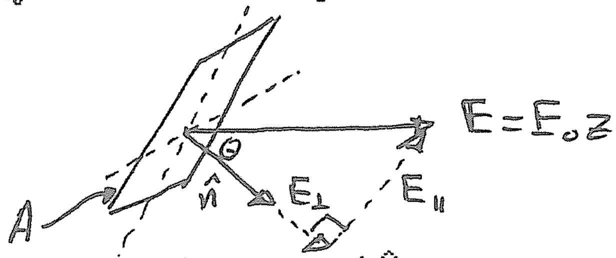
For a constant electric field $\vec{E} = E_0\hat{z}$, (with A $\hat{z}$):



$$\phi_e = |\vec{E}|A$$

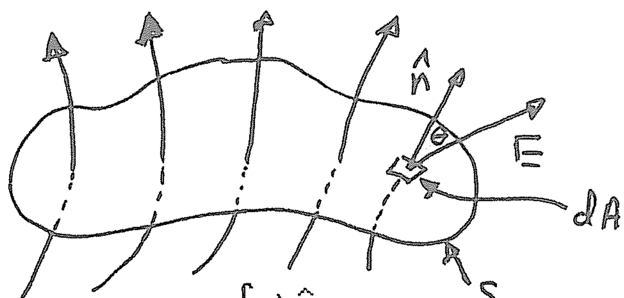


For constant $\vec{E} = E_0\hat{z}$ and A at some angle θ to $\hat{z}$:



$$\phi_e = |\vec{E}|A\cos\theta = (\vec{E}\cdot\hat{n})A$$

Generally, for $\vec{E}$ varying in both magnitude and angle over an arbitrary surface,



$$\phi_e = \int_S \vec{E}\cdot\hat{n}\, dA$$

---

3) Gauss's Law (!!!)

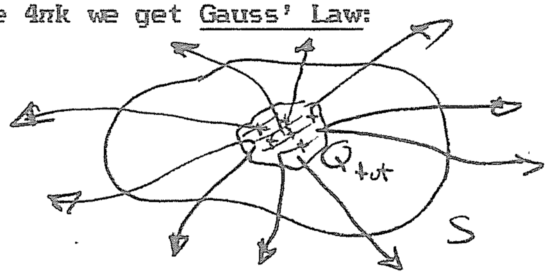
Recall that when we discussed "lines of force" we decided that, for a point charge, the strength of the field was roughly proportional to the number of field lines per unit area of the surface perpendicular to the field at a given point. That was because the field strength dropped off as $1/r^2$ and the area increased as $(4\pi)r^2$. From this (and the rule for lines of force that states that the number of lines of force entering or leaving a charge is proportional to the magnitude of the charge) we can deduce that:

$$|E| \sim (\#\text{lines/unit area}) \text{ so}$$

$$|E|A \sim (\#\text{lines/area}) \times (\text{area}) \sim \#\text{lines} \propto \text{total charge, or:}$$

$$\phi_e = \oint_S \vec{E}\cdot\hat{n}\, dA \propto \text{total charge in S.}$$

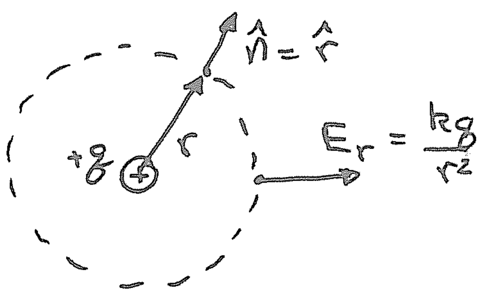
Selecting (we will prove this correct) the constant of proportionality to be $4\pi k$ we get Gauss' Law:



$$\phi_e^S = \oint_S \vec{E}\cdot\hat{n}\, dA = 4\pi k Q_{\text{enclosed by S}} .$$

In words: The electric flux through a closed surface S is equal to a constant ($4\pi k$) times the total charge enclosed by S. Note that this is independent of the shape of S (as long as it is closed) and the distribution of charge inside S.

To see how this works (and verify the constant of proportionality) we will apply Gauss' Law to a point charge with a spherical Gaussian surface (S) with a radius r drawn symmetrically around the charge.



$$\phi_e^S = \int_S \vec{E}\cdot\hat{n}\, dA = \int_S \frac{kq}{r^2}\,\hat{r}\cdot\hat{n}\, dA = \frac{kq}{r^2}\int_S dA = \frac{kq}{r^2}\,4\pi r^2 =$$

$$\equiv 4\pi kq.$$

Thus the constant is indeed $4\pi k$, and we see that the thing that allows us to apply Gauss' law is the ability to evaluate the flux integral. This can be done only when $\vec{E}$ is constant and perpendicular to a simple surface whose area we can evaluate. As we shall see, there are three such geometries, and when we've learned them, there is nothing more we can do (without resorting to a computer). (Next time! Points, lines and planes of charge!)

EDJ 5. Gauss' Law (continued).

N.B

Recall from last time the definition of electric flux:

$$\phi = \int_S \vec{E}\cdot\hat{n}\, dA$$

ϵ_0 is "permittivity of free space"

and Gauss's Law:

$$\phi_e^S = \oint_S \vec{E}\cdot\hat{n}\, dA = 4\pi k Q_{\text{enclosed by S}} = \frac{Q}{\epsilon_0}$$

or $\frac{1}{\epsilon_0} = 4\pi k$

In words: The electric flux through a closed surface S is equal to a constant ($4\pi k$) times the total charge enclosed by S.

LEARN THIS LAW!

Today: We learn the THREE geometries of charge distribution that allow us to evaluate the flux through a symmetric surface. In these three cases we can use Gauss' law to evaluate the electric field! The method for all three cases is the same. We:

a) DRAW the charge distribution.

b) DRAW the appropriate GAUSSIAN SURFACE (this is a surface with the symmetry of the charge distribution as noted below).

c) EVALUATE the flux integral.

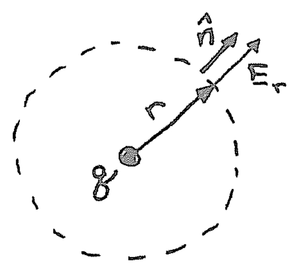
d) EVALUATE the charge inside the Gaussian surface drawn.

e) SOLVE for $\vec{E}$ algebraically.

The three cases are:

1) Spherically symmetric charge distributions (point charge, sphere of charge, spherical shell of charge).

a),b)



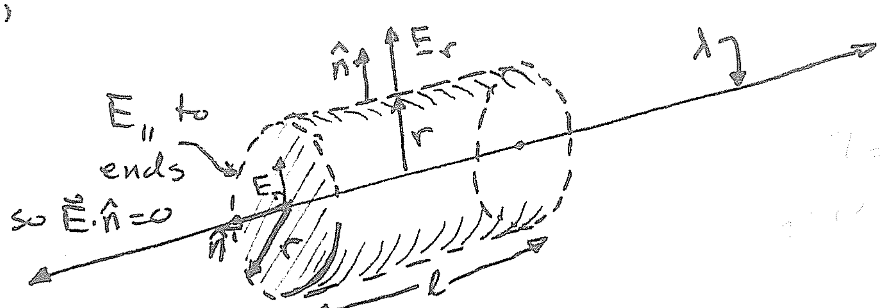
c) $\phi_e^S = \oint_S \vec{E}\cdot\hat{n}\, dA = \int_S E_r dA = E_r \int_S dA = E_r 4\pi r^2.$

d) $4\pi k Q_{enc}$ e. g. $\left[= 4\pi kq \right]$ or $\left[= 4\pi k \int_V \rho(\vec{r}_0) d^3 r_0 \right].$

e) $E_r 4\pi r^2 = 4\pi kq$ ===> $E_r = \frac{kq}{r^2}$.

2) Cylindrically symmetric charge distributions ("infinite" line of charge, cylinder of charge, or cylindrical shell of charge).

a),b)



c) $\phi_e^S = \oint_S \vec{E}\cdot\hat{n}\, dA = \int_{End} (0)dA + \int_{Cyl} E_r dA = E_r 2\pi r\ell.$

d) $4\pi kQ_{enc}$ e. g. $\left[= 4\pi k(\lambda\ell) \right]$ or $\left[= 4\pi k\int_V \rho(\vec{r}_0)d^3r_0 \right].$

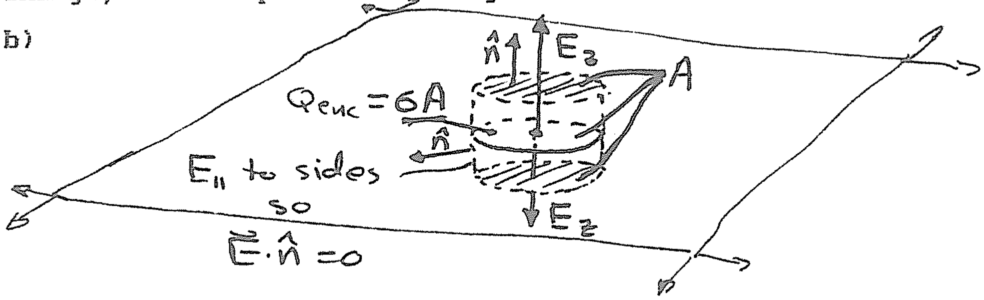
e) $E_r 2\pi r\ell = 4\pi k(\lambda\ell)$ ===> $E_r = \dfrac{2k\lambda}{r}.$

(<u>MUCH</u> easier than integral method!)

---

3) Planar charge distributions ("infinite" plane of charge, slab of charge, several planes of charge).

a),b)



c) $\phi_e^S = \oint_S \vec{E}\cdot\hat{n}\, dA = \int_{Sides} (0)dA + \int_{Ends} E_z dA = 2E_z A.$

d) $4\pi kQ_{enc}$ e. g. $\left[= 4\pi k(\sigma A) \right]$ or $\left[= 4\pi k\int_V \rho(\vec{r}_0)d^3r_0 \right].$

e) $2E_z A = 4\pi k(\sigma A)$ ===> $E_z = 2\pi k\sigma.$

---

4) Final note on GAUSSIAN SURFACES. These are 1) Spheres; 2) Cylinders (of length ℓ with <u>closed</u> ends); or 3) "pillboxes", with <u>closed</u> ends and sides <u>perpendicular</u> to the plane of charge. The idea behind the construction of a given Gaussian surface is that a) The field should be normal to and constant on the contributing surface and b) it should be parallel to all other surfaces. The surface must be <u>closed</u>, but this method of construction eliminates end effects.

Lecture 5. Gauss' Law (continued).

1) Clean up loose ends from last lecture. Recap the definition of <u>electric flux</u>:

$$\phi = \int_S \vec{E}\cdot\hat{n}\, dA \qquad \epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N\text{-}m^2}$$

and <u>Gauss's Law</u>:

$$\phi_e^S = \oint_S \vec{E}\cdot\hat{n}\, dA = 4\pi k Q_{\text{enclosed by } S} = \frac{Q_{enc}}{\epsilon_0} \quad \Rightarrow \frac{1}{\epsilon_0} = 4\pi k$$

<u>LEARN THIS LAW!</u>

Today (following closely the EDJ!) we learn three geometries of charge distribution that allow us to evaluate the flux through a symmetric surface. In these three cases we can use Gauss' law to evaluate the electric field!

---

2) The method for all three cases is the same. We:

a) DRAW the charge distribution.

b) DRAW the appropriate GAUSSIAN SURFACE (this is a surface with the symmetry of the charge distribution as noted below).

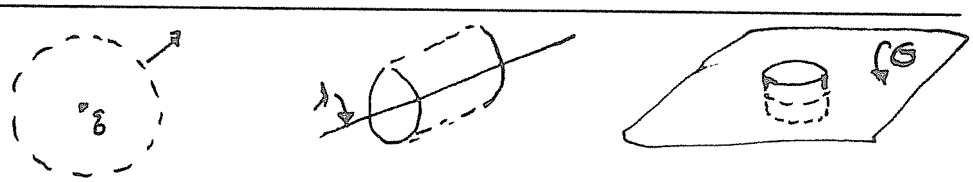
c) EVALUATE the flux integral.

d) EVALUATE the charge inside the Gaussian surface drawn.

e) SOLVE for $\vec{E}$ algebraically.

---

3) Discuss GAUSSIAN SURFACES. These are i) Spheres; ii) Cylinders (of length ℓ with <u>closed</u> ends); or iii) "pillboxes", with <u>closed</u> ends and sides <u>perpendicular</u> to the plane of charge. The idea behind the construction of a given Gaussian surface is that a) The field should be normal to and constant on the contributing surface and b) it should be parallel to all other surfaces. The surface must be <u>closed</u>, but this method of construction eliminates end effects.



The three cases are:

4) Spherically symmetric charge distributions (point charge, sphere of charge, spherical shell of charge).

a),b),c) $\phi_e^S = \oint_S \vec{E}\cdot\hat{n}\, dA = \int_S E_r dA = E_r \int_S dA = E_r 4\pi r^2.$

d) $4\pi k Q_{enc}$ e. g. $\left[= 4\pi kq \right]$ or $\left[= 4\pi k \int_V \rho(\vec{r}_0) d^3 r_0 \right]$.

e) $E_r 4\pi r^2 = 4\pi kq \implies E_r = \frac{kq}{r^2}$.

5) Cylindrically symmetric charge distributions ("infinite" line of charge, cylinder of charge, or cylindrical shell of charge).

a),b),c) $\phi_e^S = \oint_S \vec{E}\cdot\hat{n}\, dA = \int_{End} (0) dA + \int_{Cyl} E_r dA = E_r 2\pi r \ell.$

d) $4\pi k Q_{enc}$ e. g. $\left[= 4\pi k(\lambda\ell) \right]$ or $\left[= 4\pi k \int_V \rho(\vec{r}_0) d^3 r_0 \right]$.

e) $E_r 2\pi r\ell = 4\pi k(\lambda\ell) \implies E_r = \frac{2k\lambda}{r}$.

(<u>MUCH</u> easier than integral method!)

---

6) Planar charge distributions ("infinite" plane of charge, slab of charge, several planes of charge).

a),b),c) $\phi_e^S = \oint_S \vec{E}\cdot\hat{n}\, dA = \int_{Sides} (0) dA + \int_{Ends} E_z dA = 2E_z A.$

d) $4\pi k Q_{enc}$ e. g. $\left[= 4\pi k(\sigma A) \right]$ or $\left[= 4\pi k \int_V \rho(\vec{r}_0) d^3 r_0 \right]$.

e) $2E_z A = 4\pi k(\sigma A) \implies E_z = 2\pi k\sigma$.

---

7) If time, discuss sphere of uniform charge density distribution or multiple spherical shells or cylindrical ditto or planar ditto.

EDJ 6. Conductors in electrostatic equilibrium.

Recall from last time Gauss's Law:

$$\phi_e^S = \oint_S \vec{E}\cdot\hat{n}\, dA = 4\pi k Q_{\text{enclosed by } S} .$$

and its application to spheres, cylinders and slabs of charge.

Today

1) We define Conductors to be substances with large numbers of essentially free, mobile charge carriers. These carriers are usually electrons (in, for example, metals).

a) ~1 conduction electron atom.

b) Conduction electrons are free to move.

c) They respond very quickly to applied electric fields.

---

2) Properties of conductors (at electrostatic equilibrium!)

a) There is no electric field inside a conductor.

b) All surplus or redistributed charge lies on the surface of a conductor.

c) The electric field close to the surface (outside!) a conductor is related to the surface charge distribution σ by

$$E_n = 4\pi k\sigma = \frac{\sigma}{\epsilon_0}$$

and is (as indicated) normal to the surface of the conductor.

---

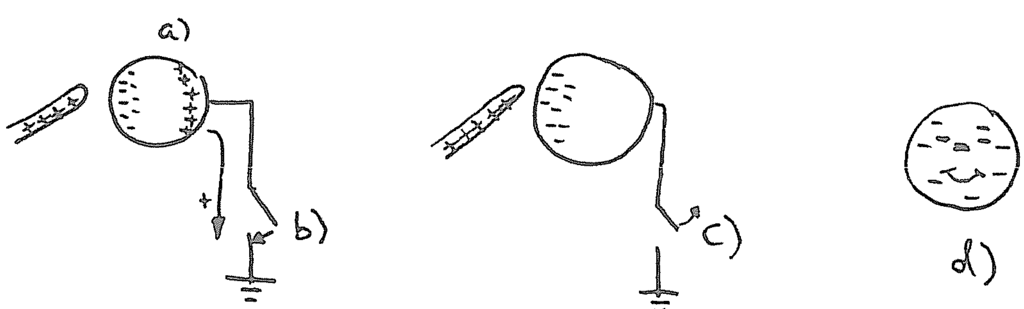
3) Charging by induction.

a) Bring charged object close to conductor(s)

b) Remove repelled charge via ground or second conducting object

c) Remove ground path or object

d) Remove charged object. The conductor is now also charged.



Lecture 6. Conductors in electrostatic equilibrium.

Recall from last time Gauss's Law:

$$\phi_e^S = \oint_S \vec{E}\cdot\hat{n}\, dA = 4\pi k Q_{\text{enclosed by } S} .$$

and its application to spheres, cylinders and slabs of charge.

---

1) Today we define Conductors to be substances with large numbers of essentially free, mobile charge carriers. These carriers are usually electrons (in, for example, metals) and the conductor usually contributes approximately one conduction electron per atom. This means that there are

$$(6\times10^{23}\,\frac{\text{electrons}}{\text{mole}})\times(1.6\times10^{-19}\,\frac{\text{Coulombs}}{\text{electron}}) = 9.6\times10^{4}\text{C} \approx 10^{5}\text{C}$$

of free charge in one mole of a metal. This is a tremendous amount of charge (most laboratory charges are measured in nC or μC). It is the amount of charge that flows through a power line carrying one ampere (Coulomb/second) in 27.7 hours! As a consequence, a conductor has an "inexhaustable" supply of conduction electrons, and we can treat them as a mobile fluid that can respond "instantly" to any applied electric field.

---

2) From this we can readily deduce another property of conductors in electrostatic equilibrium. That is: There is no electric field inside a conductor at equilibrium.

We can understand this by imagining that there were a non-zero electric field in a conductor. Then there would be a force $\vec{F} = -e\vec{E}$ acting on its conduction electrons. But they are free to move, and so they move toward the source of the field. Eventually, they hit the edge of the conductor, and build up in a surface layer of charge. This surface layer of (negative) charge produces an equal and opposite field inside the conductor that exactly cancels the field there, at which point charge ceases to move, and equilibrium is established.

---

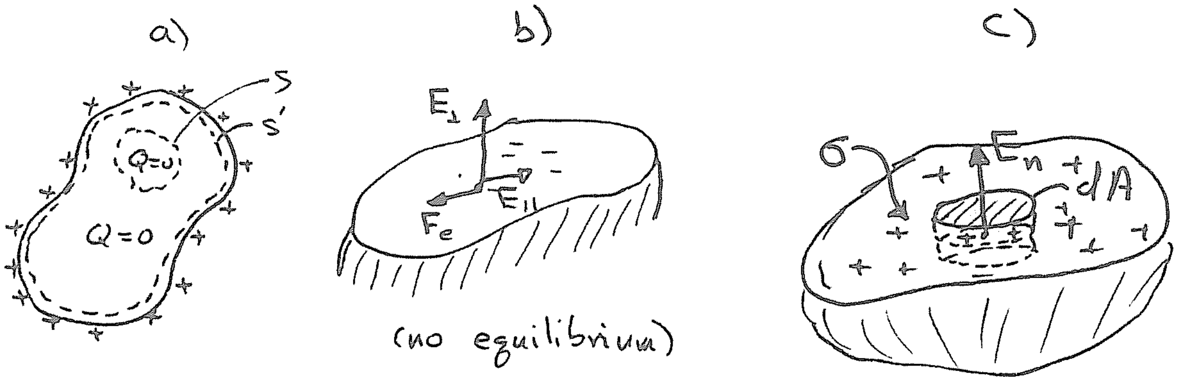
3) Once we realize that there is no electric field inside a conductor at equilibrium, there are several consequences of this we can explore with Gauss' law.

a) Surplus charge placed on a conductor always resides on the surface (otherwise G's law says that we would have an electric field inside).

b) The electric field outside a conductor (but very close to its surface) points perpendicular (or normal) to its surface. (If it didn't, then there would be a component of E parallel to the surface. This would exert a force on the electrons near the surface and they would rearrange along the surface until the parallel component of E disappeared).

c) Therefore (from (c) and G's law) $\vec{E}$ in the vicinity of the surface of a conductor always exactly equals $4\pi k\sigma$ normal to the surface.

Pictures for a), b), c)



$$E_n dA = \sigma dA (4\pi k)$$
$$E_n = \sigma/\epsilon_0 = 4\pi k\sigma$$

---

4) Charging by induction.

a) Bring charged object close to conductor(s)

b) Remove repelled charge via ground or second conducting object

c) Remove ground path or object

d) Remove charged object.

---

5) End up with demonstrations of Gauss' Law or begin Potential.

Today

1) Recall that the electric force is conservative. We can then evaluate the Potential Energy of two point charges separated by a distance r:

$$U(r) = -\Delta W\Big|_{\infty}^{r} = -\int_{\infty}^{r} \vec{F}\cdot d\vec{r} = -\int_{\infty}^{r} F_r\,dr = -\int_{\infty}^{r} \frac{kq_1q_2}{r^2}\,dr$$

$$U(r) = \frac{kq_1q_2}{r}$$

---

2) We define the Electric Potential produced by a single point charge q to be (in analogy with the relation between force and field):

$$V(r) = \lim_{q_0 \to 0} \frac{U(r)}{q_0} = \frac{kq}{r}$$

for a point charge q.

---

3) More generally, if $\Delta U = U_b - U_a$ is the difference in potential energy of a charge q_0 at point a and the same charge at point b,

$$\Delta V = \frac{\Delta U}{q_0} = V_b - V_a = -\int_{a}^{b} \vec{E}\cdot d\vec{r}$$

is the potential difference between the points a and b.

---

4) Characteristics of Potential (or potential difference):

a) It is a scalar quantity (NOT a vector).

Its (SI) units are Volts.

$$1\ \text{Volt} = 1\ \frac{\text{Joule}}{\text{Coulomb}} = 1\ \frac{\text{Newton-Meter}}{\text{Coulomb}}.$$

c) ΔV is directly proportional to the charge producing it.

d) ΔV depends on the distance from source charges to the point of observation, not their direction.

d) Only potential differences are physically meaningful. This is because only differences in potential energy are physically meaningful.

e) The potential difference is the work per unit charge that would be done to move a charge beteen the initial and final locations at constant speed.

f) We define as a useful energy unit the electron-Volt (eV):

$$1 \text{ eV} = (1.6\times10^{-19}\text{C})\times(1\text{V}) = 1.6\times10^{-19}\text{Joules}.$$

---

5) The potential of a point charge is

$$V(r) = \frac{kq}{r}$$

and carries the sign of the charge q.

---

6) The field is related to the potential by

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}.$$

---

7) The potential of a collection of point charges is the (scalar) sum of the potentials of each charge!

$$V_{total}(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|}.$$

---

do examples if more time.

(a) two point chgs ± ∓ x → y↑

$V = \int \frac{\rho\, d^3r_0}{|\vec{r}-\vec{r}_0|}$ (b) ring

(c) disk.

Lecture 7. Electric Potential.

Today

1) Recall that the electric force is conservative. We can then evaluate the Potential Energy of two point charges separated by a distance r:

$$U(r) = -\Delta W\Big|_{\infty}^{r} = -\int_{\infty}^{r} \vec{F}\cdot d\vec{\ell} = -\int_{\infty}^{r} F_r\,dr = -\int_{\infty}^{r} \frac{kq_1q_2}{r^2}\,dr$$

$$U(r) = \frac{kq_1q_2}{r}$$

---

2) We define the Electric Potential produced by a single charge q to be (in analogy with the relation between force and field):

$$V(r) = \lim_{q_0 \to 0} \frac{U(r)}{q_0}.$$

It follows that

$$V(r) = \lim_{q_0 \to 0} -\int_{\infty}^{r} \frac{\vec{F}}{q_0}\cdot d\vec{\ell} = -\int_{\infty}^{r} \vec{E}\cdot d\vec{\ell} = -\int_{\infty}^{r} \frac{kq}{r^2}\,dr,$$

$$\text{or} \quad V(r) = \frac{kq}{r}$$

for a point charge q.

---

3) More generally, if $\Delta U = U_b - U_a$ is the difference in potential energy of a charge q_0 at point a and the same charge at point b,

$$\Delta V = \frac{\Delta U}{q_0} = V_b - V_a = -\int_{a}^{b} \vec{E}\cdot d\vec{\ell}$$

is the potential difference between the points a and b.

---

4) Characteristics of Potential (or potential difference):

a) It is a scalar quantity (like energy), not a vector. Furthermore, it depends only on the charge q and the scalar distance between the charge and the point of observation. This makes it easy to work with!

b) Its (SI) units are called Volts. From the above, it is evident that

$$1 \text{ Volt} = 1\,\frac{\text{Joule}}{\text{Coulomb}} = 1\,\frac{\text{Newton-Meter}}{\text{Coulomb}}.$$

c) The potential (or potential difference) depends only on the charges producing it and points in space. Like a field, it is a quantity that is characteristic of a point of space rather than a directly measurable force or energy. It too is a "solution" to the problem of action at a distance.

d) Only potential differences are physically meaningful. This is because only differences in potential energy are physically meaningful. We can set the zero of potential to be any number we like without affecting the physics. In general, for a point charge (or any bounded charge distribution) we will choose the zero to be at infinity. For infinite charge distributions (like the line or plane) we can only talk about potential differences between two specific points (as we shall see).

e) The potential (like the potential energy) is positive or negative depending on whether the two charges have the same sign or opposite signs. We use the convention that the potential of a point charge is

$$V(r) = \frac{kq}{r}$$

and carries the sign of the charge q.

f) The potential is the work per unit charge that would be done to bring a charge from infinity to the position $\vec{r}$ at constant speed.

g) Another useful and meaningful unit is the amount of energy gained or lost by an atomic charge (the charge on an electron or proton) when it falls through a potential difference of one volt. We call this energy unit the electron-Volt (eV):

$$1 \text{ eV} = (1.6x10^{-19}\text{C})x(1\text{V}) = 1.6x10^{-19}\text{Joules.}$$

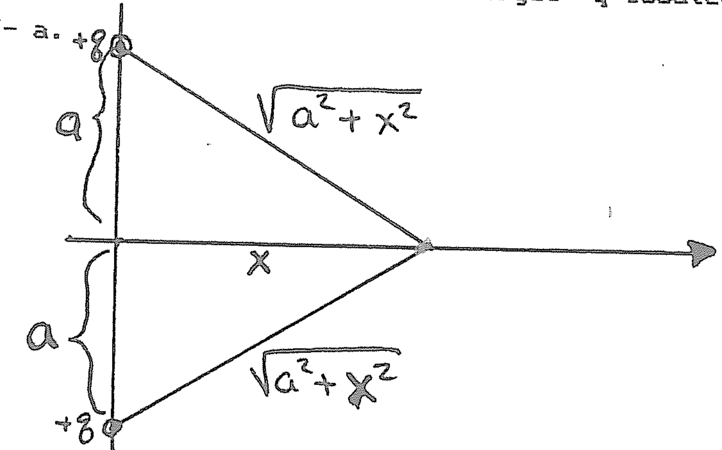
This unit of energy (usually with a kilo, mega, giga, or tera in front of it is frequently used to describe the energy of moving elementary particles or the interaction energies of individual atoms. i.e.-- a 10 MeV (million electron-Volt) accelerator, a 50 keV x-ray, etc. I do expect you to be equally comfortable with Joules and electron-Volts as units in which to work different problems.

The superposition principle for electric fields also holds for electric potentials. To find the potential of a collection of point charges, we add the (scalar) potentials of each charge:

$$V_{total}(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|}.$$

Example:

a) The potential on the x-axis of two charges +q located on the y-axis at +/- a.



Solution: The potential or a point charge at a distance r from the charge is $\frac{kq}{r}$. So the potential from the top charge at x is just:

$V_{top}(x) = \frac{kq}{(x^2 + a^2)^{1/2}}$. The potential of the bottom charge is also $V_{bottom}(x) = \frac{kq}{(x^2 + a^2)^{1/2}}$. Since potential is a scalar, to find the total we just add them (no vector arithmetic!). The total potential is then:

$$V_{total}(x) = \frac{2kq}{(x^2 + a^2)^{1/2}}.$$

Pretty easy, huh!

b) Note in the problem above, if charges are +q and -q, $V(x) = 0$!

c) Note that just as $F_x = -\frac{\partial U}{\partial x}$, $E_x = -\frac{\partial V}{\partial x}$. So we can find the electric field from the potential.

$$E_x(x) = -\frac{\partial V(x)}{\partial x} = \frac{2kqx}{(x^2 + a^2)^{3/2}},$$

just like we got (with a lot more work) from vector decomposition!

Note $$E_y(x) = -\frac{\partial V(x)}{\partial y} = 0.$$

EDJ 8. Electric Potential.

Recall from Monday:

1) $\Delta V = \frac{\Delta U}{q_0} = (V_b - V_a)$ (by definition).

2) $V(r) = \frac{kq}{r}$ (for a point charge q).

3) $V_{total}(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|}$ (for a collection of charges).

Today:

1) The field is related to the potential by

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}.$$

(Recall relation between Force and Potential.)

---

2) Equipotential surfaces: Surfaces drawn perpendicular to the lines of force. They are surfaces of constant potential. They are analogous to contour lines on a map (which are gravitational equipotential lines!)

---

3) Since $\vec{E} = 0$ inside a conductor and along its surfaces, a conductor in electrostatic equilibrium is at a constant potential.

$$\left[E_x = -\frac{\partial V(\text{constant})}{\partial x} = 0 = E_y = E_z.\right]$$

---

4) For continuous collection of charge,

$$V_{tot}(\vec{r}) = \int_{\mathbb{R}^3} \frac{k\,\rho(r_0)}{|\vec{r} - \vec{r}_0|}\, d^3r_0 = \int_{\mathbb{R}^3} \frac{kdQ}{R}.$$

<u>Examples Du Jour</u>

a) Two Point charges revisited. Find V(x), $\vec{E}(x)$, etc. using the rules above.

b) A Ring of charge. Find V(x) on axis of ring. Then find $\vec{E}(x)$ from gradient rule.

c) A Disk of charge. Find V(x) on axis of disk. Then find $\vec{E}(x)$. Then find limiting forms of V(x) as $x \to 0$ and $x \to \infty$.

---

5) Last, if we have time, is to discuss electric potential and Gauss' Law.

Lecture 8. Electric Potential.

Recall that

$$V(r) = \frac{kq}{r}$$

for a point charge q and that, more generally, if $\Delta U = U_b - U_a$ is the difference in potential energy of a charge q_0 at point a and the same charge at point b,

$$\Delta V = \frac{\Delta U}{q_0} = V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{l}$$

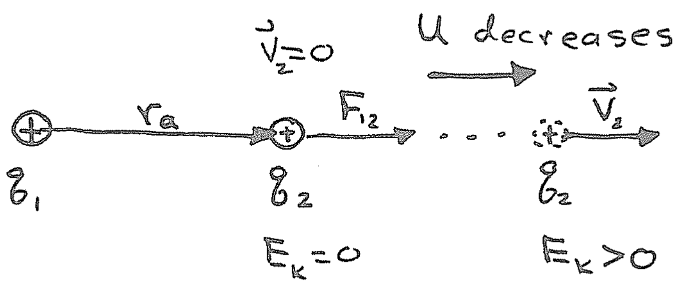
is the potential difference between the points a and b.

Today:

1) The field is related to the potential by

$$E_x = -\frac{\partial V}{\partial x}, \text{ etc.}$$

We can understand this in terms of the relationship between Force and potential energy. Suppose we examine the force between two positive charges separated by a distance r_a:



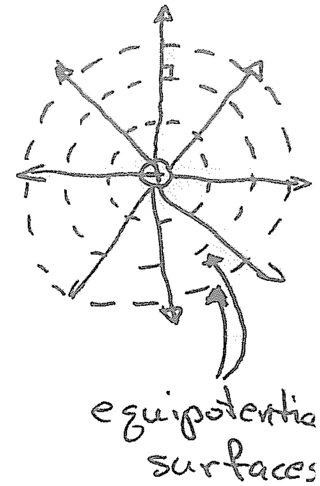
$$\vec{F}_{12} = \frac{kq_1q_2}{r_a^2}\hat{r}$$

The force pushes q_2 away from q_1. Thus, if we release q_2 at rest, it accelerates as it moves away from q_1. Its kinetic energy therefore increases. Conservation of energy tells us that its potential energy therefore decreases (because its total energy must remain the same). We see, then, that forces point in the direction that the potential energy decreases.

Field is just force/unit charge. Potential is just potential energy/unit charge. So fields point in the direction that the potential decreases.

---

2) This lets us define Equipotential Surfaces as surfaces drawn perpendicular to the field (lines of force). Since the potential changes only when one moves along a field line, the potential is constant on these surfaces. They are analogous to contour lines on a map, which are actually the intersection of gravitational equipotential surfaces (that surround the earth) with the surface of the earth.



---

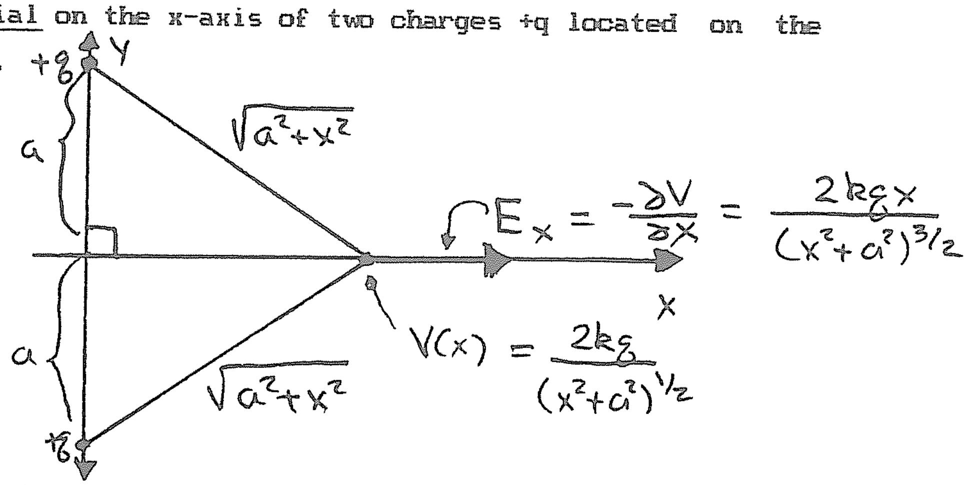
3) Recall that $\vec{E}$ is zero inside a conductor. This tells us that a conductor in electrostatic equilibrium is at a constant potential. (Because the derivative of a constant is zero, i.e.--

$$E_x = -\frac{\partial V_{constant}}{\partial x} = 0 = E_y = E_z \text{).}$$

This is important in circuit design because all parts of a circuit that are connected by a continuous conducting path are thus at the same potential (assuming the conductor has negligible resistance).

Examples

4) The potential on the x-axis of two charges +q located on the y-axis at +/- a.



Solution: The potential or a point charge at a distance r from the charge is $\frac{kq}{r}$. So the potential from the top charge at x is just:

$V_{top}(x) = \frac{kq}{(x^2 + a^2)^{1/2}}$. The potential of the bottom charge is also $V_{bottom}(x) = \frac{kq}{(x^2 + a^2)^{1/2}}$. Since potential is a scalar, to find the total we just add them (no vector arithmetic!). The total potential is then:

$$V_{total}(x) = \frac{2kq}{(x^2 + a^2)^{1/2}}.$$

Pretty easy, huh!

b) Note in the problem above, if charges are $+q$ and $-q$, $V(x) = 0$!

c) Note that just as $F_x = -\frac{\partial U}{\partial x}$, $E_x = -\frac{\partial V}{\partial x}$. So we can find the electric field from the potential.

$$E_x(x) = -\frac{\partial V(x)}{\partial x} = \frac{2kqx}{(x^2 + a^2)^{3/2}},$$

just like we got (with a lot more work) from vector decomposition!

Note:

$$E_y(x) = -\frac{\partial V(x)}{\partial y} = 0.$$

---

6) This leads us, naturally enough, to guess that to find the potential at a point due to a continuous charge distribution we add up the contributions from each little differential chunk of charge.

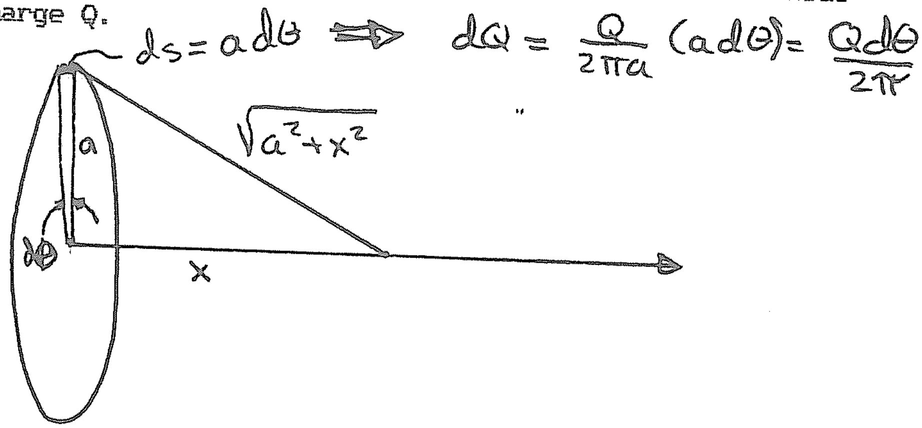
$$dV = \frac{kdq}{|\vec{r} - \vec{r}_0|}, \text{ or}$$

$$V_{total}(\vec{r}) = \int_V \frac{k\,\rho(\vec{r}_0)}{|\vec{r} - \vec{r}_0|}\, d^3r_0$$

where $dq = \rho dV$ or $dq = \sigma dA$ or $dq = \lambda d\ell$ as appropriate. This, too, is simpler than the expression for the field.

More Examples

7) Find the potential on the axis of a ring of charge with radius a and total charge Q.



Solution: As usual, $\lambda = \frac{Q}{2\pi a}$, and the charge in the little chunk of length $ds = ad\theta$ is $dq = \lambda ds = \frac{Qd\theta}{2\pi}$. The potential of that little piece is

$$dV = \frac{k\,dq}{(x^2 + a^2)^{1/2}} = \frac{kQd\theta}{2\pi(x^2 + a^2)^{1/2}}.$$

Again, there is no vector arithmetic this time.

$$V(x) = \int_{\emptyset}^{2\pi} dV = \frac{kQ}{(x^2 + a^2)^{1/2}} \; !$$

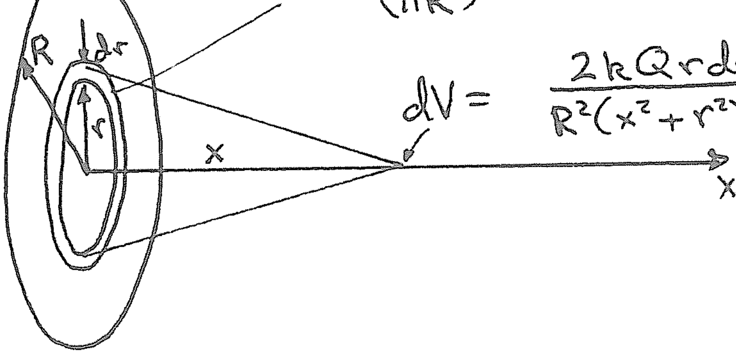
This is obvious, in retrospect. The entire ring is at the same distance $\sqrt{(x^2 + a^2)}$ from the point x. The potential depends only on the distance and charge, not the direction. So the potential of the whole ring is the same as the potential of a point charge Q at the same distance.

b) Exercise: Verify that the field on the axis is $E_x = -\frac{\partial V}{\partial x}$.

---

8) Find the potential on the axis of a disk of charge (radius R, charge Q).

$$dQ = \left(\frac{Q}{\pi R^2}\right) 2\pi r\,dr = \frac{2Qr\,dr}{R^2}$$



$$dV = \frac{2kQr\,dr}{R^2(x^2+r^2)^{1/2}}$$

Solution: Again, treat the disk as a collection of rings. The charge on a ring of differential thickness dr is

$$dq = \sigma dA = \left[\frac{Q}{\pi R^2}\right]\left[2\pi r dr\right] = \frac{2Qrdr}{R^2}.$$

Its potential is

$$dV = \frac{kdq}{(x^2 + r^2)^{1/2}} = \frac{2kQrdr}{R^2(x^2 + r^2)^{1/2}},$$

and so the total potential is

$$V(x) = \int_{\emptyset}^{R} \frac{2kQrdr}{R^2(x^2 + r^2)^{1/2}} = \frac{2kQ}{R^2}(x^2 + r^2)^{1/2}\Big|_{\emptyset}^{R}$$

$$V(x) = \frac{2kQ}{R^2}\left[(x^2 + R^2)^{1/2} - x\right].$$

or

$$V(x) = 2\pi k\sigma \left[(x^2 + R^2)^{1/2} - x\right].$$

d) Exercise: Show that $E_x = -\frac{\partial V}{\partial x}$.

---

e) also find $\lim_{x \to 0} V(x) = -2\pi k \sigma x$ (+ constant)

$$\lim_{x \to \infty} V(x) = \frac{kQ}{x}$$

EDJ 9. Electric Potential and Gauss' Law.

Today

1) Spherical shells.

Find Field (from G's law)

Find potential (by integrating $E_r dr$) in each distinct region.

Note V = constant where E = Ø.

---

2) Cylindrical shells (coaxial cable)

Find field (GL)

Find potential (by integrating $E_r dr$) and note must take potential <u>difference</u> $V(r_b) - V(r_a)$ because $V(Ø) = \infty$ and $V(\infty) = \infty$.

Note that V = constant where E = Ø.

---

3) Potential difference between parallel plates (capacitor)

Find field (GL)

Find potential by integrating $E_z dz$ from Ø to d (again, $V(\infty) = \infty$ but at least V(Ø) can equal Ø.

---

4) Find the potential both inside and outside a sphere of uniform charge density ρ_0.

Find field (GL)

Find potential by integrating $E_r dr$ from ∞ to r

---

5) Find the potential of a spherical conductor with radius R and total charge Q.

Find field (GL and definition of conductor).

Find potential of conductor by integrating in from ∞ to R.

---

6) Charge sharing is a way of creating large potentials (MV). A Van de Graaff generator uses this principle. Since

$$V = \frac{kQ}{R}$$

where R is the radius of the sphere, the maximum field that can be built up is

$$E_{max} = 3MN/C = 3MV/m.$$

But

$$E_{max} = \frac{kQ}{R^2} = V_{max}R$$

which tells us the maximum potential that can be put on a sphere before breakdown.

---

7) When a charge is placed on a nonspherical conductor, it spreads out to make it equipotential. But the field is still strongest near points on the surface with a small radius of curvature, i.e.--sharp points. In fact,

$$E_{max} = V_{max}R_{sharpest}$$

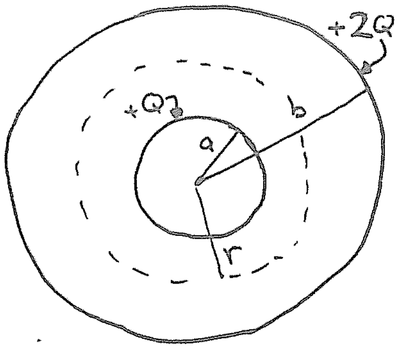
is still approximately valid. The ionization of the air due to strong electric fields and subsequent flow of current is called corona discharge.

Van de Graaff Demo, if time

Lecture 9. Electric Potential and Gauss' Law.

---

1) Spherical shells. Q on shell of radius a, 2Q on shell of radius b.



i) Find Field (from G's law)

$$E_r 4\pi r^2 = 4\pi k(3Q) \qquad r > b$$

$$E_r = \frac{k3Q}{r^2} \qquad r > b$$

$$E_r = \frac{kQ}{r^2} \qquad a < r < b$$

$$E_r = 0 \qquad r < a$$

ii) Find potential (by integrating $E_r dr$) in each distinct region.

$$\Delta V = -\int_{\infty}^{r} E_r dr = \frac{3kQ}{r} \qquad r > b$$

$$\Delta V = -\int_{\infty}^{b} \frac{k3Q}{r^2} dr - \int_{b}^{r} \frac{kQ}{r^2} dr = \frac{k3Q}{b} + \frac{kQ}{r} - \frac{kQ}{b}$$

$$= \frac{k2Q}{b} + \frac{kQ}{r} \qquad a < r < b$$

$$\Delta V = -\int_{\infty}^{b} \frac{k3Q}{r^2} dr - \int_{b}^{a} \frac{kQ}{r^2} dr = \frac{k2Q}{b} + \frac{kQ}{a} \qquad a > r.$$

Note V = constant where E = 0.

---

2) Cylindrical shells (coaxial cable) Same steps.

i) Find field (GL)

ii) Find potential (by integrating $E_r dr$) and note must take potential <u>difference</u> $V(r_b) - V(r_a)$ because $V(0) = \infty$ and $V(\infty) = \infty$.

Note that V = constant where E = 0.

---

3) Potential difference between parallel plates (capacitor)

i) Find field (GL)

ii) Find potential by integrating $E_z dz$ from 0 to d (again, $V(\infty) = \infty$ but at least $V(0)$ can equal 0.

---

4) Find the potential both inside and outside a sphere of uniform charge density ρ_0.

Find field (GL)

Find potential by integrating $E_r dr$ from ∞ to r

---

5) Find the potential of a spherical conductor with radius R and total charge Q.

Find field (GL and definition of conductor).

Find potential of conductor by integrating in from ∞ to R.

---

6) Charge sharing is a way of creating large potentials (MV). A Van de Graaff generator uses this principle. Since

$$V = \frac{kQ}{R}$$

where R is the radius of the sphere, the maximum field that can be built up is

$$E_{max} = 3MN/C = 3MV/m.$$

But

$$E_{max} = \frac{kQ}{R^2} = V_{max}R$$

which tells us the maximum potential that can be put on a sphere before breakdown.

---

7) When a charge is placed on a nonspherical conductor, it spreads out to make it equipotential. But the field is still strongest near points on the surface with a small radius of curvature, i.e.--sharp points. In fact,

$$E_{max} = V_{max}R_{sharpest}$$

is still approximately valid. The ionization of the air due to strong electric fields and subsequent flow of current is called corona discharge.

Van de Graaff Demo, if time

Today

1) A capacitor is a device for storing electric charge and energy in a circuit. It is made up of two conductors separated by an insulator. When the plates are charged up by, say, a battery, they develop a potential difference between them as charge is moved from one conductor to the other. We define the capacitance C by

$$C = \frac{Q}{\Delta V}$$

and, since the potential difference is always directly proportional to the charge, the capacitance depends only on the geometry of the conductors.

2) Units: SI units are

1 farad = 1 Coulomb/volt.

BUT these units are VERY LARGE on a practical scale (a 1 farad capacitor would be the size of a desk or so. . .) and so we generally use microfarads (μF) or picofarads (pF) in actual applications.

3) Examples: Note in the following examples that the calculated capacitance always depends only on the geometry of the conductors! The method for calculating the capacitance in these examples is:

i) Find the field in between the charged objects with, e.g.--Gauss' law.

ii) Find the potential difference between the two (equipotential) charged conductors by integrating the field from a point on one to the other. We only need the magnitude of ΔV, not its sign (though you should know the sign and its meaning!).

iii) Find the capacitance $C = Q/\Delta V$. ΔV should always have a Q in it, so what remains should have an ε_0 on top and depend only on the geometry of the conductors.

a) Parallel Plate capacitor: Given two plates charged to +Q and -Q respectively, with area A and separated by a distance d, the capacitance is:

$$V = Ed = \frac{\sigma d}{\varepsilon_0} = \frac{Qd}{\varepsilon_0 A}$$

so

$$C = \frac{Q}{V} = Q/Qd/\varepsilon_0 A = \frac{\varepsilon_0 A}{d} .$$

Since $\varepsilon_0 = 8.85x10^{-12}$ F/m (from the dimensions of the result) if the plates are 1m x 1m and are separated by 10^{-3}m the capacitance is

$$C = (8.85 \times 10^{-12}\frac{F}{m}) \times (1m^2) \times (10^3 m^{-1}) = 8.85 \times 10^{-9} \approx .01\mu F \ (!)$$

Cylindrical Capacitor: Show for cylindrical shells of radii a and b of length L and carrying a charge +Q and -Q that the capacitance is

$$C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}$$

Spherical Capacitor: Show for spherical shells of radii a and b carrying a charge +Q and -Q that the capacitance is

$$C = 4\pi\varepsilon_0 \frac{ab}{b - a}$$

---

4) Adding Capacitors.

The symbol for capacitors is

Then

$$\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

(for capacitors in SERIES) and

$$C_{tot} = C_1 + C_2 + C_3$$

(for capacitors in PARALLEL).

Lecture 10. Capacitance

1) A capacitor is a device for storing electric charge and energy in a circuit. It is made up of two conductors separated by an insulator. When the plates are charged up by, say, a battery, they develop a potential difference between them as charge is moved from one conductor to the other. We define the capacitance C by

$$C = \frac{Q}{\Delta V}$$

and, since the potential difference is always directly proportional to the charge, the capacitance depends only on the geometry of the conductors.

2) Units: SI units are

1 farad = 1 Coulomb/volt.

BUT these units are VERY LARGE on a practical scale (a 1 farad capacitor would be the size of a desk or so. . .) and so we generally use microfarads (μF) or picofarads (pF) in actual applications.

---

3) Examples: Note in the following examples that the calculated capacitance always depends only on the geometry of the conductors! The method for calculating the capacitance in these examples is:

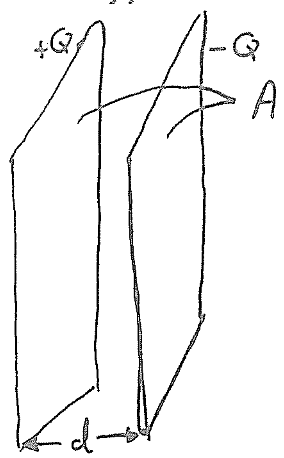
i) Find the field in between the charged objects with, e.g.--Gauss' law.

ii) Find the potential difference between the two (equipotential) charged conductors by integrating the field from a point on one to the other. We only need the magnitude of ΔV, not its sign (though you should know the sign and its meaning!).

iii) Find the capacitance $C = Q/\Delta V$. ΔV should always have a Q in it, so what remains should have an ε_0 on top and depend only on the geometry of the conductors.

---

a) Parallel Plate capacitor: Find the capacitance of two plates charged to +Q and -Q respectively, with area A and separated by a distance d.



Solution:

i) $\phi^S = 2E_x \Delta A = \frac{Q\Delta A}{\varepsilon_0 A} \longrightarrow E_x = 2\left[\frac{Q}{2\varepsilon_0 A}\right] = \frac{Q}{\varepsilon_0 A}$

ii) $V = -\int_0^d E_x dx = |-E_x d| = \frac{Qd}{\varepsilon_0 A}$

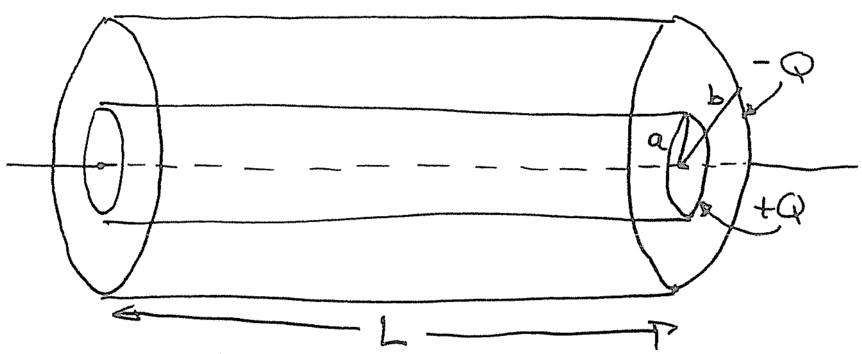
iii) $C = \frac{Q}{V} = Q/Qd/\varepsilon_0 A = \frac{\varepsilon_0 A}{d}$.

Since $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m (from the dimensions of the result) if the plates are 1m x 1m and are separated by 10^{-3}m the capacitance is

$C = (8.85 \times 10^{-12} \frac{F}{m}) \times (1m^2) \times (10^3 m^{-1}) = 8.85 \times 10^{-9} \approx .01\mu F$ (!)

Note that in order to make a 1 Farad capacitor we would need plates 1km x 1km ($A = 10^6 m^2$) separated by a distance of .01 mm (10^{-5}m). That's a pretty big capacitor!

Cylindrical Capacitor: Find the capacitance of cylindrical shells of radii a and b of length L and carrying a charge +Q and -Q.



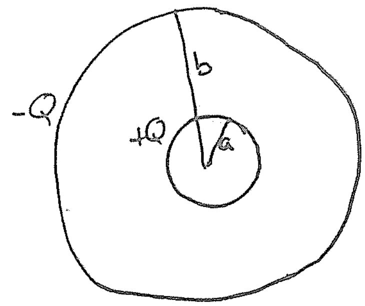
Solution: From Gauss's law,

i) $\phi^S = E_r 2\pi r \ell = \frac{Q\ell}{\varepsilon_0 L} \longrightarrow E_r = \frac{Q}{2\pi\varepsilon_0 L}\frac{1}{r}$

ii) $$\Delta V = -\int_a^b E_r\,dr = -\frac{Q}{2\pi\varepsilon_0 L}\ln\left[\frac{b}{a}\right]$$

iii) $$C = \frac{Q}{\Delta V} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}$$

Spherical Capacitor: Find the capacitance of two spherical shells of radii a and b carrying a charge +Q and −Q.



Solution:

i) $$\phi^S = E_r 4\pi r^2 = \frac{Q}{\varepsilon_0} \longrightarrow E_r = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}$$

ii) $$\Delta V = -\int_a^b E_r\,dr = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{b} - \frac{1}{a}\right]$$

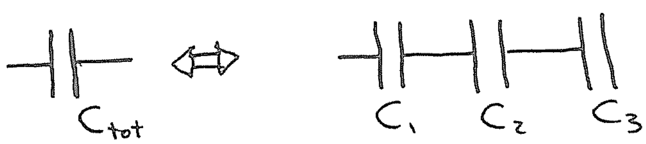
iii) $$C = \frac{Q}{\Delta V} = \left|4\pi\varepsilon_0\frac{ab}{a-b}\right| = 4\pi\varepsilon_0\frac{ab}{b-a}$$

---

4) Adding Capacitors.

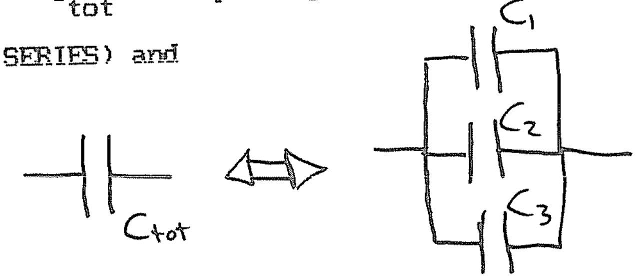
The symbol for capacitors is —||— .

Then



$$\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

(for capacitors in SERIES) and



$$C_{tot} = C_1 + C_2 + C_3$$

(for capacitors in PARALLEL).

Recall

1) $C = Q/V$

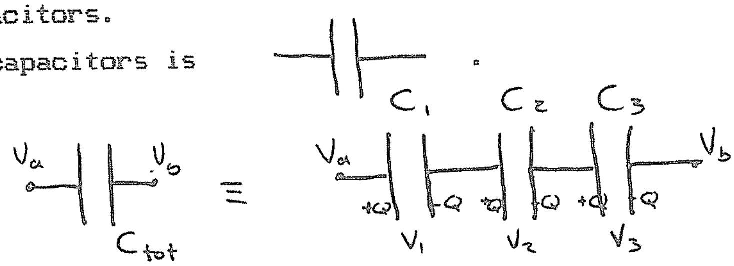
2) $C = \dfrac{\varepsilon_0 A}{d}$ (for parallel plate capacitor).

Today

2) Adding Capacitors.

The symbol for capacitors is —||— .

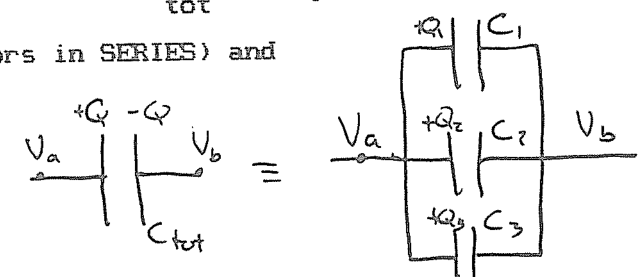
Then



$$V_b - V_a = \frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

$$\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

(for capacitors in SERIES) and



$$V_b - V_a = \frac{Q_1}{C_1} = \frac{Q_2}{C_2} = \frac{Q_3}{C_3} = \frac{Q_1 + Q_2 + C_3}{C}$$

$$Q = Q_1 + Q_2 + Q_3$$

$$C_{tot} = C_1 + C_2 + C_3$$

(for capacitors in PARALLEL).

---

3) Energy stored in a Capacitor.

$$dU = Vdq = \frac{q}{C}dq$$

$$U = \int dU = \int_0^{Q_0} \frac{q}{C}dq = \frac{1}{2}\frac{Q_0^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}Q_0 V$$

4) Electrostatic Field energy.

In terms of the field E_0 in a (parallel plate) capacitor,

$$U = (1/2)CV_0^2 = (1/2)(\varepsilon_0 A/d)(E_0 d)^2 \quad \text{or}$$

$$U = \frac{1}{2}\varepsilon_0 E_0^2 Ad = \frac{1}{2}\varepsilon_0 E_0^2 V$$

(with V = volume of capacitor Ad). We define electric field energy density as

$$\eta = \frac{U}{V} = \frac{1}{2}\varepsilon_0 E_0^2$$

This is the energy stored in the electric field per unit volume. It is this mechanism (as we shall see) that is responsible for the energy carried by light waves.

Recall

1) $C = Q/V$, $C = \dfrac{\varepsilon_0 A}{d}$ (for parallel plate capacitor).

2) $U = \dfrac{1}{2}CV^2 = \dfrac{1}{2}QV = \dfrac{1}{2}\cdot\dfrac{Q^2}{C}$ (energy stored in a capacitor)

Today

3) Electrostatic Field energy.

In terms of the field E_0 in a (parallel plate) capacitor,

$$U = (1/2)CV_0^2 = (1/2)(\varepsilon_0 A/d)(E_0 d)^2 \quad \text{or}$$

$$U = \frac{1}{2}\varepsilon_0 E_0^2 Ad = \frac{1}{2}\varepsilon_0 E_0^2 V$$

(with V = volume of capacitor Ad). We define electric field energy density as

$$\eta = \frac{\Delta U}{\Delta V} = \frac{1}{2}\varepsilon_0 E_0^2 (\vec{r})$$

This is the energy stored in the electric field per unit volume. It is this mechanism (as we shall see) that is responsible for the energy carried by light waves.

Example: What is the electric field energy stored on a Spherical conductor with radius R and total charge Q?

a) $U = \dfrac{1}{2}\cdot QV = \dfrac{1}{2}\cdot\dfrac{kQ^2}{R}$ <u>or</u>

b) $dU = \eta dV = \dfrac{1}{2}\varepsilon_0 |E|^2 (4\pi r^2 dr) \longrightarrow U = \dfrac{1}{2}\cdot\dfrac{kQ^2}{R}$ also.

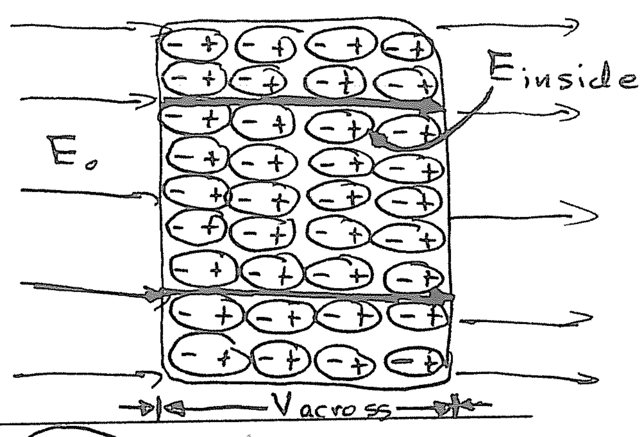
---

4) <u>Dielectrics.</u> A dielectric is defined to be an insulator that reduces an applied external electric field inside. The <u>dielectric constant</u>, K, is related to the electric field and the potential by

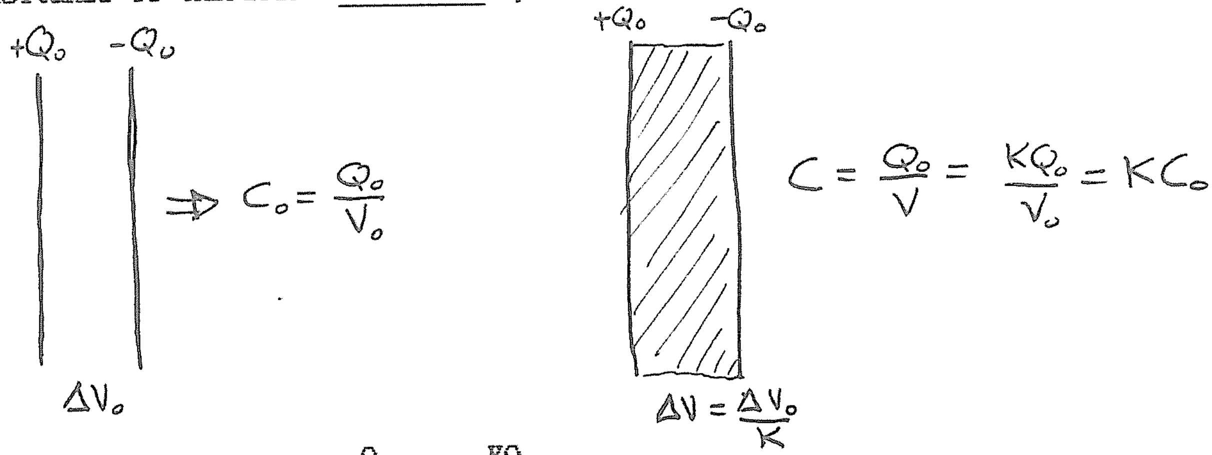
a) $E_{inside} = \dfrac{E_0}{K}$

b) $V_{across} = \dfrac{V_0}{K}$.

E_0 <u>partially</u> <u>cancelled</u> inside.

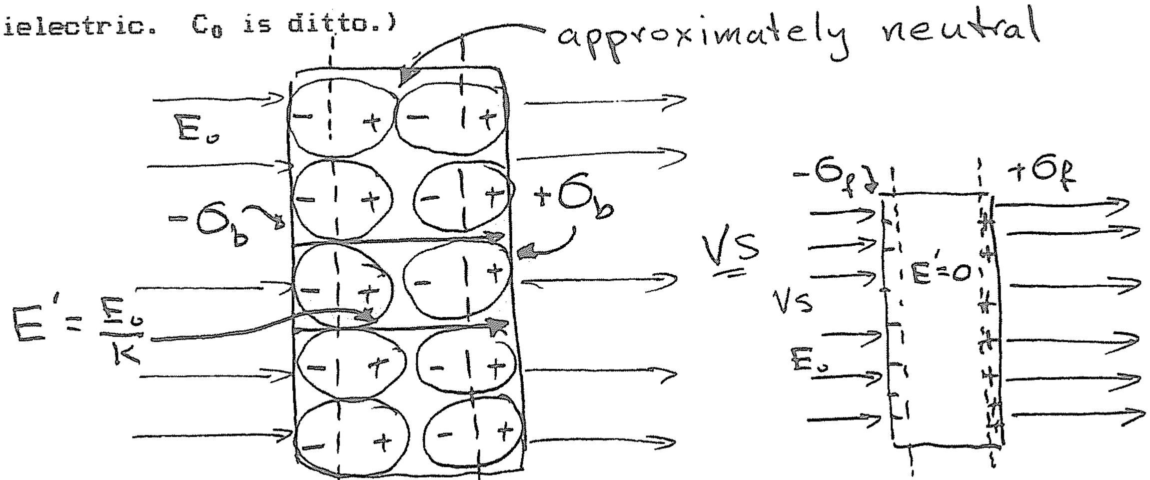


If a capacitor is filled with a dielectric substance between the plates (or shells, etc.), the field, and hence potential difference, between the plates is <u>reduced</u> by 1/K. Since the capacitance is the ratio between charge and potential difference, the capacitance is therefore <u>increased</u> by a factor of K.



c) $C_{\text{with dielectric}} = \frac{Q_0}{V} = \frac{KQ_0}{V_0} = KC_0.$

(V_0 is the potential difference between the plates in the absence of the dielectric. C_0 is ditto.)



Finally, the field inside a dielectric is reduced because of a small surface polarization charge induced by the electric field. This surface polarization charge is called the "bound" surface charge density σ_b and is related to the "free" surface charge density that would be built up on an equivalent conductor by

d) $E = \frac{E_0}{K} = E_0 - E'$

$$E' = E_0\left(1 - \frac{1}{K}\right) = E_0\left[\frac{K - 1}{K}\right]$$

or $\left(E' = \frac{\sigma_b}{\epsilon_0},\ E_0 = \frac{\sigma_f}{\epsilon_0}\right)$

$$\sigma_b = \sigma_f\left[\frac{K - 1}{K}\right]$$

5) Three functions of a dielectric in Capacitor design.

a) Increases capacitance (K>1)

b) Mechanically separates plates (which are strongly attracted together, as they have equal and opposite charges and are very close together).

c) Helps prevent dielectric breakdown. Dielectric breakdown for air occurs at field strengths of around 3MV/m. Dielectric breakdown for porcelain (for example) occurs at around 24MV/m.

Finally, a note to interested readers of these notes. You are responsible for reading the text. Therefore you are responsible for a qualitative understanding of electrostatic precipitators, xerography, Benjamin Franklin's work, and stuff like that out of the stories at the end of chapters.

(No additional lecture notes)

1) We <u>define</u> current to be the <u>flow of charge</u>. Specifically, it is the quantity of charge that passes a given point (or goes through a given surface!) per unit time.

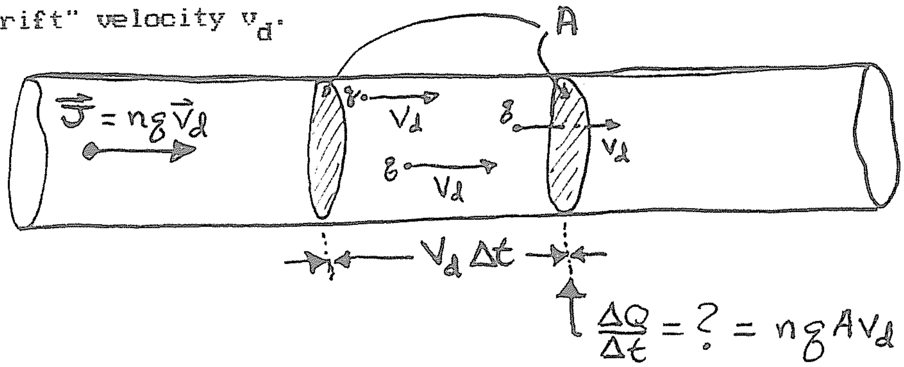
$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}.$$

Units (SI) of current are therefore

1 Ampere = 1 Coulomb/ 1 second.

---

2) Suppose one has a conductor with n free charge carriers per unit volume, each moving to the right as drawn below with some average "drift" velocity v_d.



Then in a time Δt, all the charge carriers in the volume $Av_d\Delta t$ will pass the surface at the point marked. If each charge carrier carries a charge q (usually + or − e) then the total charge that passes that point will be

$$\Delta Q = nqAv_d\Delta t,$$

so the current will be

$$I = \Delta Q/\Delta t = nqv_dA.$$

---

3) The <u>current density</u> is the current per unit area that crosses a given surface. In the picture above it is

$$\vec{J} = \frac{I}{A} = nq\vec{v}_d.$$

Since v_d is properly a vector, and since the current flows most strongly in the direction of v_d, we can make the current density a vector quantity. When we do this, the current becomes the <u>flux of the current density</u> (I told you we would see more fluxes).

$$I = \int_A \vec{J}\cdot\hat{n}dA = nqA|v_d|$$

---

f) since $EL = \Delta V = IR = (AJ)R = (AJ)\frac{L}{A\sigma}$

$\vec{J} = \sigma\vec{E}$.

---

6) Energy in electric circuits.

$$\Delta W = \Delta Q(V_a - V_b) = \Delta Q \Delta V$$

so

$$\frac{\Delta W}{\Delta t} = \frac{\Delta Q}{\Delta t}\Delta V = I\Delta V = IV$$

This is the power.

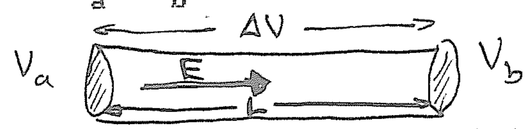
$$P = IV = I^2R = \frac{V^2}{R}.$$

---

4) Ohm's Law. A conductor carrying a current is not in electrostatic equilibrium. The conductor therefore can (and does!) have an electric field inside it. This electric field is needed to push the electrons through the conductor against the resistance to flow created by "bouncing" off of the atoms in the conductor.

Since the electrons have a force exerted on them continually by the electric field, and move a certain distance through the conductor, work is done moving them along. This work is done by the electric force, and can therefore be viewed as the result of moving across a potential difference

$$\Delta V = V_a - V_b = EL \quad \text{(for a wire of length L)}$$



An experimentally observed phenomenon is that, for most materials at most temperatures, the current flow in a wire is directly proportional to the potential difference across it. The constant of proportionality is called the resistance.

$$\Delta V = IR$$

This is Ohm's Law.

---

5) Relations between field, current, potential, resistance and all that.

a) $R = \dfrac{\Delta V}{I}$ (Units: 1 Ohm = 1 Volt-second/Coulomb)

b) $R = \rho \dfrac{L}{A}$ (ρ is the resistivity of a conductor)

c) $\sigma = \dfrac{1}{\rho}$ (σ is the conductivity of a conductor)

d) $R = \dfrac{L}{\sigma A}$ (resistivity = inverse of conductivity)

e) Resistivity increases (approximately linearly) with temperature (for many substances).

Recall

1) The electric current $I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$.

2) $I = \Delta Q/\Delta t = nqv_d A$. (In a n, q, v_d, A conductor).

3) The current density is the vector current per unit area:
$\vec{J} = \frac{I}{A} = nq\vec{v}_d$. Then

4) $I = \int_A \vec{J}\cdot\hat{n}dA$ (current is flux of current density)

Today

1) Ohm's Law.

An experimentally observed phenomenon is that, for most materials at most temperatures, the current flow in a wire is directly proportional to the potential difference across it. The constant of proportionality is called the resistance.

$$\Delta V = IR$$

This is Ohm's Law.

---

2) Relations and definitions:

a) $R = \frac{\Delta V}{I}$ (Units: 1 Ohm = 1 Volt-second/Coulomb)

b) $R = \rho\frac{L}{A}$ (ρ is the resistivity of a conductor)

c) $\sigma = \frac{1}{\rho}$ (σ is the conductivity of a conductor)

d) $R = \frac{L}{\sigma A}$ (resistivity = inverse of conductivity)

e) Resistivity increases (approximately linearly) with temperature (for many substances). $\rho = \rho_{20}[1 + \alpha(t - 20°C)]$ (α is slope of line)

f) since $EL = \Delta V = IR = (AJ)R = (AJ)\frac{L}{A\sigma}$

$\vec{J} = \sigma\vec{E}$.

Explain Water analogy: P = V, F = I, R = R, Ball = C etc.

---

6) Energy in electric circuits.

$$\Delta W = \Delta Q(V_a - V_b) = \Delta Q\Delta V$$

so

$$P = \frac{\Delta W}{\Delta t} = \frac{\Delta Q}{\Delta t}\Delta V = I\Delta V = IV$$

This is the power dissipated (as heat or work) in any element of an electric circuit.

---

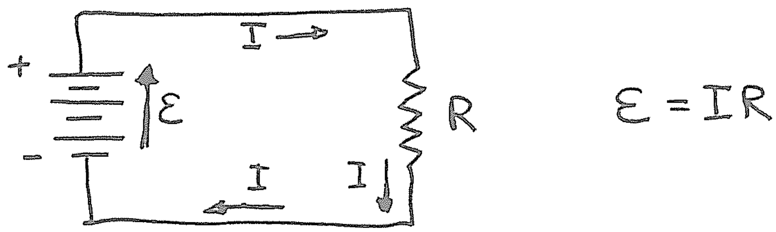
7) In the case of resistors and resistance, the power dissipated appears as heat (Joule heating) and is related to voltage, current and/or resistance by

$$P = IV = I^2R = \frac{V^2}{R}.$$

Resistance and Joule heating are the analogue, in electrical circuits and motion of electric charge, of friction in ordinary mechanical systems. It (irreversibly) turns potential (energy) into heat.

---

8) A battery (or other device for generating a potential difference) is called (generically) an electromotive force (emf). The symbol employed for the potential generated by an emf is $\mathcal{E}$ and is drawn



The power supplied by an emf to an electric circuit is (using identical arguments to those above

$$P = \frac{\Delta W}{\Delta t} = \frac{\Delta Q\mathcal{E}}{\Delta t} = \mathcal{E}I$$

Frequently an emf produces heat when it is functioning, due to its own internal resistance. This energy must be included when balancing the energy flow of a circuit.

---

9) Assorted True Facts You Are Responsible For.

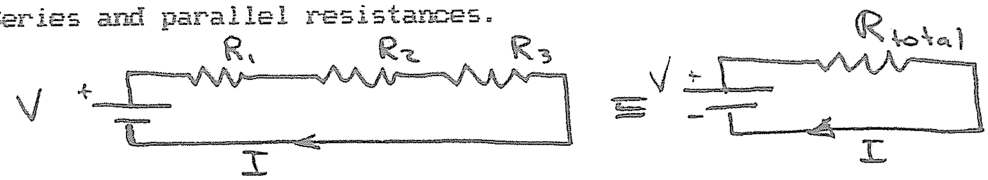
1) Classical conduction model "correctly" predicts Ohm's law (v_d directly proportional to electric field). But, it gives wrong order of magnitude for resistivity and/or conductivity and gives completely incorrect temperature dependence of resistance.

2) Correct (Quantum mechanical) conduction picture views electrons as waves scattered through periodic lattice of conductor. In addition to predicting Ohm's law, this yields correct temperature dependence down to very low temperatures. A more extended analysis correctly predicts superconductivity.

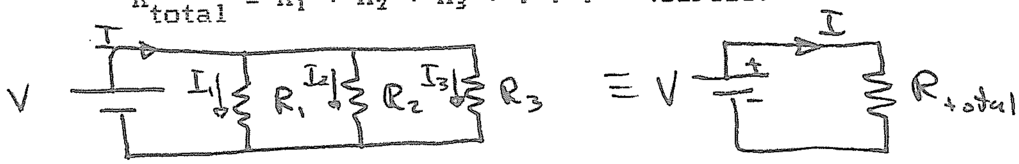
3) Superconductivity occurs in many substances as they approach absolute zero in temperature. It is a purely quantum mechanical phenomenon. When a superconducting substance is cooled below a critical temperature, its electrical resistance becomes zero. (Not "very small". Zero!)

4) There are three (rather arbitrary) categories into which materials may be classified. These are conductors, insulators, and semiconductors. They are differentiated by the presence, absenge, or size of a band gap. If a large gap occurs between the top of the valence band and the bottom of the conduction band, a substance is an insulator. If a small gap separates the top of the valence band and the bottom of the conduction band, a substance is a semiconductor. If the top of the valence band occurs inside (i.e.--with no gap) a band full of unoccupied states (conduction band) due to overlap or filling, a substance is a conductor.

1) Series and parallel resistances.



$$R_{total} = R_1 + R_2 + R_3 + \ldots \qquad \text{(Series)}$$



$$R_{total}^{-1} = R_1^{-1} + R_2^{-1} + R_3^{-1} + \ldots \qquad \text{(Parallel)}$$

2) Kirchoff's Laws

a) The sum of potential differences around any closed circuit loop is zero. (This is energy conservation in disguise).

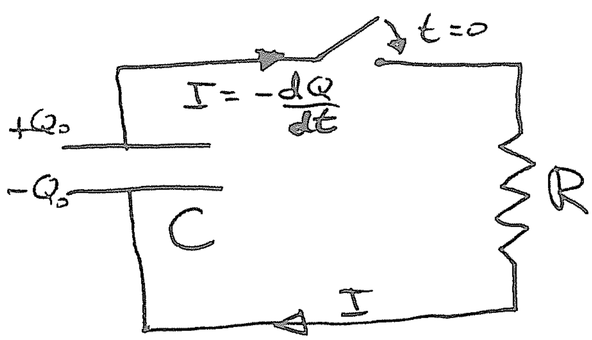
$$\sum_{loop} \Delta V_i = 0$$

b) The sum of currents leading into (+) and out of (-) any circuit junction is zero. (This is charge conservation in disguise).

$$\sum_{junction} I_i = 0$$

3) RC-Circuits

a) Discharging a capacitor (initially charged to Q_0.)



$$\frac{Q_0}{C} = IR \qquad \text{with } I = -\frac{dQ}{dt}$$

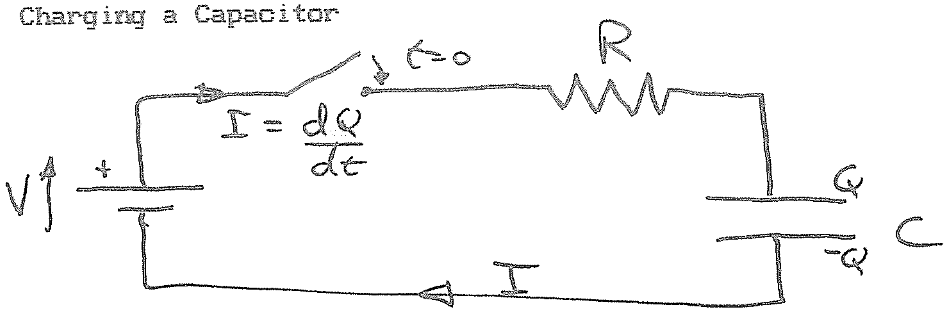
or

$$\frac{dQ}{dt} + \frac{Q}{RC} = 0$$

with solution $\qquad Q(t) = Q_0 e^{-t/RC}.$

(The <u>time constant</u> of an RC circuit is called $\tau_c = RC$. It is the time required for the charge to decay to $1/e = .36788$ -- about one third -- its original value.)

b) Charging a Capacitor



$$V - IR - \frac{Q}{C} = 0 \quad (\text{with } I = \frac{dQ}{dt}).$$

or

$$V = \frac{dQ}{dt} + \frac{Q}{RC}.$$

This is an inhomogeneous first order, linear differential equation. We solve it by adding a constant to the solution to the homogeneous differential equation (see a) and evaluating the constants to match the initial conditions.

$$Q(t) = Q_c + Q_h e^{-t/RC}$$

(we substitute this into the D. E. above)

$$V = -\frac{Q_h}{C}e^{-t/RC} + (\frac{Q_h}{C}e^{-t/RC} + \frac{Q_c}{C})$$

$$= \frac{Q_c}{C}, \quad \text{so}$$

$$Q_c = CV, \quad \text{and}$$

$$Q(t) = CV + Q_h e^{-t/RC}.$$

At time $t = 0$, $Q(0) = 0$ (the capacitor is initially uncharged).

Thus

$$Q_h = -Q_c = -CV, \quad \text{and}$$

$$Q(t) = CV(1 - e^{-t/RC})$$

is the complete solution to the inhomogeneous equation above.

Note:

a) $I(t) = \frac{dQ}{dt} = \frac{V}{R}e^{-t/RC}$

b) $V_C(t) = \frac{Q(t)}{C} = V(1 - e^{-t/RC})$

c) $V_R(t) = I(t)R = Ve^{-t/RC}$ (so that $V = V_C(t) + V_R(t)$!)

etc,

Power, energy, and all that.

$$P = VI, \quad \text{so}$$

Discharging,

$$P_C = \frac{Q(t)}{C} I(t) = + \frac{Q_0^2}{RC^2} e^{-2t/RC} \quad \text{(adding energy \underline{to} circuit)}$$

$$P_R = I(t)^2 R = - \frac{Q_0^2}{RC^2} e^{-2t/RC} \quad \text{(taking energy \underline{from} circuit)}$$

And so on. Exercise: Show that as $t \to \infty$, the total energy dissipated by the resistor equals the total energy initially on the capacitor, $\frac{1}{2} Q_0^2/C$. hint-- $U(t) = \int_0^t P(t)dt$

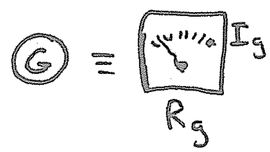
Recall

$Q(t) = Q_0 e^{-t/RC}$ (discharging)

$Q(t) = Q_f(1 - e^{-t/RC})$ (charging)

Today

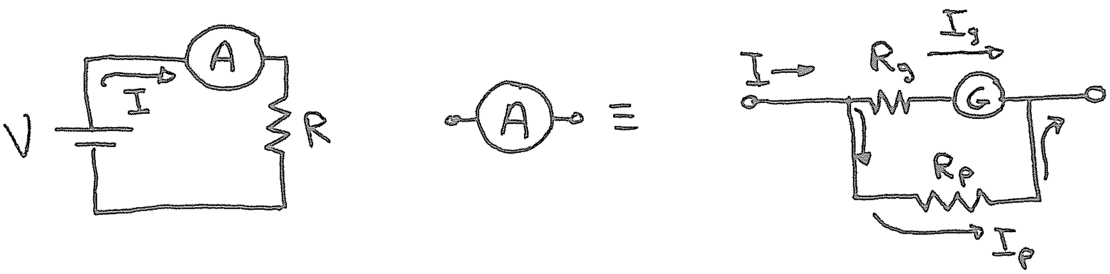
1) A <u>Galvanometer</u> is a device for measuring small currents. It has a scale (and a maximum current corresponding to maximum scale deflection) and an internal resistance (I_g and R_g respectively).



---

2) A galvanometer can be used to build

a) An Ammeter



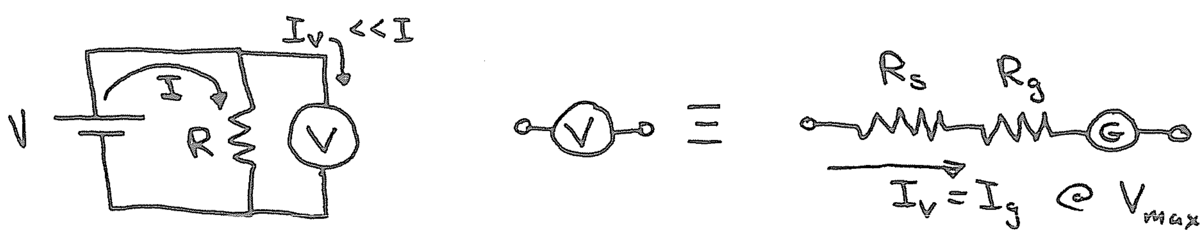
Solution: $R_p \ll R_g$, and $I_g R_g = I_p R_p$. Also, $I = I_p + I_g \approx I_p$.

Then:
$$R_p = \frac{I_g R_g}{I_p} = \frac{I_g R_g}{I_{tot}}$$

(example, to make 5A full scale ammeter, if $R_g = 20\Omega$ and $I_g = 5\times10^{-4}$ A, $I = I_p = 5$A, $R_p = 2\times10^{-3}\Omega$).

---

b) Voltmeter



Solution: $R_s \gg R_g$ and $I_g(R_s + R_g) = V_{max}$ so

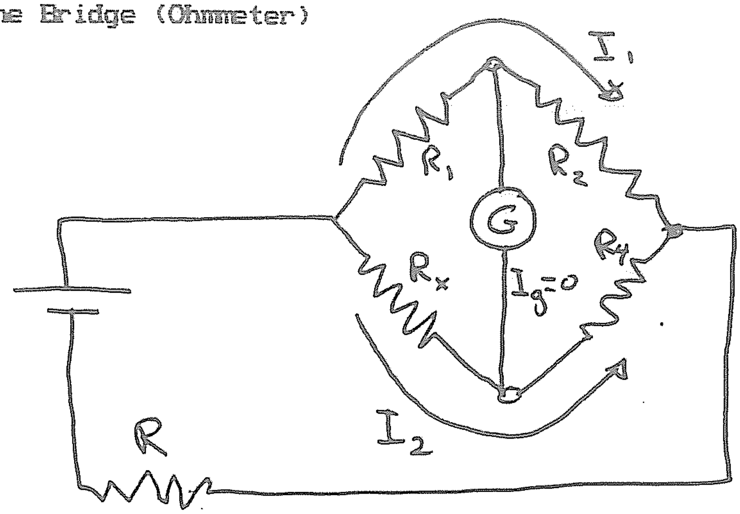
$$R_s + R_g = \frac{V_{max}}{I_g} \text{ or}$$

$$R_s = \frac{V_{max}}{I_g} - R_g \approx \frac{V_{max}}{I_g}$$

(example: $R_g = 20\Omega$, $I_g = 5\times10^{-4}$A $\longrightarrow R_s = 20k\Omega$ for $V_{max} = 10V$).

---

c) Wheatstone Bridge (Ohmmeter)



Solution. Adjust R_1 and R_2 (usually complementary slide resistors) so that $I_g = 0$. Then $V_1 = V_x$ and $V_2 = V_3$ and

$$I_1 R_1 = I_2 R_x$$

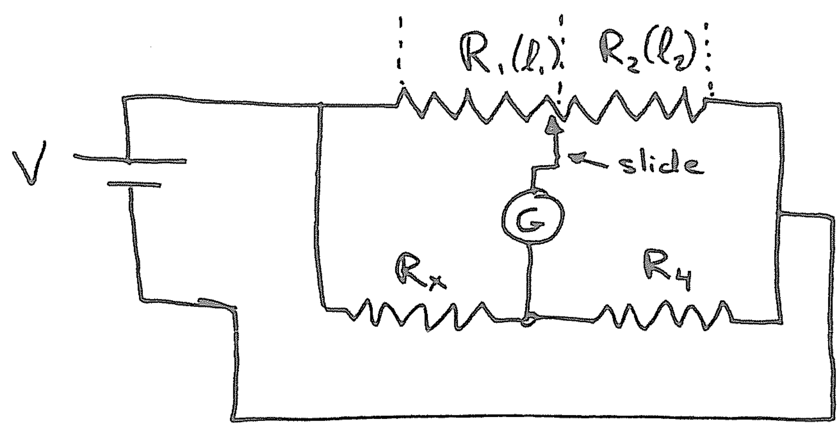
$$I_1 R_2 = I_2 R_4$$

and dividing we get $$\frac{R_1}{R_2} = \frac{R_x}{R_4}$$

or $$R_x = R_4 \frac{R_1}{R_2}.$$

If R_1 and R_2 are complementary slide resistors, so that $R_1 = \ell_1 \rho/A$ and $R_2 = \ell_2 \rho/A = (L-\ell_1)\rho/A$, then

$$R_x = R_4 \frac{\ell_1}{\ell_2} = R_4 \frac{\ell_1}{(L - \ell_1)}.$$



Magnetism

True Facts time. Moving charges exert a force on each other that is different from the (coulomb) force exerted by stationary charges on each other. The "new" force thus introduced is called the Magnetic Force. As usual, the magnetic force is to much of a hassle to treat directly all the time, so we invent the magnetic field (or magnetic induction) $\vec{B}$ such that it describes the force on a single moving charge.

Unfortunately (for you the student, not for mother nature) when we go into the laboratory to figure out how the magnetic force on a charge q and field are related, we get some very peculiar results.

a) The force is proportional to the charge q.

b) The force is proportional to the speed of the charge v. (?!?)

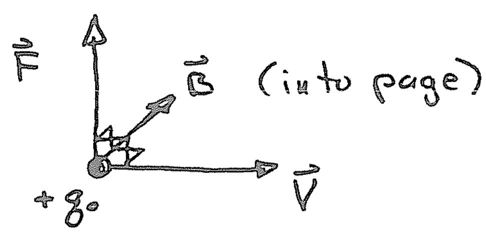
c) The magnitude and direction of the force depend on the direction of the velocity $\vec{v}$ of the charge as follows. If the velocity points in a certain direction (parallel to $\vec{B}$, as it turns out) then the force is zero. If the velocity makes an angle θ with this line, then the force as proportional to $\sin\theta$ and points in a direction perpendicular to both the field and the velocity (????)

d) The force on negative charges is the opposite of the force on positive charges.

Yuck-o, right? We summarize these empirical results as

$$\vec{F} = q(\vec{v} \times \vec{B})$$

(where $|\vec{v} \times \vec{B}| = |\vec{v}||\vec{B}|\sin\theta$, i.e.--the vector (cross) product). $\vec{B}$ is the magnetic field, also called the magnetic induction (for obscure and unimportant historical reasons) and its relative geometry is drawn below:



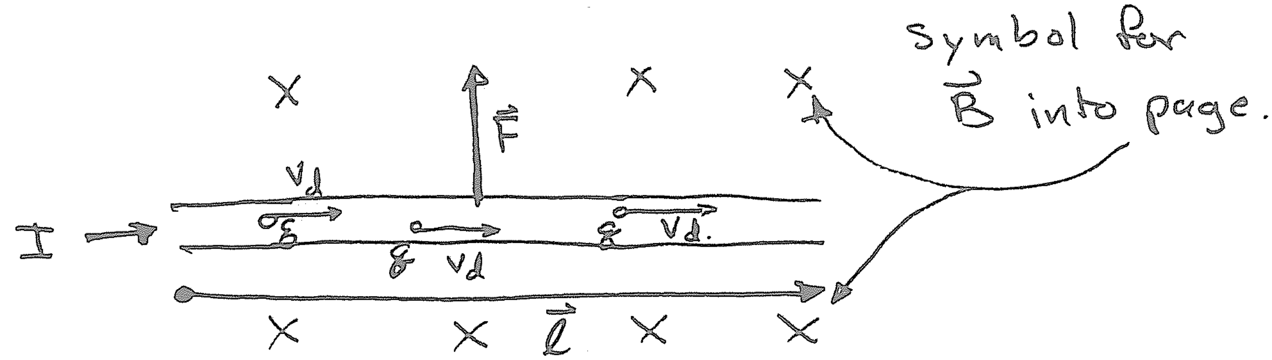
When a wire carries a current (made up of moving charges), it therefore experiences a magnetic force. It is

$$\vec{F} = (q\vec{v}_d \times \vec{B})nA\ell = I(\vec{\ell} \times \vec{B}).$$

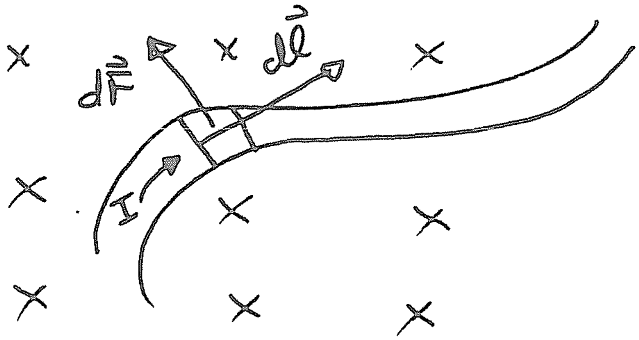
But this is only correct for a straight wire. For a curved wire, we have to add all the little (differential, locally straight) pieces that make up the curve, each with force

$$d\vec{F} = I(d\vec{\ell} \times \vec{B}).$$

We will only be able to do this integral in a <u>very</u> few cases.



or



Magnetism

True Facts time. Moving charges exert a force on each other that is different from the (coulomb) force exerted by stationary charges on each other. The "new" force thus introduced is called the Magnetic Force. As usual, the magnetic force is to much of a hassle to treat directly all the time, so we invent the magnetic field (or magnetic induction) $\vec{B}$ such that it describes the force on a single moving charge.

Unfortunately (for you the student, not for mother nature) when we go into the laboratory to figure out how the magnetic force on a charge q and field are related, we get some very peculiar results.

a) The force is proportional to the charge q.

b) The force is proportional to the speed of the charge v. (?!?)

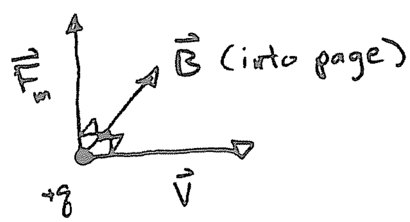
c) The magnitude and direction of the force depend on the direction of the velocity $\vec{v}$ of the charge as follows. If the velocity points in a certain direction (parallel to $\vec{B}$, as it turns out) then the force is zero. If the velocity makes an angle θ with this line, then the force as proportional to $\sin\theta$ and points in a direction perpendicular to both the field and the velocity (????)

d) The force on negative charges is the opposite of the force on positive charges.

Yuck-o, right? We summarize these empirical results as

$$\vec{F} = q(\vec{v} \times \vec{B})$$

(where $|\vec{v} \times \vec{B}| = |\vec{v}||\vec{B}|\sin\theta$, i.e.--the vector (cross) product). $\vec{B}$ is the magnetic field, also called the magnetic induction (for obscure and unimportant historical reasons) and its relative geometry is drawn below:

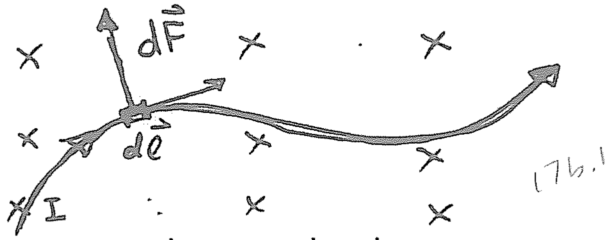


17a.1

When a wire carries a current (made up of moving charges), it therefore experiences a magnetic force. It is

$$\vec{F} = (q\vec{v}_d \times \vec{B})nA\ell = I(\vec{\ell} \times \vec{B}).$$

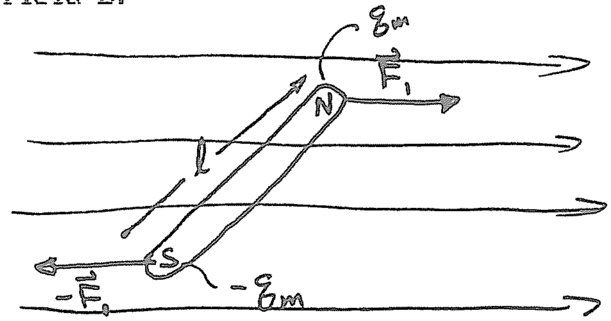
But this is only correct for a straight wire. For a curved wire, we have to add all the little (differential, locally straight) pieces that make up the curve, each with force



$$d\vec{F} = I(d\vec{\ell} \times \vec{B}).$$

We will only be able to do this integral in a <u>very</u> few cases.

Why magnetic fields (where are the magnets)? Because magnetic fields exert forces on magnets (and magnets exert magnetic forces on moving charges). Suppose we have a bar magnet of length $\vec{\ell}$ in a uniform magnetic field $\vec{B}$.



If we make up a "charge" corresponding the the electric charge,

$$\vec{F}_m = q_m\vec{B}$$

(where q_m is called the <u>pole strength</u> of a permanent bar magnet) then, since a bar magnet has two equal and opposite poles,

$$F_m = (q_m - q_m)\vec{B} = 0$$

and $$\vec{\tau} = \vec{\ell} \times \vec{F} = \vec{\ell} \times q_m\vec{B} = q_m\vec{\ell} \times \vec{B} = \vec{m} \times \vec{B}$$

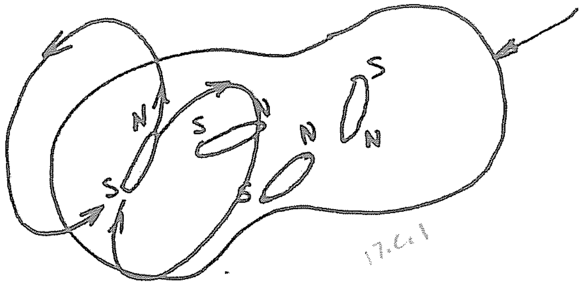
where $$\vec{m} = q_m\vec{\ell}$$

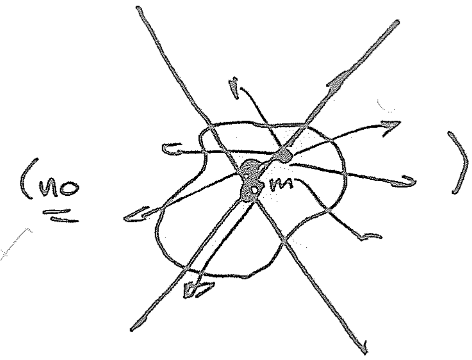
is called the <u>magnetic moment</u>.

Gauss' Law for magnetostatic fields

Maxwell's Equation #2

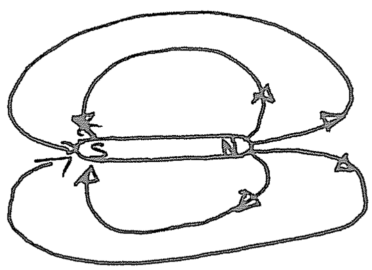
$$\oint_S \vec{B}\cdot\hat{n}\,dA = 4\pi k_m q_m = 0$$



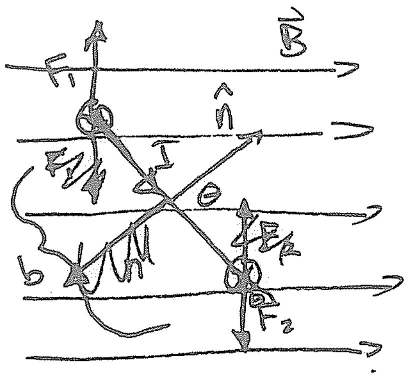


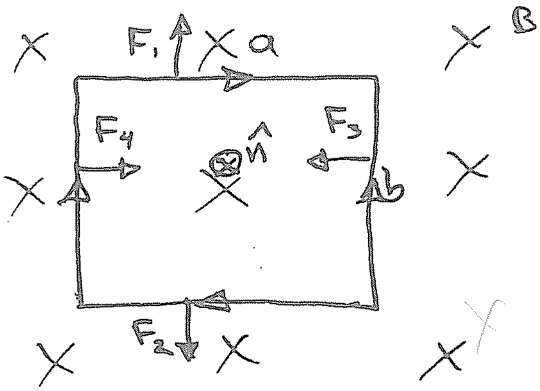
Meaning: While we can define magnetic "pole strength" or "magnetic charge" and make up Gauss' Law for magnetism, above, the zero in the equation says There Are No Magnetic Monopoles In Nature. Yet. In other words, we NEVER see (or more properly, have never seen) a "north" magnetic pole (the equivalent of a positive magnetic charge) without a "south" pole attached. Magnetic fields are always (at least) dipole fields.

We can draw the magnetic equivalent of "lines of force", called lines of magnetic induction. They look like the field of an electric dipole for a bar magnet.



Example: Find the torque on a (rectangular) current loop placed in a magnetic field.





Solution: $F_1 = F_2 = IaB$, $F_3 = F_4 = IbB$ (with directions shown above. F_3 and F_4 cancel, as do F_1 and F_2, but F_1 and F_2 exert a net torque

$$|\vec{\tau}| = |F_2||b|\sin\theta \text{ as drawn.}$$
$$= IabB\sin\theta$$

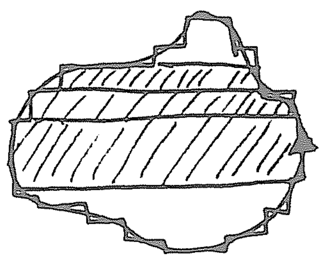
or

$$\vec{\tau} = IA\hat{n} \times \vec{B} = \vec{m} \times \vec{B}$$ (into page on left above)

with

$$\vec{m} = IA\hat{n}.$$

$\hat{n}$ is a unit vector perpendicular to the plane of the loop pointing in the direction given by the right-hand rule (the direction your right thumb points when your fingers curl around in the direction of the current). This result is actually quite general, because a plane figure can always be represented by little rectangular chunks distributed around the curve.



In general, the magnetic moment of such a figure with N loops is

$$\vec{m} = NIA\hat{n}$$

where $\hat{n}$ is a unit vector whose direction is determined by the right-hand rule.

Motion of a Point Charge in a (Uniform) Magnetic field.

1) Since $\vec{F}_m$ is perpendicular to $\vec{v}$, magnetostatic fields do no work.

2) If $\vec{v}$ is perpendicular to $\vec{B}$, a charged particle moves in a circle (under a centripetal force).

3) If $\vec{v}$ has a component parallel to $\vec{B}$, a charged particle moves in a helical path along the B-direction.

4) The fundamental equation governing the motion of point charges in a uniform magnetic field is (for the part of $\vec{v}$ perpendicular to the field only)

$$qvB = \frac{mv^2}{r}.$$

From this equation doth all others follow Know it.

For example, the radius of a given particle's orbit is

5) $$r = \frac{mv}{qB},$$

its angular frequency (or angular velocity) is

6) $$\omega = \frac{v}{r} = \frac{qB}{m}$$

(this is called the Cyclotron Frequency) and the period of its orbit is

7) $$T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB}.$$

You are responsible for understanding and reproducing the following applications/examples:

8) Cyclotron: Based on ΔV alternating at ω above (cyclotron frequency).

9) Region of crossed fields: Forms a velocity selector such that $|\vec{v}| = E/B$.

10) Mass spectrograph: Used to determine unknown masses of atomic/chemical constituents. Based on

$$\frac{m}{q} = \frac{B^2r^2}{2V}.$$

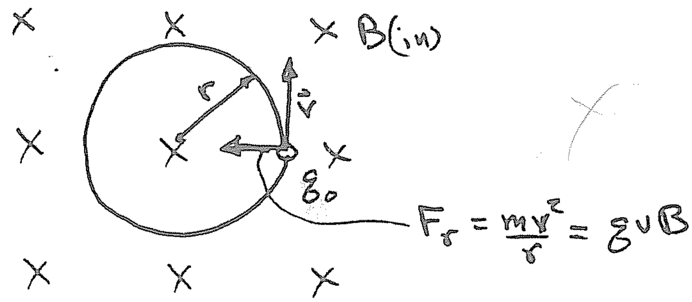
11) <u>The Hall effect</u>: Either measures n (charge carrier density) or the magnetic field (if n is known) by measuring ΔV across a conducting strip placed in a magnetic field. Just a region of crossed fields inside a conductor.

$$\Delta V = \frac{IBd}{Aqn}$$

Motion of a Point Charge in a (Uniform) Magnetic field.

Since the magnetic force on a moving charge is perpendicular to its direction of motion, magnetic fields do no work !!! The magnetic force changed the direction of motion of a charged particle, not its speed.

If the velocity vector of a charged particle is perpendicular to the magnetic field, then the particle moves in a circular orbit:



(because F is perpendicular to v and both lie in a plane perpendicular to B). The basic equation of motion satisfied is then one of centripetal force

$$qvB = \frac{mv^2}{r}.$$

From this equation doth al' others follow Know it.

For example, the radius of a given particle's orbit is

$$r = \frac{mv}{qB},$$

its angular frequency (or angular velocity) is

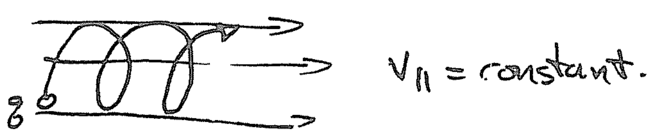
$$\omega = \frac{v}{r} = \frac{qB}{m}$$

and the period of its orbit is

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB}.$$

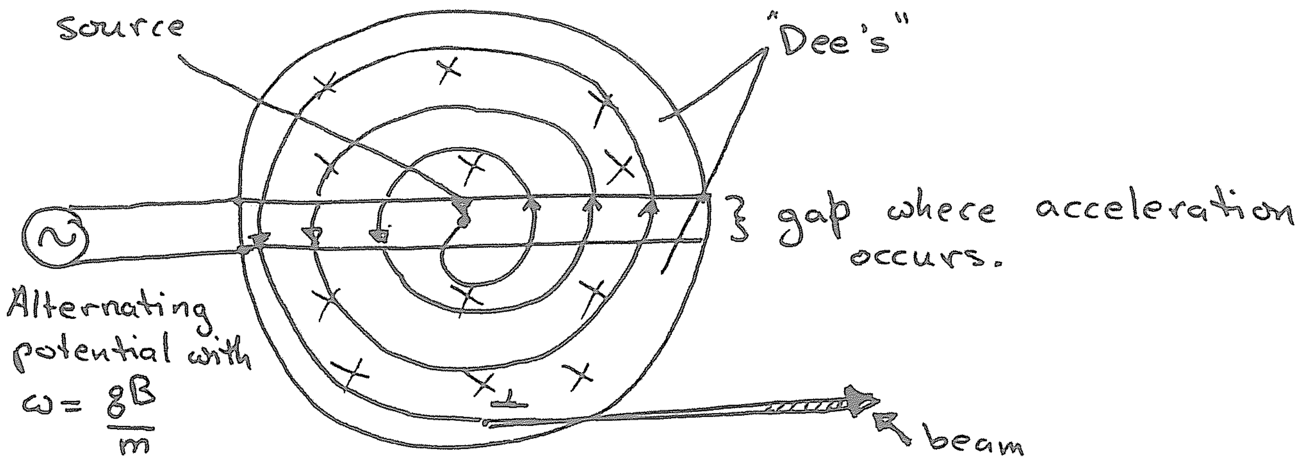
Note that the frequency of a given particle's orbit does not depend on its velocity or the radius of its orbit! The frequency ω is called the cyclotron frequency, for reasons that will become apparent when we "do" the cyclotron.

If the velocity of the charged particle is not perpendicular to B, then (since $F_{||}$ is 0), the particle describes a helix, i.e.-- it moves uniformly along B but moves in circles perpendicular to B.



Cyclotron

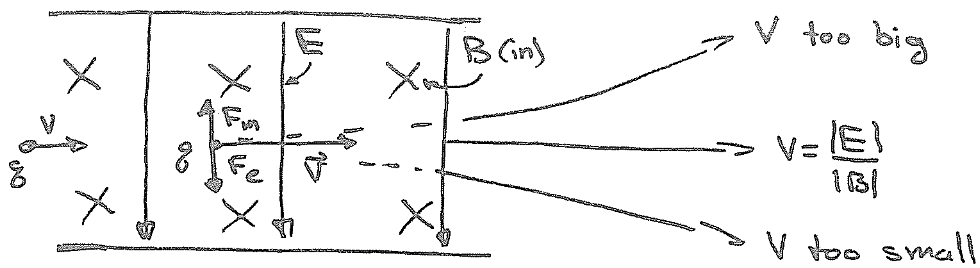
A cyclotron is a device for accelerating charged particles (protons, deuterons, alpha particles, etc.) to kinetic energies in the 1-30 MeV range. It works by tricking a proton (e.g.) into "falling" across the same, small potential (with the potential difference) lots of times until a large kinetic energy is built up.



A particle begins at the source, in the center of the cyclotron. It spirals outward from the center, passing between the "D" 's once every half cycle. Since the time it takes to make a revolution is independent of its speed or the radius of its orbit, by adjusting the frequency of an alternating potential difference between the D's to be the same as the (cyclotron) frequency of its orbit, the particle falls with the potential at every point and speeds up as it spirals out.

A particles final speed is limited only by a) the strength of the magnetic field, b) the radius of the cyclotron, and most importantly, by c) relativity. When the particle is moving sufficiently rapidly, the frequency shifts relativistically and the cyclotron condition no longer holds.

Region of crossed fields



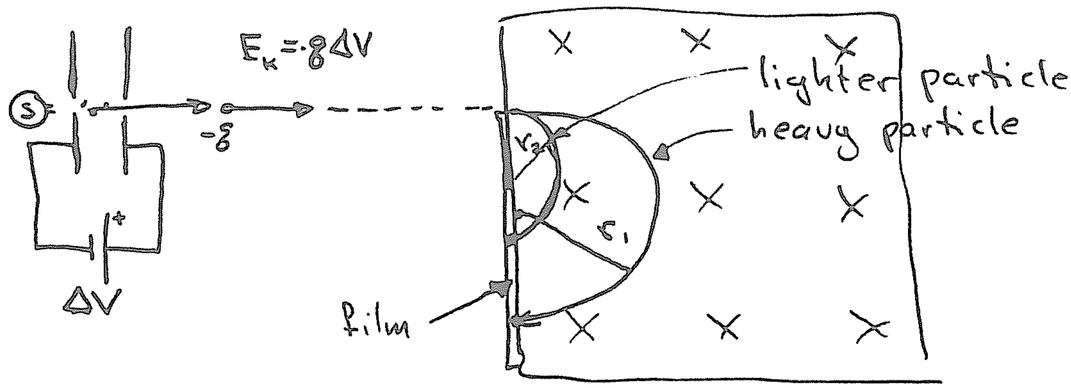
This is a velocity selector. The Electric force (down) is always qE. The magnetic force, on the other hand, depends on the velocity and is qvB (up) when the fields are perpendicular to one another as shown. The two forces balance when

$$qE = qvB,$$

i.e.-- when
$$|v| = \frac{E}{B}.$$

Particles (with any non-zero charge charge!) with speeds unequal to this speed will be deflected either up or down. Note that since q cancels in the selection equation, a particle can have <u>any</u> mass and <u>any</u> positive or negative charge and will still be undeflected only if $|v| = E/B$.

Mass spectrograph



A mass spectrograph is commonly used to measure the atomic and molecular masses of ions produces by heating and "sparking" a source (from, say, an organic chemistry experiment). As usual, we start with

$$qvB = \frac{mv^2}{r},$$

so
$$\frac{m}{q} = \frac{Br}{v}.$$

Thus the ratio of charge to mass of a particle determines the radius of its orbit. But it is easy to get charged particles with the same kinetic energy, difficult to get charged particles with a single velocity, so

$$\frac{1}{2} mv^2 = qV$$

is the kinetic energy of a particle after it falls across a potential V as shown. Substituting 2qV for mv^2 in the first equation above, we get

$$qvB = \frac{2qV}{r}$$

or
$$v = \frac{2V}{Br}$$

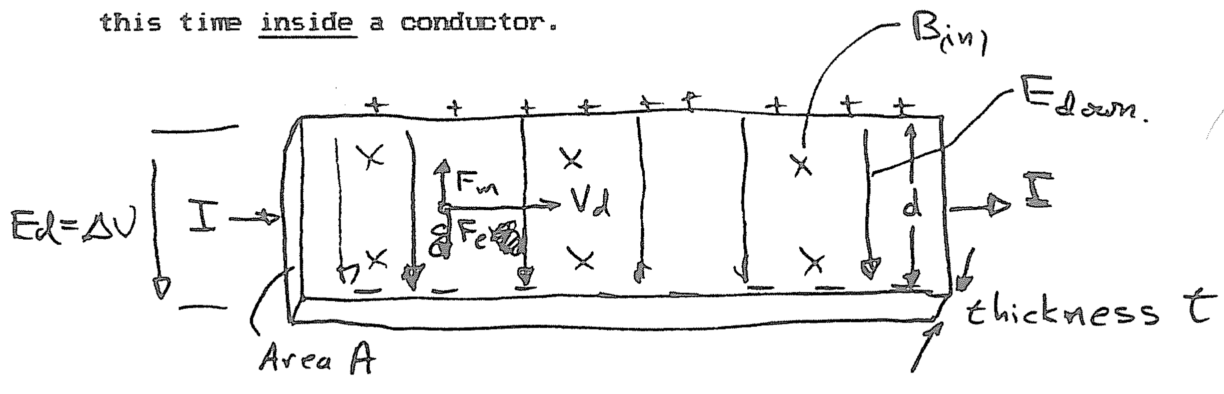
Putting this in for v in the second equation above we obtain

$$\frac{m}{q} = \frac{B^2r^2}{2V}.$$

Thus by measuring r (via a photographic film, etc.) we can determine m/q. By including a reference molecule or otherwise calibrating our spectrograph, we can then determine the masses of the constituents of the purple glop we made in experiment 23 in Organic! All Right!

The Hall effect.

The Hall effect is just the region of crossed fields again, this time inside a conductor.



When we put a conducting strip carrying a current in a magnetic field as shown, the magnetic force pushes the charges making up the current up or down. But they can't pass the edge of the strip, so there they accumulate. In so doing, they produce an electric field (shades of the parallel plate capacitor!) that builds up until the net force on a particle in the current is zero.

$$qE = qv_dB$$

or

$$E = v_dB.$$

But, the potential difference across the strip is then

$$\Delta V = Ed = v_dBd!$$

But $J = I/A = nqv_d$ so

$$v_d = \frac{I}{Aqn},$$

and

$$\Delta V = \frac{IBd}{Aqn}$$

or

$$n = \frac{IBd}{Aq\Delta V}.$$

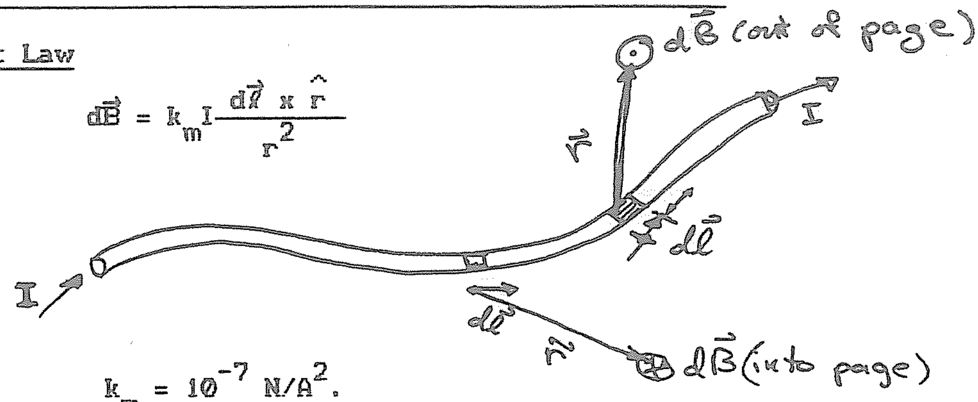
The Hall effect can then be used to measure the number of charge carriers per unit volume of a conductor in terms of the potential difference across it (ΔV can be measured very precisely) or it can be used to build a magnetic field strength detector or whatever!

Note Well: The magnetic field in this example seems to produce an EMF across the strip! How peculiar! Not really. . .
As we shall soon see, this leads us quite naturally to another of Maxwell's equations. . .

---

1) The Biot-Savart Law

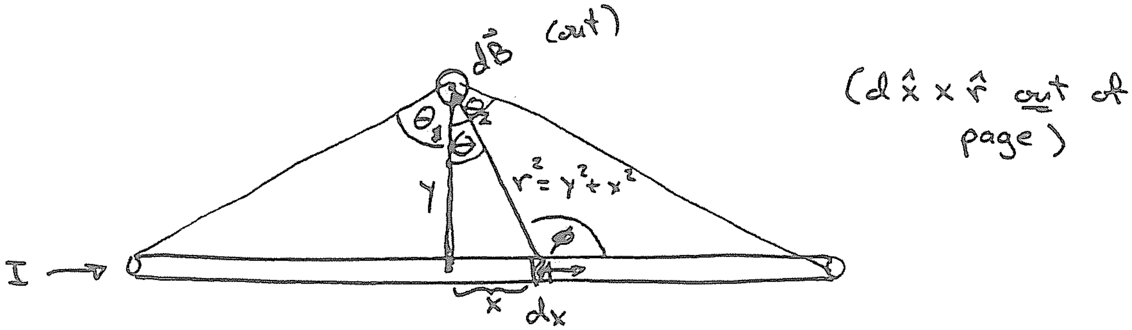
$$d\vec{B} = k_m I \frac{d\vec{\ell} \times \hat{r}}{r^2}$$



where $k_m = 10^{-7}$ N/A^2.

We also define a constant $\mu_0 = 4\pi k_m = 4\pi \times 10^{-7}$ N/A^2 called the permeability of free space (analogous to ϵ_0, the permittivity of free space).

Application: Magnetic field (induction) of a long, straight current.



$$dB = k_m I \frac{dx}{r^2} \sin\phi = k_m I \frac{dx \cos\theta}{r^2}$$

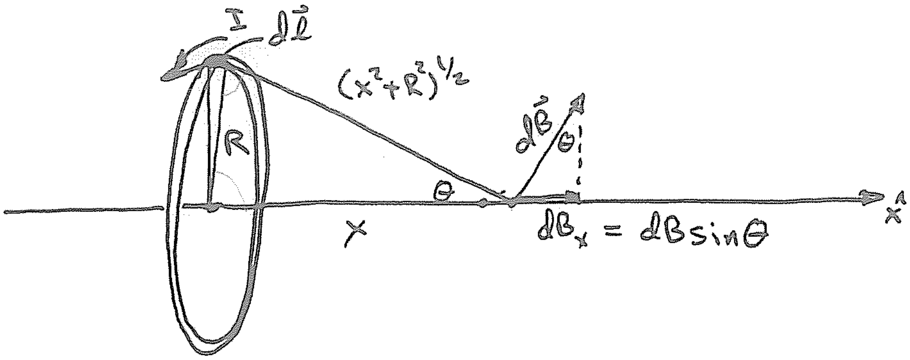
$$x = y \tan\theta \longrightarrow dx = y \sec^2\theta \, d\theta = (r^2/y) d\theta$$

so
$$B = \int_{\theta_1}^{\theta_2} k_m \frac{I}{y} \cos\theta \, d\theta = k_m \frac{I}{y}\left[\sin\theta_2 - \sin\theta_1\right]$$

or, if $\theta_1 = -90^\circ$ and $\theta_2 = 90^\circ$,
$$B = 2k_m \frac{I}{y} = \frac{\mu_0 I}{2\pi y}.$$

(direction given by r.h.r.)

Magnetic field on axis of circular current loop.



$$|dB| = k_m I \frac{(d\vec{l} \times \hat{r})}{r^2} = k_m I \frac{d\ell}{x^2 + R^2}$$

$$dB_x = dB \sin\theta = k_m I \frac{R d\ell}{(x^2 + R^2)^{3/2}}$$

$$B_x = 2\pi k_m \frac{IR^2}{(x^2 + R^2)^{3/2}}.$$

Definition of Ampere and Coulomb.

If two very long parallel wires one meter apart carry equal currents, the current in each is defined to be one ampere if the force per unit length on each wire is 2×10^{-7} N/m.

Note that coincidentally enough, the ratio

$$\frac{k}{k_m} = 9 \times 10^{16} m^2/sec^2 = c^2! \text{ (speed of light squared)}$$

1 Coulomb is the charge that flows through a wire carrying 1 Ampere in 1 second.

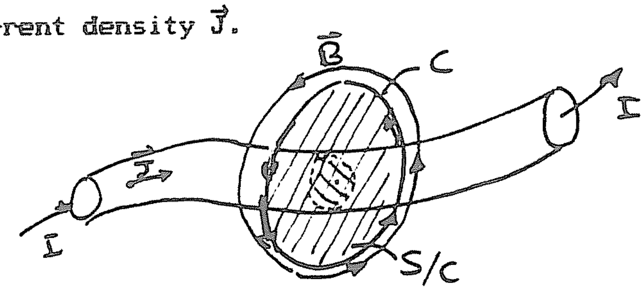
Ampere's Law (Our third Maxwell's equation!)

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{(through C)}}$$

or, more appropriately

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \int_{S/C} \vec{J}\cdot\hat{n}\, dA$$

in terms of the current density $\vec{J}$.



Ampere's law (like Gauss' Law) lets us calculate the magnetic field in certain symmetric cases almost trivially. They are

1) Long straight wire.
2) Solenoid.
3) Toroidal Solenoid

and we'll cover them next time. . .

Units of magnetic field. From the magnetic force law

$$\vec{F} = q(\vec{v} \times \vec{B}),$$

the SI units of B are

$$1 \text{ Tesla} = 1\frac{N}{A\text{-}m}.$$

But it turns out that the SI unit is rather large for typical currents (recall $k_m = 10^{-7}$ N/A^2) so the commonly used unit of magnetic field strength is the Gauss, where

$$1 \text{ Tesla} = 10^4 \text{Gauss}.$$

Remember to convert fields in Gauss to Tesla before doing a SI calculation!

Ampere's Law (Our third Maxwell's equation!)

$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 I_{(\text{through } C)}$$

Ampere's law (like Gauss' Law) lets us calculate the magnetic field in certain symmetric cases almose trivially. They are

1) Long straight wire.
2) Solenoid.
3) Toroidal Solenoid

---

1) The magnetic field of a long straight wire is given by Ampere's law as follows.

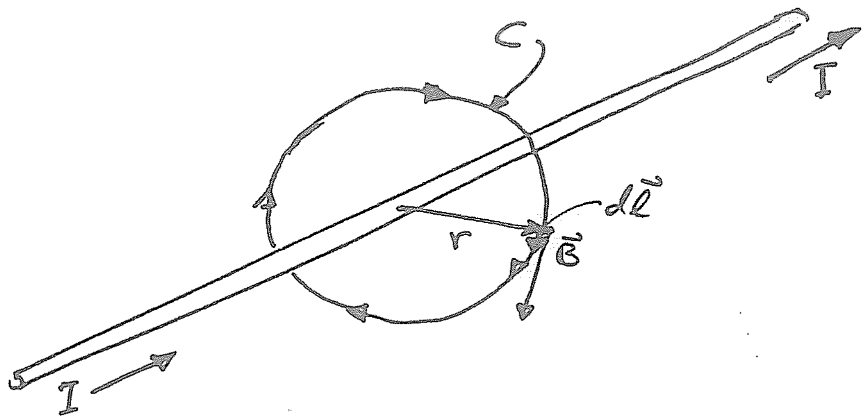
a) Draw an Amperian Path (C). This is a curve C drawn such that $\vec{B}$ is constant in magnitude and tangent to the curve at each point.

b) Evaluate $\oint_C \vec{B}\cdot d\vec{\ell}$ for the curve C. Since B is constant it comes out of the integral (recall G's law). The dot product goes away because $\vec{B}$ is tangent to $d\vec{\ell}$. The integral of $d\ell$ is just the length of the curve C.

c) Finally, figure out how much current goes through C. This may be the flux of the current density $\vec{J}$ (inside a wire) or it may be a matter of counting up the current on our fingers, or we may have to use the Maxwell Displacement current (see below), but we can do it. This is the only "tricky" part of the problem, just as it was for Gauss' Law.

d) Use algebra and the right hand rule to obtain the answer.

So for the wire,



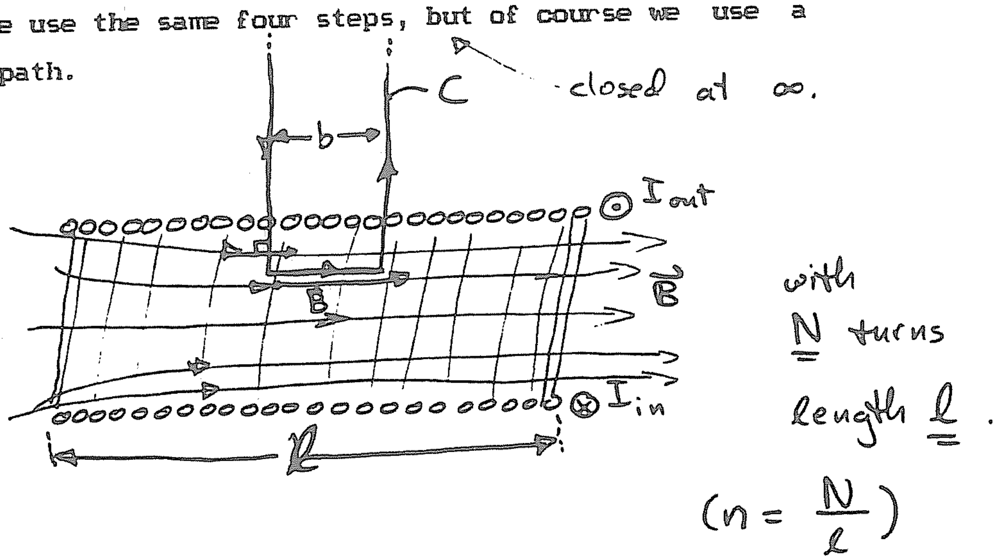
a), b) $$\oint_C \vec{B}\cdot d\vec{\ell} = B2\pi r$$

c) $$= \mu_0 I,$$

d) $$B = \frac{\mu_0 I}{2\pi r}$$ (direction by R.H.R.).

---

2) Solenoid. We use the same four steps, but of course we use a unique amperian path.



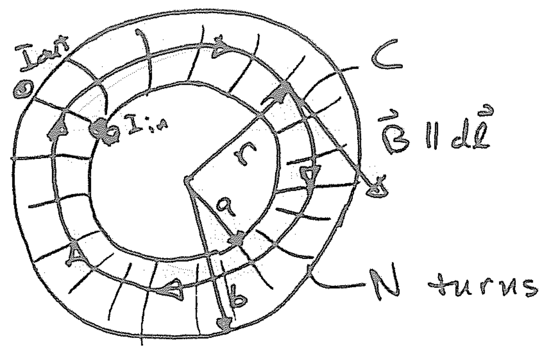
a), b) $$\oint_C \vec{B}\cdot d\vec{\ell} = Bb$$

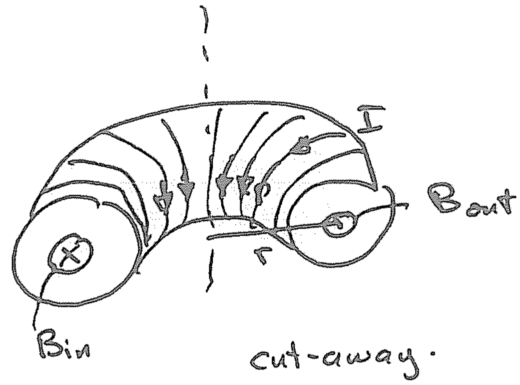
c) $$= \mu_0 I \frac{N}{\ell} b$$

d) $$B = \mu_0 I \frac{N}{\ell} = \mu_0 I n$$ (r.h.r, etc.)

---

3) Toroidal Solenoid. Ditto.





a), b) $$\oint_C \vec{B}\cdot d\vec{\ell} = B2\pi r$$

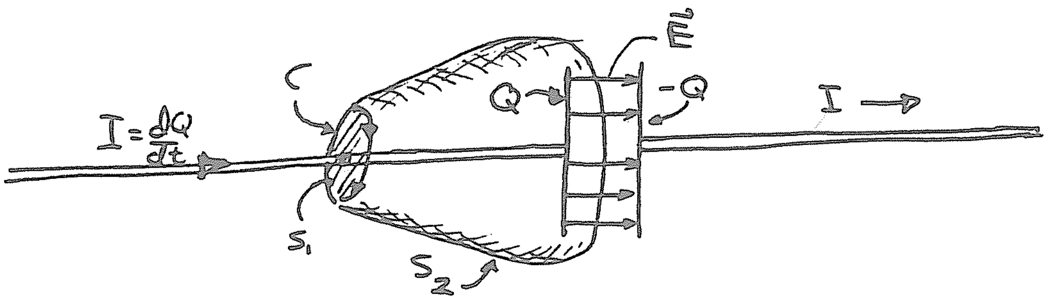
c) $$= \mu_0 NI \quad \text{if } a<r<b$$

$$= 0 \quad \text{otherwise.}$$

d) so $$B = \frac{\mu_0 IN}{2\pi r} \quad \text{(r.h.r).}$$

Finally, Maxwell's Displacement current.

Ampere's law is incorrect as Ampere wrote it. There is a contradiction that arises from applying it to the current flowing onto a parallel plate capacitor as shown that arises because of the mathematical meaning of the words "through C" (which mean through an <u>arbitrary</u> surface S bounded by C)! The problem is that the current through C as drawn below goes through S_1 but not through S_2, even though they both bound the same curve! Maxwell fixed this by applying Gauss' law to the closed surface formed by S_1+S_2, and generalizing the term "current" so that one always gets the same answer, regardless of which surface one uses.



$$\oint_{S_1+S_2} \vec{E}\cdot\hat{n}\, dA = \frac{Q}{\epsilon_0},$$

so
$$\frac{d}{dt}\epsilon_0 \int_{S_1+S_2} \vec{E}\cdot\hat{n}dA = \frac{dQ}{dt} = I$$

but ϕ_E through S_1 is zero (since E is confined between the plates)

so
$$I_{S_1} = \frac{dQ}{dt} = \frac{d}{dt}\oint_{S_2} \vec{E}\cdot\hat{n}dA = I_{disp}$$

or I_{disp} is the <u>Maxwell displacement current</u>. For this one little term, Maxwell got his name on all four equations. That is because it unified the fields of Electricity and Magnetism and showed light to be an electromagnetic phenomenon. This was (and is) important!

So, Ampere's law should correctly read

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0\left[I_{S_k} + \epsilon_0\frac{d\phi_E^{S_k}}{dt}\right].$$

Application: Find the Magnetic field inside and outside a circular parallel plate capacitor.

Solution: Find the field, and then the flux, in terms of the current I, through a circular(!) amperian path. Take the time derivative and solve for the displacement current. Solve (as usual) for B. B should vary linearly with r inside the capacitor, and should vary like 1/r (c.f.--line of current) outside.

Electric currents are produced by changing magnetic fields. This is a direct consequence of the magnetic force law, which states that magnetic fields exert forces on moving charges. Equivalently, moving fields exert forces on stationary charges. If these forces are interpreted to be electric forces, then we find experimentally that

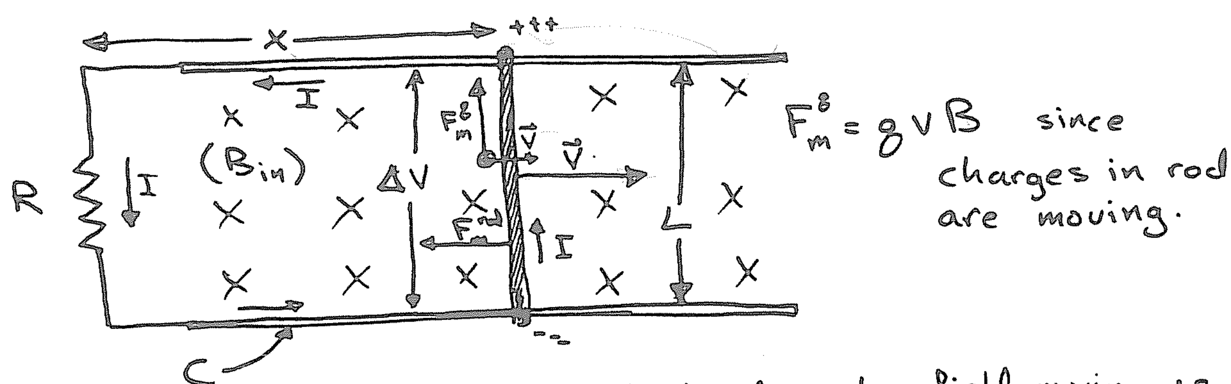
$$\mathcal{E} = \oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_{S/C} \vec{B}\cdot\hat{n}\,dA$$

This is Faraday's Law, which relates the electric field to a changing magnetic flux, just as Ampere's law with Maxwell's displacement current related the magnetic field to a changing electric flux! This is also the fourth (and last) Maxwell equation, and completes our study of electromagnetism. Sort of.

Lenz's Law is the minus sign in Faraday's law, which means: "The direction of the induced current in C is that which produces a magnetic moment (due to the current in C) which opposes the change in the flux." It is required, as we shall see, in order that energy be conserved.

What is induced EMF?

Let's look at a "rod on rails"



$$\Delta W = qBLv \quad \text{(work done by field moving +q across rod - distance L)}$$

$$\Delta V = \frac{\Delta U}{q} = -\frac{\Delta W}{q} = -BLv$$

but $v = \frac{dx}{dt}$, so

$$\Delta V = -\frac{d}{dt}BLx = -\frac{d}{dt}BA = -\frac{d\phi_m}{dt},$$

which is just Faraday's law. In other words, the induced EMF is just a force which arises due to the relative motion of the flux lines of the magnetic field and charges in the conducting path C.

Since there is an "EMF" in the loop, $I = \Delta V/R$ or

$$I = -\frac{BLv}{R}.$$

Since there is a current in the rod, there is a magnetic force

$$\vec{F}_m^{rod} = I(\vec{l} \times \vec{B}) = IBL = -\frac{B^2L^2\vec{v}}{R}$$

acting on the rod that <u>slows it down</u>. This is necessary because the rod starts with a certain amount of kinetic energy and energy is continually dissipated as joule heat in the resistor. What would happen if the force pointed the other way?

Power: In the problem above,

$$P_{rod} = Fv = \frac{B^2L^2v^2}{R}$$

$$P_{resistor} = I^2R = \frac{B^2L^2v^2}{R^2}R = P_{rod}$$

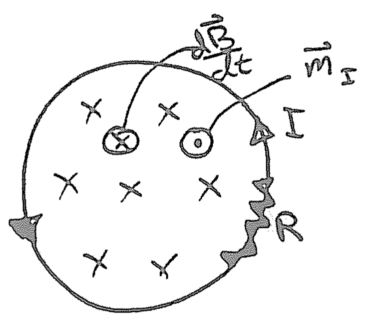
i.e.--the rate at which work is done on the rod equals the rate at which energy is lost in the resistor. This is a machine that dissipates the kinetic energy of the rod as heat, or a magnetic induction "brake".

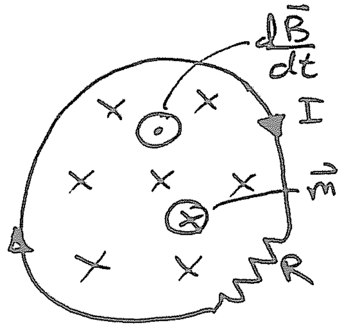
$$\mathcal{E} = \oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int_{S/C} \vec{B}\cdot\hat{n}dA$$

Faraday's Law

Lenz's Law is the minus sign in Faraday's law, which means: "The direction of the induced current in C is that which produces a magnetic moment (due to the current in C) which opposes the change in the flux." It is required, as we shall see, in order that energy be conserved.

In application:





$B = B't$ increases

$\frac{d\vec{B}}{dt}$ into page, $\vec{m}$ out of page.

$B = B_0 - B't$ decreases

$\frac{d\vec{B}}{dt}$ out of page, $\vec{m}$ into page

Suppose B is increasing initially according to the rule $B = B't$ and points into the page. Then

$$\phi_m = B'At$$

$$\frac{d\phi_m}{dt} = A\frac{dB}{dt} = B'A$$

and $\quad \Delta V = -B'A, \quad I = \frac{-B'A}{R}, \quad m = IA = \frac{-B'A^2}{R}.$

We find the direction on I from Lenz's Law as follows. The flux through C is increasing. Therefore $\vec{m}$ (the magnetic moment of the induced current loop) must point opposite to $\vec{B}$ to oppose the change in flux (it tries to cancel out part of the increase in flux). The current must flow counter-clockwise (CCW) in order for $\vec{m}$ to point opposite to $\vec{B}$, out of the page.

If, after some time, we choose to turn down the magnetic field (still pointing into the page!) according to the rule $B = B_0 - B't$,

$$\phi_m = B_0A - B'At$$

$$\frac{d\phi_m}{dt} = -B'A,$$

and $\quad \Delta V = B'A, \quad I = \frac{B'A}{R}, \quad m = IA = \frac{B'A^2}{R}.$

The direction of I (from Lenz's Law) is now clockwise (CW) because the magnetic moment $\vec{m}$ must point in the same direction as the diminishing magnetic field $\vec{B}$ in order to try to maintain the decreasing flux as long as possible.

Suppose, in the example above, $B = B_0 \cos\omega t$. Then

$$\frac{dB}{dt} = -B_0\omega \sin\omega t,$$

and

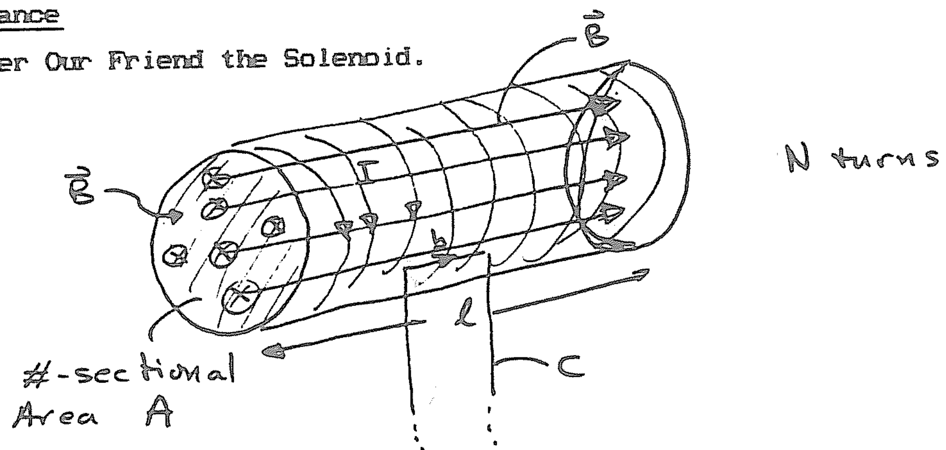
$$\Delta V = -\frac{d\phi_m}{dt} = -\frac{dB}{dt}A = B_0 A\omega \sin\omega t.$$

What is the initial direction of the current for t just greater than zero (when $\vec{B}$ through the loop is decreasing)? Answer: CW.

Understand flux linkage and eddy currents. We'll do transformers (not the toys!) in detail later.

Inductance

Consider Our Friend the Solenoid.



$$\oint_C \vec{B}\cdot d\vec{\ell} = Bb = \mu_0 I \frac{N}{\ell} b,$$

or

$$B = \mu_0 I \frac{N}{\ell}.$$

Then ϕ_m (due to the current I in the solenoid) is

$$\phi_m = NBA = \mu_0 I A \frac{N^2}{\ell}.$$

We see that the flux through the solenoid due to its own current is always proportional to I. This encourages us to define the self-inductance of the solenoid as

$$L = \frac{\phi_m}{I} = \mu_0 A \frac{N^2}{\ell}.$$

This is a quantity analogous to capacitance in that it can be used to find the induced emf in the solenoid produced by a change in the current I,

$$\Delta V = -\frac{d\phi_m}{dt} = -L\frac{dI}{dt}.$$

Like the capacitance, L depends only on the geometry of the circuit (in this case a solenoid).

More generally, a current loop has many contributions to its flux. They include its self-flux, produced by current in the loop itself and mutual flux produced by magnetic fields resulting from the flow of current in other, nearby circuits. The magnetic field produced by the each other circuit is (c.f.--Biot-Savart) proportional to the current in that circuit, and so we can write

$$\phi_m^i = L_i I_i + \sum_j M_{ij} I_j$$

where L_i is the self-inductance of the ith circuit element and M_{ij} is called the mutual inductance between the jth circuit element and the ith one. For example,

---

The Mutual inductance of a long straight wire and a rectangular current loop.

(see two pgs. on).

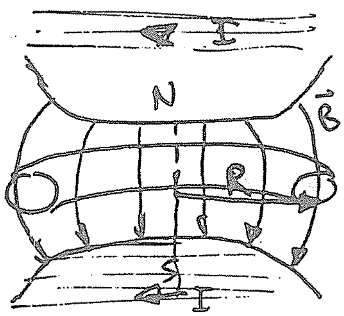
Show that $M = \frac{\mu_0 c}{2\pi} \ln\frac{b}{a}$.

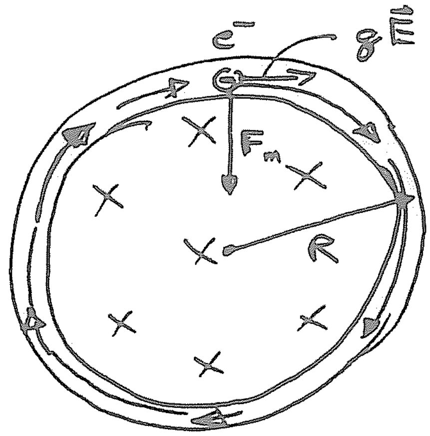
In terms of the mutual and self inductance, the emf is

$$\Delta V^i = -L_i\frac{dI_i}{dt} - \sum_j M_{ij}\frac{dI_j}{dt}.$$

Usually (not always) we will be concerned with self-inductance L more than mutual inductance. We begin by examining the role of inductors in electric circuits next time.

The Betatron. Beta particles are electrons, so this is a simple electron accelerator. The electrons accelerated by such a device are used (for example) to make x-rays.





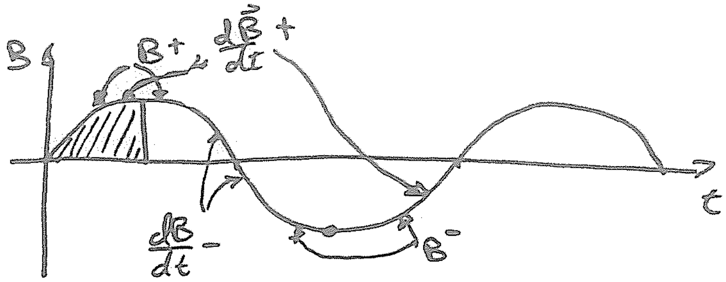
From
$$\frac{mv^2}{R} = qvB$$
we get
$$|\vec{p}| = mv = qRB$$
where B is the magnetic field at the beam pipe radius R. As long as this relation is satisfied, a charged particle will move in a circle of radius R and stay inside the beam pipe.

From
$$|\mathcal{E}| = \oint_C \vec{E}\cdot d\vec{l} = E\cdot 2\pi R = \frac{d\phi_m}{dt}$$
$$= \pi R^2 \frac{dB_{av}}{dt}$$
or
$$E = \frac{R}{2}\frac{dB_{av}}{dt}$$
we get
$$\frac{d|\vec{p}|}{dt} = qE = \frac{1}{2} qR\frac{dB_{av}}{dt}.$$

This says that the rate of change of the momentum of a charged particle moving in a circle of radius R is proportional to half the rate of change of the averge flux (related to the average field through R).

The two conditions are compatible if
$$B\ (\text{at } R) = \frac{1}{2} B_{average}\ (\text{through } R).$$

As long as this is true, a particle will simultaneously be accelerated (due to induced EMF) and will be bent in a circle of constant radius (since B increases as necessary to maintain this condition).

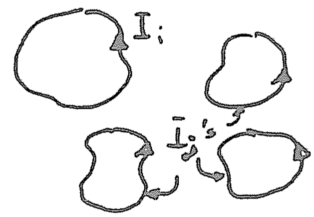


We cannot increase B without bound. In order to get pulses of beta particles (electrons) out on a regular basis, we vary B sinusoidally as shown. We only accelerate electrons in the first quarter cycle. Why?

---

2) Recall

$$\phi_m^i = L_i I_i + \sum_j M_{ij} I_j$$



or

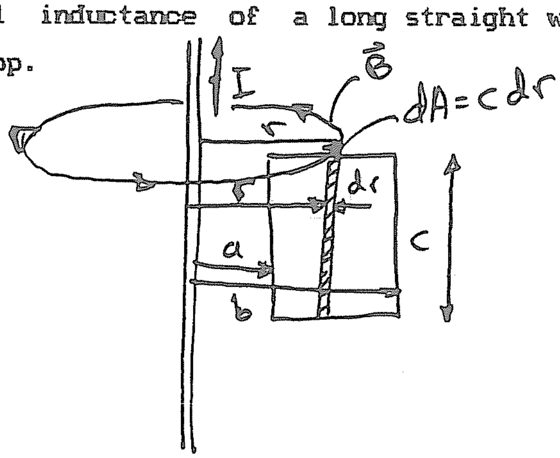
$$\mathcal{E}_m^i = -L_i \frac{dI_i}{dt} - \sum_j M_{ij} \frac{dI_j}{dt}$$

where L is the self-inductance of a circuit element and M is the mutual inductance between circuit elements. The SI units of inductance are Henries,

$$1 \text{ Henry} = 1 \frac{\text{V-sec}}{\text{Amp}}$$

Last time we calculated the self inductance of a solenoid. This time we will calculate. . .

The Mutual inductance of a long straight wire and a rectangular current loop.



(If $\vec{B}$ is increasing, which way does current flow in loop? ccw)

$$d\phi_m = BdA = \frac{\mu_0 I}{2\pi r} cdr$$

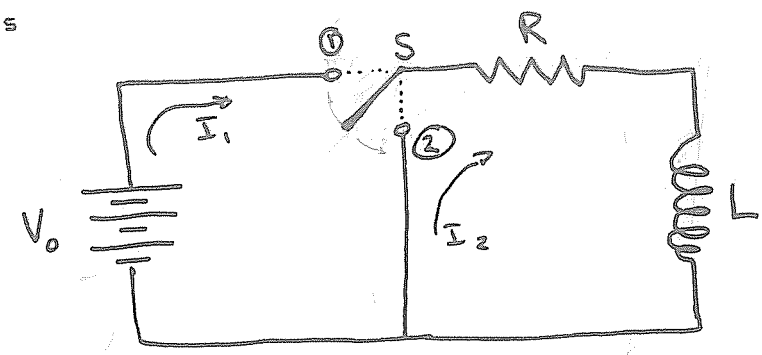
so

$$\phi_m = \frac{\mu_0 c}{2\pi} I \ln\frac{b}{a}$$

and therefore

$$M = \frac{\phi_m}{I} = \frac{\mu_0 c}{2\pi} \ln\frac{b}{a}.$$

LR-circuits



1) $V_0 = IR + L\frac{dI}{dt}$, so we solve for

$$I(t) = \frac{V_0}{R}(1 - e^{-tR/L})$$

2) $IR + L\frac{dI}{dt} = 0$, so we solve for (exponential decay)

$$I(t) = I_0 e^{-tR/L}.$$

(Time constant of LR circuit is $\tau_c = \frac{L}{R}$).

Magnetic Energy

$$\frac{dU_m}{dt} = LI\frac{dI}{dt},$$

so

$$U_m = \frac{1}{2}LI_f^2.$$

The magnetic field energy density is (c.f.-solenoid)

$$\eta_m = \frac{1}{2\mu_0}B^2$$

so that the total electromagnetic field energy density is

$$\eta = \eta_e + \eta_m = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2.$$

$$L^{solenoid} = \frac{\mu_0 N^2 A}{\ell} = \left[\frac{BNA}{I}\right]$$

w/ $B = \frac{\mu_0 I N}{\ell}$

$$U_m^{solenoid} = \frac{\mu_0 N^2 I_f^2 A}{2\ell}$$

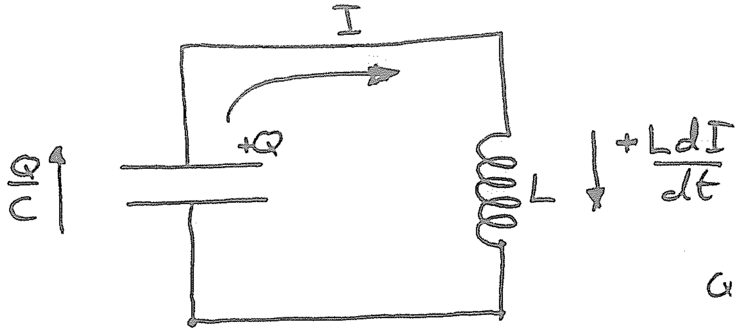
$$= \frac{1}{2} B I_f A N$$

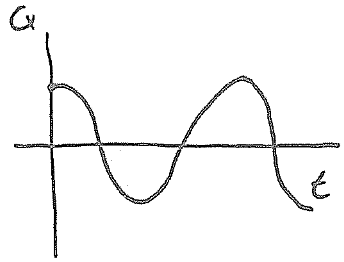
$$= \frac{1}{2} B \frac{\mu_0 N}{\ell} I_f \frac{(A\ell)}{\mu_0}$$

$$= \frac{1}{2\mu_0} B^2 (A\ell) = \frac{1}{2\mu_0} B^2 (Vol).$$

or $\eta_m = \frac{U_m}{(Vol)} = \frac{1}{2\mu_0} B^2$

LC circuits





$$L\frac{dI}{dt} + \frac{Q}{C} = L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0,$$

so

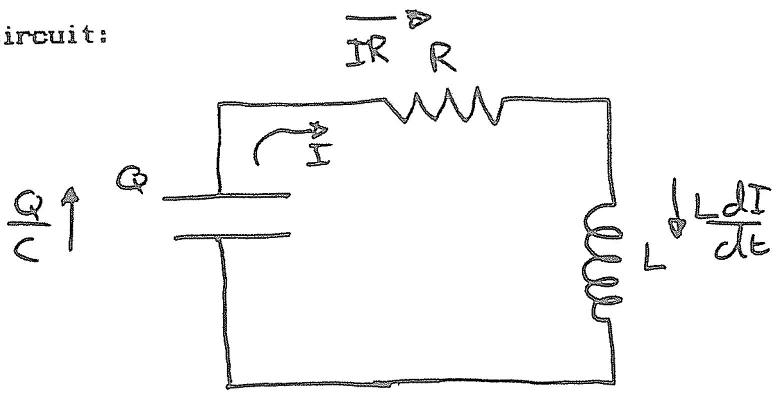
$$Q(t) = Q_0\cos(\omega_0 t + \delta)$$

with

$$\omega_0 = \frac{1}{\sqrt{LC}} .$$

This is a harmonic oscillation of charge, where L ~ mass m and C ~ spring constant k.

But, real oscillators don't run forever, they are damped by, e.g.--friction. Similarly, LC circuits don't have zero resistance and don't run forever. Joule heating damps out the charge oscillation in time. (R acts like friction). So we consider the LRC circuit:



$$L\frac{dI}{dt} + RI + \frac{Q}{C} = 0,$$

or, in terms of Q

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0.$$

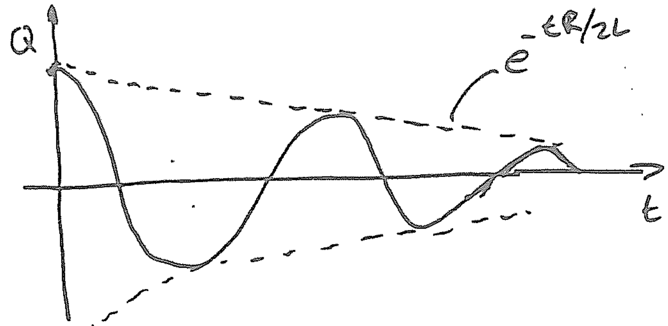
The solution of this is (try $Q(t) = Q_0 e^{\alpha t}$ and solve characteristic for α)

$$Q(t) = Q_0 e^{-tR/2L} \cos(\omega' t + \delta)$$

where

$$\omega' = \omega_0 \left[1 - \left(\frac{RC}{2\omega_0}\right)^2\right]^{1/2}$$

and looks like



From chapter 29, learn

1) Paramagnetism

2) Diamagnetism

3) Ferromagnetism.

well enough to identify a material's property from a few pieces of information.

---

solution:

$$Q = Q_0 e^{\alpha t}$$

so

$$\alpha^2 Q(t) + \alpha \frac{R}{L} Q(t) + \frac{1}{LC} Q(t) = 0$$

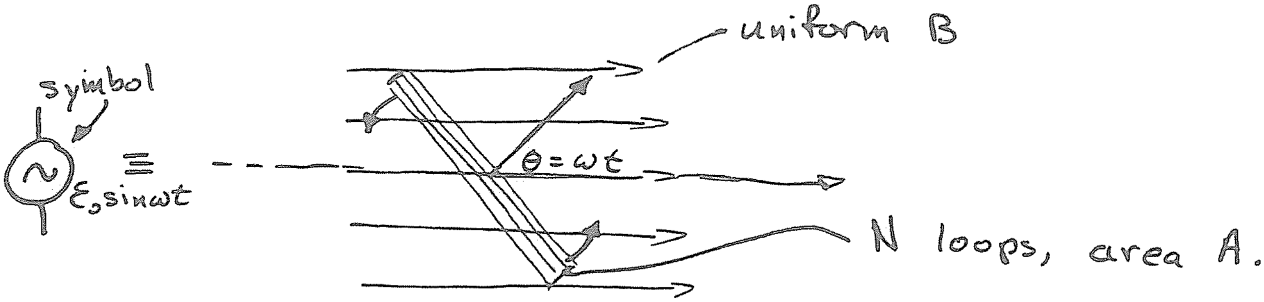
or $\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0$ is characteristic

$$\therefore \quad \alpha = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2} = \frac{-R}{2L} \pm i \frac{1}{\sqrt{LC}} \sqrt{1 - \frac{CR^2}{4L}}$$

$$= \frac{-R}{2L} \pm i\omega_0 \sqrt{1 - \left(\frac{CR}{2\omega_0}\right)^2}$$

$$\therefore \quad Q(t) = Q_0 e^{-\frac{Rt}{2L}} e^{\pm i\omega' t} = Q_0 e^{-\frac{Rt}{2L}} \cos(\omega' t + \delta) \quad \text{w/ } \omega' = \omega_0 \sqrt{1 - \left(\frac{RC}{2\omega_0}\right)}$$

1. AC generator (Based on Faraday's Law).



The angle $\theta = \omega t$. So, $\int_A \vec{B}\cdot\hat{n}dA = \int_A B\cos(\omega t)\,dA = BA\cos(\omega t)$ per loop. Thus Faraday's law reads

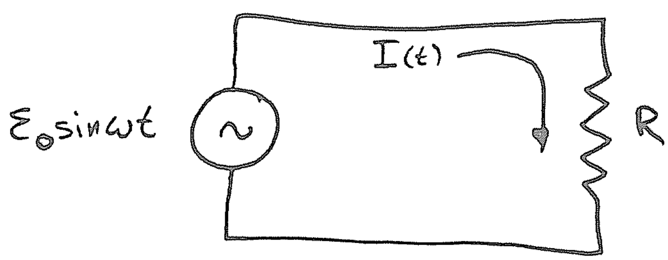
$$\mathcal{E} = -\frac{d}{dt}BAN\cos(\omega t) = BAN\omega\sin(\omega t) = \mathcal{E}_0\sin(\omega t).$$

This is how one makes an AC generator that produces a sinusoidally varying voltage using permanent magnets.

---

2) Now that we know how to make an AC voltage, we have to discover how all of the components (i.e.--resistors, capacitors, and inductors) behave when we apply an alternating voltage across them. We will learn this, of course, from Kirchhoff's laws.

a) Resistors. Are easy.

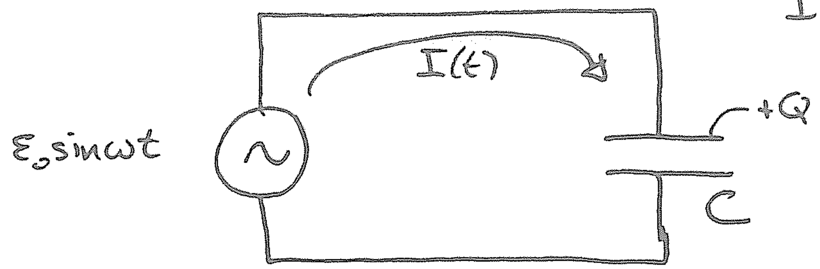


$$\mathcal{E}_0\sin\omega t - IR = 0, \text{ so}$$
$$I(t) = \frac{\mathcal{E}_0}{R}\sin\omega t.$$

Note that the potential difference across a resistor is <u>in phase</u> with the current through the resistor.

b) Capacitor.

$I = +\frac{dQ}{dt}$

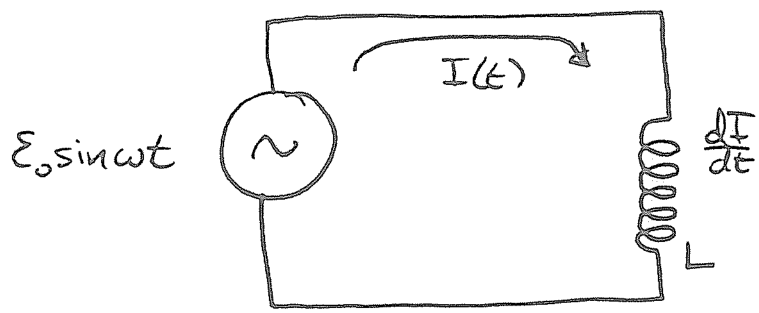


$$\mathcal{E}_0 \sin\omega t - \frac{Q}{C} = 0, \text{ so (since } I = \frac{dQ}{dt}\text{)}$$

$$I(t) = \mathcal{E}_0 \omega C \cos\omega t = I_0 \sin(\omega t + 90^\circ).$$

Note that the current (I) "through" a capacitor <u>leads</u> the potential difference across the capacitor by 90°.

c) Inductors.



$$\mathcal{E}_0 \sin\omega t - L\frac{dI}{dt} = 0, \text{ so (integrating to } I(t)\text{)}$$

$$I(t) = -\frac{\mathcal{E}_0}{\omega L} \cos\omega t = \frac{\mathcal{E}_0}{\omega L} \sin(\omega t - 90^\circ).$$

Note that the current through the inductor <u>lags</u> the potential difference across the inductor by 90°.

We can summarize these rules as:

a) I and ΔV_R are <u>in phase</u>.

b) I leads ΔV_C by 90°.

c) I lags ΔV_L by 90°.

A pnemonic device to remember the last two is ELI the ICE man. E is ahead of I in L (ELI, in order). and I is ahead of E in C (ICE, in order) (Thanks to Meredith, to whom brownie points are gratefully rendered. . .)

It is useful to define two new quantities that behave like resistance in the equations for I_C and I_L.

$$I_C = \frac{E_0}{1/\omega C} \sin(\omega t + 90^\circ) = \frac{E_0}{X_C} \sin(\omega t + 90^\circ)$$

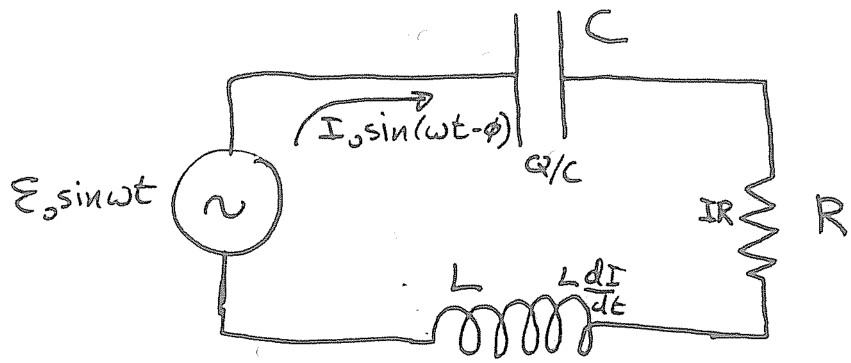
and

$$I_L = \frac{E_0}{\omega L} \sin(\omega t - 90^\circ) = \frac{E_0}{X_L} \sin(\omega t - 90^\circ)$$

where $X_C = 1/\omega C$ is called the capacitive reactance (and has units of ohms) and $X_L = \omega L$ is called the inductive reactance (ditto).

EDJ 25. AC Circuits

1. LRC circuit. A driven, damped oscillator.



$$\mathcal{E}_0 \sin\omega t - \frac{Q}{C} - IR - L\frac{dI}{dt} = 0.$$

or

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = \frac{\mathcal{E}_0}{L}\sin\omega t$$

is the differential equation we have to solve. As usual, the solution to this (inhomogeneous) equation is a particular solution (which just happens to be)

$$I(t) = I_0 \sin(\omega t - \phi)$$

plus a solution to the homogeneous equation, which we solved in chapter 28. But the homogeneous solution was damped out at the rate $e^{-Rt/2L}$; consequently we can wait a long time and this "transient" solution will go away.

This is a roundabout way of saying that we will take it as given that the steady state current through the LRC circuit is

$$I(t) = I_0 \sin(\omega t - \phi).$$

Note that, since this is a series circuit, the same current goes through all of the elements in the circuit.

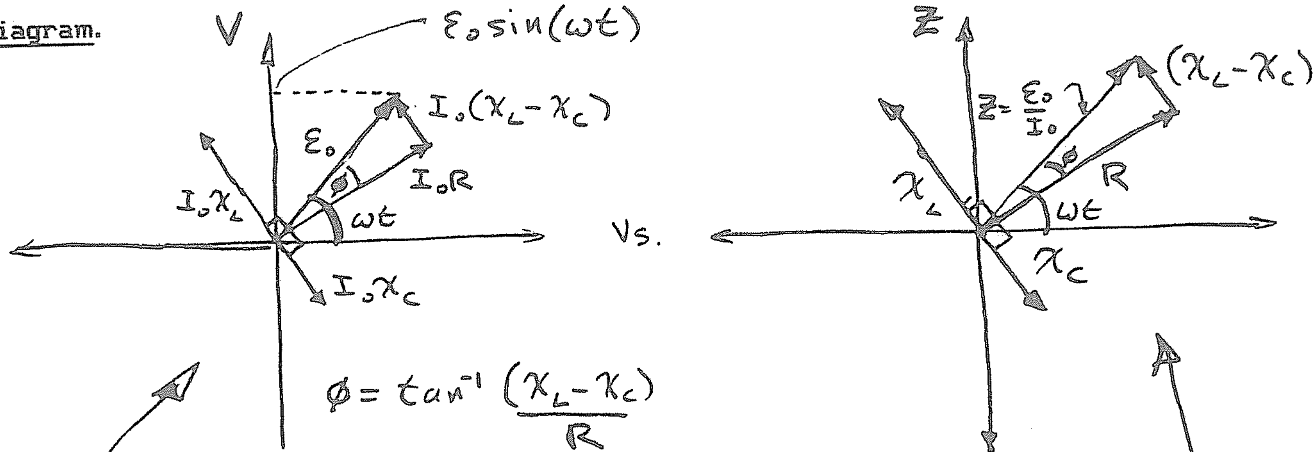
We wish to both verify that this is a solution and understand the properties of the solution. The easiest way to do this is substitute I(t), Q(t) from I(t), and dI/dt into Kirchhoff's law above.

$$\mathcal{E}_0 \sin\omega t = I_0 R \sin(\omega t - \phi) + I_0 X_L \sin(\omega t - \phi + 90^\circ) + I_0 X_C \sin(\omega t - \phi - 90^\circ).$$

where we have used the fact that the potential difference across an inductor leads the current (and the opposite for capacitors) and the definitions of capacitive and inductive reactance,

$$X_C = 1/\omega C \quad \text{and} \quad X_L = \omega L.$$

This equation is always true for some angle ϕ. Unfortunately, it is not easy to see what ϕ is as a function of X_C, X_L and R. We can easily determine it, however, and understand many other things about the solution by picturing the results above in a <u>phasor diagram</u>.



In this diagram, the y-axis represents voltage (across the EMF or any of the three circuit elements). Since each of these elements is a number times the sine of some angle, we think of it as a <u>vector</u> in this picture. The potential difference across a given component is just the y-component of the vector. The y-components of a vector always add up (according to Kirchhoff's law above) if the vectors themselves add up. So we draw the picture in such a way that the vector potential differences across L, C and R add up to the applied vector EMF. From this we easily see what ϕ must be in terms of the given quantities,

$$\phi = \tan^{-1} \frac{(X_L - X_C)}{R}.$$

If we divide this entire phasor diagram by I_0, we obtain a similar vector diagram whose components all have the units of resistance. This allows us to define a resistance-like quantity, called the <u>impedance</u>, for the circuit as a whole in such a way that "Ohm's law" is satisfied.

$$\mathcal{E}_0 = I_0 Z$$

From this diagram, we see that the impedance is given by

$$Z = \left[(X_L - X_C)^2 + R^2\right]^{1/2} = E_0/I_0.$$

Note the following. Z is always greater than or equal to R. It is equal to R (and minimum) only when $X_L = X_C$. Then

$$\omega_0 L = \frac{1}{\omega_0 C}, \text{ or}$$

$$\omega_0^2 = \frac{1}{LC}.$$

We recognize this as the frequency of the resonant LC circuit. This condition is called <u>resonance</u> and arises when the driving frequency of the EMF is the same as the "natural" frequency of the circuit. At resonance, the power which goes into the circuit,

$$P = E(t)I(t) = E_0 \sin\omega t \frac{E_0}{Z} \sin(\omega t - \phi)$$

$$\frac{E_0^2}{Z} \sin^2(\omega t) \cos\phi.$$

We are usually interested in the average power (averaged over many cycles). The average of $\sin^2\omega t$ is 1/2. Thus

$$P_{av} = \frac{1}{2}\frac{E_0^2}{Z}\cos\phi = E_{rms} I_{rms} \cos\phi.$$

The quantity

$$\cos\phi = \frac{R}{Z}$$

is called the power factor, and is one at resonance (when $\phi = 0^\circ$).

EDJ 26. AC Circuits

1. Recall from last time that at resonance, $X_L = X_C$ or

$$\omega_0 L = \frac{1}{\omega_0 C},$$
$$\omega_0^2 = \frac{1}{LC}.$$

This is the frequency of the resonant LC circuit. In a resonant LC circuit, the potential differences across the capacitor and inductor exactly cancel and the circuit behaves as if there is only a resistance in the circuit. This condition is called <u>resonance</u> and arises when the driving frequency of the EMF is the same as the "natural" frequency of the circuit. At resonance, the power which goes into the circuit,

$$P = \mathcal{E}(t)I(t) = \mathcal{E}_0 \sin\omega t \frac{\mathcal{E}_0}{Z}\sin(\omega t - \phi)$$
$$= \frac{\mathcal{E}_0^2}{Z}\sin^2(\omega t)\cos\phi.$$

We are usually interested in the average power (averaged over many cycles). The average of $\sin^2\omega t$ is 1/2. Thus

$$P_{av} = \frac{1}{2}\frac{\mathcal{E}_0^2}{Z}\cos\phi = \mathcal{E}_{rms} I_{rms} \cos\phi.$$

The quantity

$$\cos\phi = \frac{R}{Z}$$

is called the power factor, and is one at resonance (when $\phi = 0^\circ$). At this point we can see that $Z = R$ and $P_{av} = I_{rms}^2 R$ as noted above.

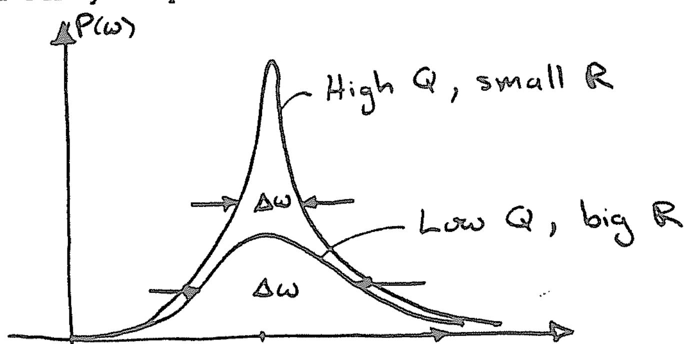
But how does the average power to the circuit vary as we move the frequency ω off of resonance (you might ask)? Well,

$$Z^2 = (X_L - X_C)^2 + R^2 = \frac{L^2}{\omega^2}(\omega^2 - \omega_0^2)^2 + R^2$$

(after some tedious algebra that <u>you should do on your own</u>), so

$$P_{av} = \frac{\mathcal{E}_{rms}^2 R\omega^2}{L^2(\omega^2 - \omega_0^2)^2 + \omega^2 R^2}.$$

This is a funny shaped curve (as a function of ω) peaked around ω_0.



Note that as long as ω is "close" to ω_0 the power is large. We wish to make the concept of "close" a little more precise. We define the half-width $\Delta\omega$ to be the width (in units of ω) of the peak at the point where the power supplied is half its peak value. After some tedious algebra (which, again, you should do on your own) one obtains the half width in terms of the given quantities as

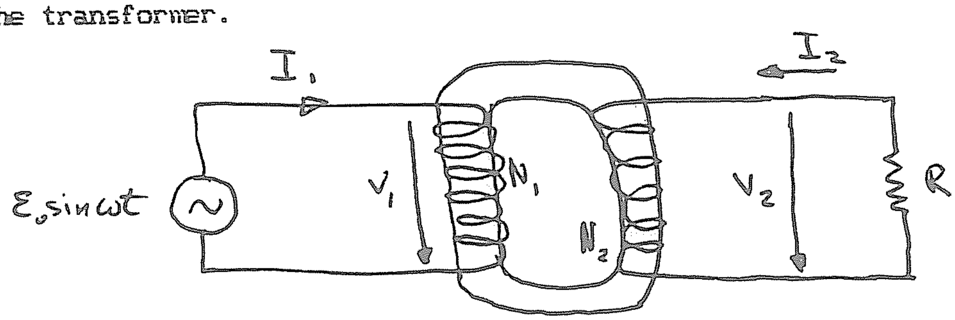
$$\Delta\omega = \frac{R}{L}.$$

We define a dimensionless number the describes the "Quality" of the resonance. The Q factor is

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{L\omega_0}{R}.$$

It indicates the relative strength and width of the resonance as qualitatively indicated on the diagram above.

The transformer.



$$V_1 = -N_1\frac{d\phi}{dt}$$
$$V_2 = -N_2\frac{d\phi}{dt}$$

or

$$V_2 = \frac{N_2}{N_1} V_1$$

(where the flux per turn is "linked" by the iron core). If we apply an EMF, then

$$\mathcal{E}_0 \sin\omega t = -V_1 \quad \text{so}$$
$$V_2 = -\frac{N_2}{N_1}\mathcal{E}_0 \sin\omega t$$

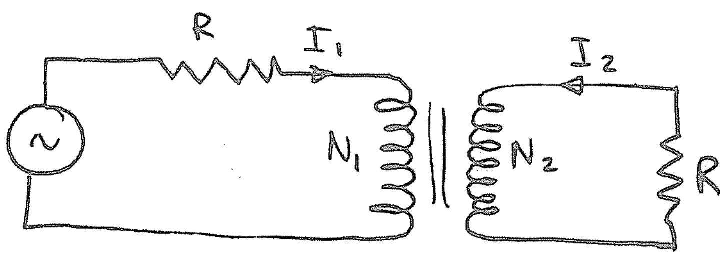
The currents are also linked by Faraday's law and inductance so that $N_1 I_1 = -N_2 I_2$.

A transformer is important in our culture not only so that you can use your hair dryer in France, but because transformers make possible the efficient transmission of power. Consider the circuit below:

In this circuit $I_1 = -\frac{N_2}{N_1} I_2$ and the ratio of the power dissipated in the two equal resistances is

$$\frac{P_1}{P_2} = \left[\frac{N_2}{N_1}\right]^2.$$

From this we see that it is desireable to transmit power at high voltage (and low current) and then use it at low voltage (and high current) because the power dissipated through equal resistive loads in the two branches of the circuit goes like the ratio of loops squared (or equivalently, like the voltages squared).



$$P_1 = I_1^2 R$$

$$P_2 = I_2^2 R = \frac{N_1^2}{N_2^2} R I_1^2$$

or $$\frac{P_1}{P_2} = \frac{N_2^2}{N_1^2}$$ so we <u>want</u>

P_1 <u>small</u> relative to P_2

or we want N_1 <u>big</u> relative to N_2

that means we want

$$V_1 = -\frac{d\phi}{dt} N_1$$ <u>big</u> compared to

$$V_2 = -\frac{d\phi}{dt} N_2$$

or $V_1 > V_2$.

First, recall the Wave Equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}.$$

To solve this equation, we guess that $y(x,t) = Y(x)T(t)$ (i.e.--that the solution is separable). Then

$$T(t)\frac{\partial^2 Y(x)}{\partial x^2} = \frac{1}{v^2}Y(x)\frac{\partial^2 T(t)}{\partial t^2},$$

or (dividing both sides by $Y(x)T(t)$)

$$\frac{\partial^2 Y(x)}{Y(x)\partial x^2} = \frac{1}{v^2}\frac{\partial^2 T(t)}{T(t)\partial t^2}.$$

Since the left side is a function of x only, and the right side a function of t only, the two sides must equal a constant in order for the equation to be true for all x and t. Or

$$\frac{\partial^2 Y(x)}{Y(x)\partial x^2} = \frac{1}{v^2}\frac{\partial^2 T(t)}{T(t)\partial t^2} = -k^2.$$

From the equation for $Y(x)$,

$$\frac{\partial^2 Y}{\partial x^2} + k^2 Y = 0.$$

From the equation for $\Gamma(t)$,

$$\frac{\partial^2 T}{\partial t^2} + \frac{k^2}{v^2}T(t) = 0.$$

But these are just "harmonic oscillator" differential equations, and we know the solutions already! For example,

$$Y(x) = Y_0\sin(kx + \delta) \quad \text{and} \quad T(t) = T_0\cos(\omega t + \phi)$$

(with $\omega = (k^2/v^2)^{1/2}$) are possible, quite general solutions. Putting them together and using a few trig identities we get a few special solutions, though there are many others:

$$y(x,t) = y_0\sin(kx - \omega t)$$

or

$$y(x,t) = y_0\sin(kx + \omega t)$$

or

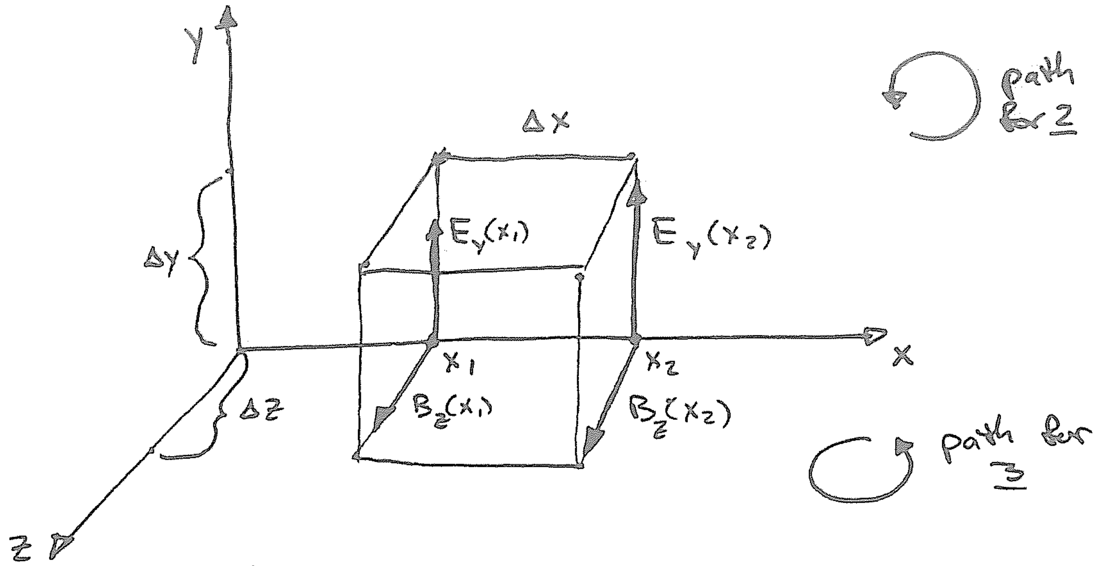
$$y(x,t) = y_0\sin(kx)\cos(\omega t)$$

or

$$y(x,t) = y_0\sin(kx)\sin(\omega t)$$

are all solutions to the wave equation, although they satisfy very different initial/boundary conditions. The first two are waves travelling to the right and left, respectively at speed v. The last two are "standing waves". In all cases,

$$k = \frac{2\pi}{\lambda}, \quad \omega = 2\pi f, \quad v = f\lambda = \frac{\omega}{k}.$$



The Wave Equation for light. Maxwell's equations.

Above is pictured an "arbitrary" cube in space that contains no charge, but does contain electric and magnetic fields that may be varying in time. We will worry only about the y component of E and the z component of B. Δx, Δy and Δz are "small"; they will become differentials in a more careful treatment.

1) From Gauss's Law for E and B,

$$\oint_S \vec{E}\cdot\hat{n}\, dA = 0 \quad \text{and} \quad \oint_S \vec{B}\cdot\hat{n}\, dA = 0$$

for the cubes, because there is no charge (electric or magnetic) inside. Therefore,

$$\int_{top} \vec{E}\cdot\hat{n}\, dA = E_y \Delta x \Delta z = -\int_{bottom} \vec{E}\cdot\hat{n}\, dA$$

and

$$-\int_{far\ side} \vec{B}\cdot\hat{n}\, dA = B_z \Delta x \Delta y = \int_{near\ side} \vec{B}\cdot\hat{n}\, dA .$$

This lets us treat only two surfaces. We don't have to worry about "losing" flux in the box in time, which simplifies Ampere's law and Faraday's law.

---

2) Faraday's law applied to top surface.

$$\oint_{C(top)} \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_{top} \vec{B}\cdot\hat{n}\, dA$$

or

$$\left[E_y(x_2) - E_y(x_1)\right]\Delta y = \frac{\partial E_y}{\partial x}\Delta x \Delta y$$
$$= -\frac{d}{dt} B_z \Delta x \Delta y$$

so

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

---

3) Ampere's Law applied to the near surface.

$$\oint_{C(near)} \vec{B}\cdot d\vec{l} = \mu_0\epsilon_0 \frac{d}{dt}\int_{near} \vec{E}\cdot\hat{n}\, dA$$

or

$$\Big[B_z(x_1) - B_z(x_2)\Big]\Delta z = -\frac{\partial B_z}{\partial x}\Delta x\Delta z$$

$$= \mu_0\epsilon_0 \frac{d}{dt} E_y \Delta x\Delta z$$

so

$$\frac{\partial B_z}{\partial x} = -\mu_0\epsilon_0 \frac{\partial E_y}{\partial t}$$

Finally, we take a space (or time) derivative or either equation, switch the order of differentiation, and use the other equation:

$$\frac{\partial}{\partial x}\frac{\partial E_y}{\partial x} = - \frac{\partial}{\partial x}\frac{\partial B_z}{\partial t}$$

or

$$\frac{\partial^2 E_y}{\partial x^2} = - \frac{\partial}{\partial t}\frac{\partial B_z}{\partial x}$$

$$= \mu_0\epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

which is the wave equation for E_y!

We know immediately, then, that

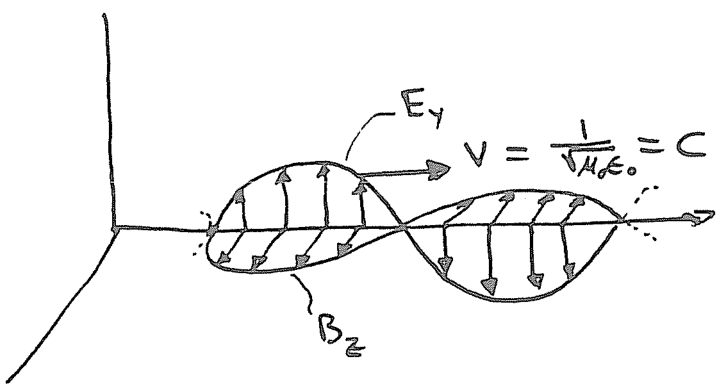
$$E_y(x,t) = E_{0y}\sin(kx - \omega t)$$

and it can be easily shown that also

$$B_z(x,t) = B_{0z}\sin(kx - \omega t).$$

---

or



Last time we derived the Wave Equation for Electromagnetic waves (light):

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0\epsilon_0\frac{\partial^2 E_y}{\partial t^2}.$$

Recalling that the constant in front of the second term is $1/v^2$, we note that

$$v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3x10^8 \frac{m}{sec} = c!$$

It is here that we really identify the coupled E and B waves with light. It is for this moment that Maxwell's equations are so named.

Recall, then, that

$$E_y(x,t) = E_{0y}\sin(kx - \omega t)$$

and

$$B_z(x,t) = B_{0z}\sin(kx - \omega t)$$

where

$$B_{0z} = \frac{E_{0y}}{c}.$$

The important thing to note here is that a) E and B are in phase and b) the amplitude of B and E are related by c.

Energy and Intensity of Electromagnetic waves.

First we <u>define</u> the <u>Poynting Vector</u>:

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}).$$

This is a vector that a) points in the direction of propagation of the wave and b) is the instantaneous intensity (energy/(area·time)) if the wave. The former is obvious. We see the latter by considering

$$\eta(x,t) = \eta_e + \eta_m$$

$$= \frac{1}{2}\epsilon_0 E_0^2\sin^2(kx - \omega t) + \frac{1}{2\mu_0}B_0^2\sin^2(kx - \omega t)$$

$$\eta(x,t) = \epsilon_0 E_0^2\sin^2(kx - \omega t).$$

Then

$$\eta = \frac{E_0^2}{\mu_0 c^2}\sin^2(kx - \omega t)$$

and $\eta c = I(x,t) = \frac{1}{\mu_0}E_0 B_0\sin^2(kx - \omega t) = |\vec{S}|$

is the instantaneous intensity as promised

Usually we are more interested in the average intensity (because $f = \frac{2\pi}{\omega}$ is typically 10^3 to 10^{15} Hz or more). If we cycle average $\sin^2(kx - \omega t)$ we get (as usual) $\frac{1}{2}$. Thus

$$\eta_{av} = \frac{1}{2}\frac{E_0^2}{\mu_0 c^2}$$

and

$$I = c\eta_{av} = \frac{1}{\mu_0}E_{rms}B_{rms} = |S_{av}|.$$

Then Power delivered to a surface is the flux of the Poynting vector,

$$P = \int_A \vec{S}\cdot\hat{n}\, dA = \text{energy/time through } A.$$

Finally, dimensionally, momentum (of a particle) is related to its energy by $|\vec{p}| = C\frac{E}{v}$. In the case of massless waves travelling at velocity c, this relation becomes exact. The momentum carried by a wave is

$$|\vec{p}| = \frac{E}{c}.$$

From this we can deduce the radiation pressure (momentum/(time·area)) from the intensity (energy/(time·area)) as

$$p_r = \frac{I}{c} = \frac{1}{c}|S_{av}|.$$

We're done!

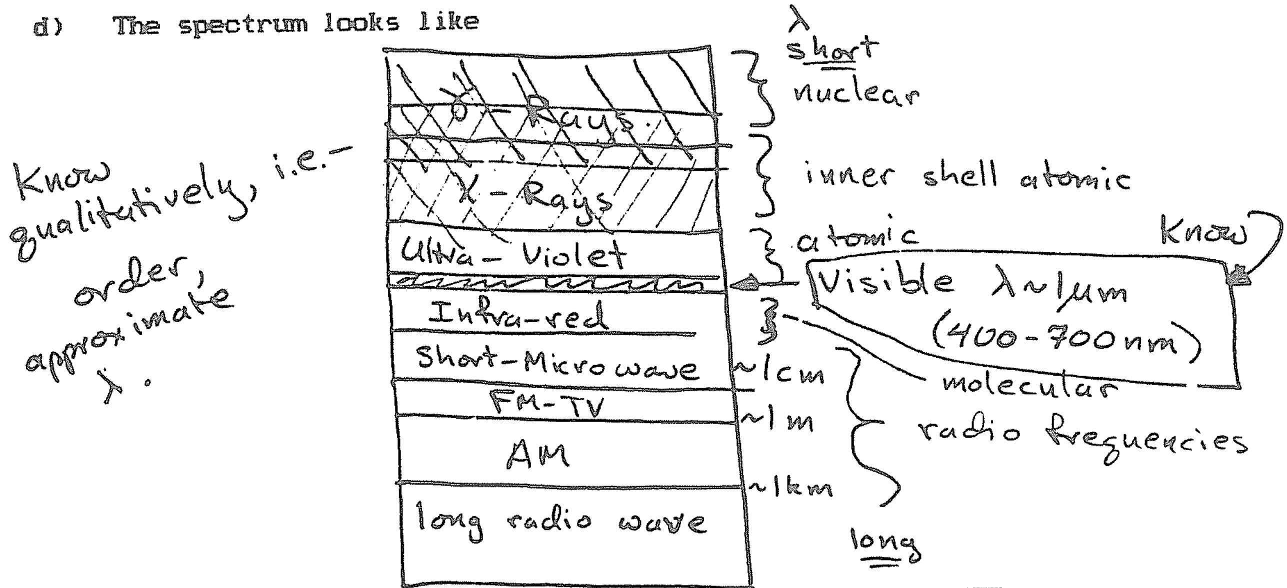
Light, as we have just seen, is a transverse electromagnetic wave. Here are some True Facts:

a) Light behaves like a "ray" or "particle" when its wavelenght is much smaller than the objects with which it interacts.

b) Light behaves like a "wave" (exhibiting interference and diffraction) when its wavelength is comparable to the size of the objects with which it interacts.

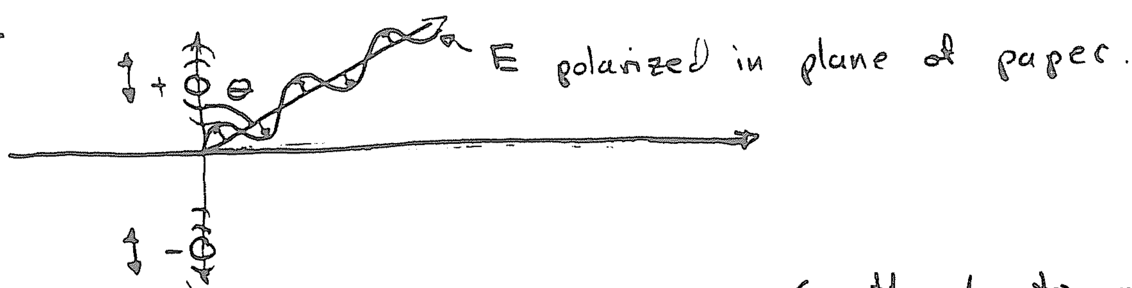
c) Light travels at $c = 3 \times 10^8$ m/sec in a vaccuum, but slows down in matter.

d) The spectrum looks like



The Production of Light.

Light is emitted by oscillating electric dipoles (among other things).



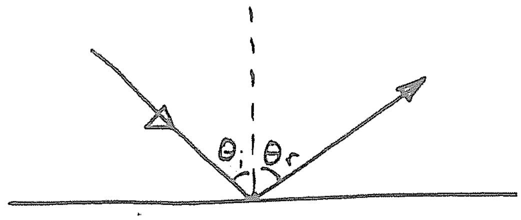
The light is emitted with intensity proportional to (mostly ⊥ to axis of dipole!)

$$I = \frac{I_0 \sin^2\theta}{r^2}.$$

This is called electric dipole radiation and is the predominant source of EM radiation in almost all frequency ranges.

More True Facts about light.

a) When light reflects from a "shiny" surface, the angle of incidence (relative to a normal drawn to the surface) equals the angle of reflection.



---

b) When light traverses a medium other than vaccuum, it slows down. (Because the medium is magnetizable and polarizable, $\epsilon_0 \longrightarrow \epsilon$ and $\mu_0 \longrightarrow \mu$). We define the index of refraction for a medium by

$$nv = c \quad \text{(with } n \geq 1\text{)}.$$

Note that since the speed changes and the frequency remains the same, the wavelength must change.

$$nv = c = f\lambda_0,$$

so

$$v = f\lambda = \frac{f\lambda_0}{n} \text{ or}$$

$$\lambda(\text{in medium}) = \lambda(\text{in vaccuum})/n.$$

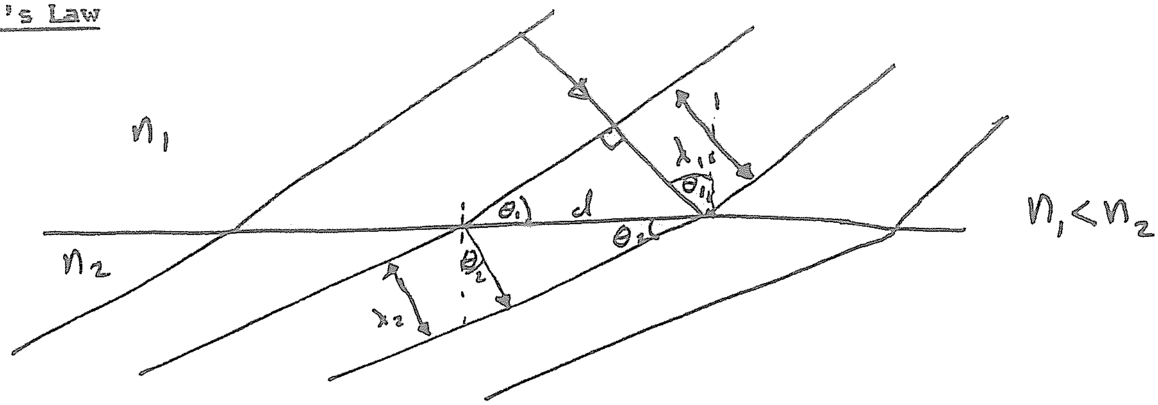
Fermat's Principle: "Light takes the path that minimizes the time of passage from one point to another."

This is a variational principle. Variational principles are important because the most important laws of physics, in either classical or quantum mechanics, are expressed/derived in terms of them including your good friend Newton's laws.

The implication of Fermat's principle is that light will travel in straight lines in a single medium, but when going from one medium to another where its speed changes, it will bend at the surface between them.

This leads us to formulate

Snell's Law



Snell's law is $n_1 \sin\theta_1 = n_2 \sin\theta_2$. Its derivation is: Observe in the figure above that if the speed of light changes (via the index of refraction) at the surface between medium one (n_1) and medium two (n_2), the wavelength, as noted, must also change. Geometrically that implies that the direction of propagation of the light must change. We see that

$$\lambda_1 = \frac{\lambda}{n_1}, \quad \lambda_2 = \frac{\lambda}{n_2}.$$

From the triangles above,

$$d\sin\theta_1 = \lambda_1 = \frac{\lambda}{n_1} \text{ and}$$

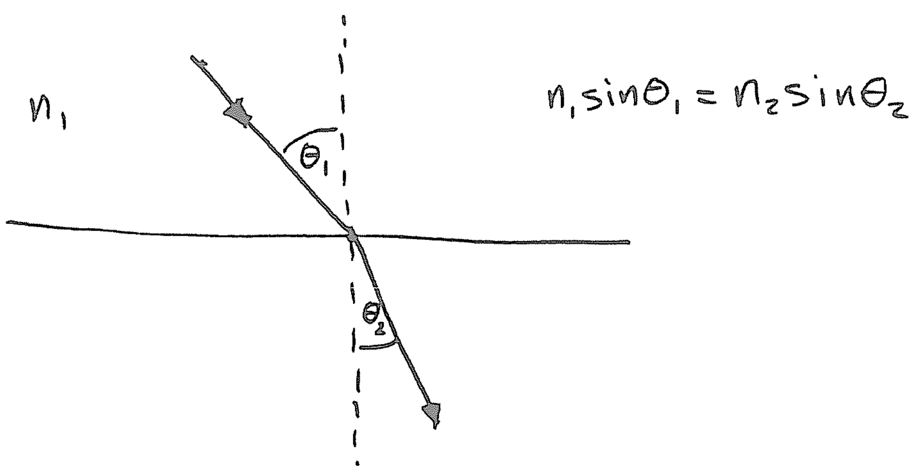
$$d\sin\theta_2 = \lambda_2 = \frac{\lambda}{n_2}.$$

Then $$n_1 \sin\theta_1 = \frac{\lambda}{d} = n_2 \sin\theta_2,$$

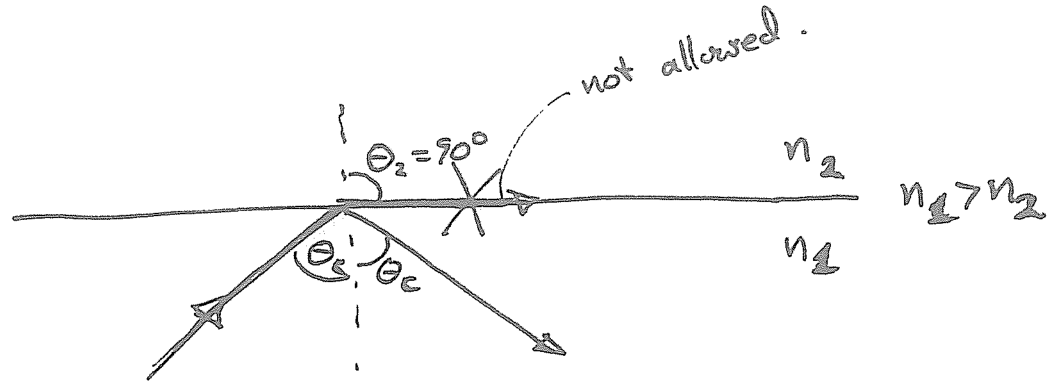
and Snell's law is proven (Q.E.D., and all that).

---

so



Critical Angle

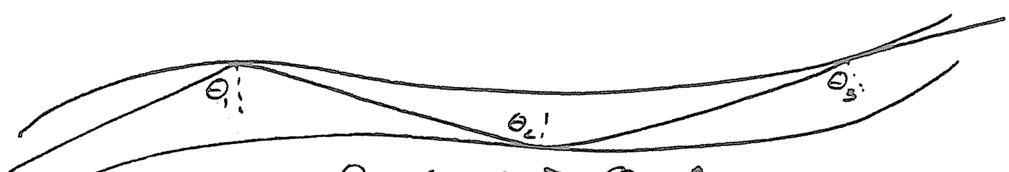


$$n_1 \sin\theta_c = n_2 \sin 90° = n_2 \Rightarrow \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

note $n_2 < n_1$!

If a light ray is going from a dense medium to a light medium ($n_1 > n_2$), then if light is incident at a particular angle, called θ_c or the <u>critical angle</u>, it will be refracted parallel to the surface. Consequently, it will never escape medium n_1. If the light ray continues to travel in n_1, it can't skim along under the surface; the angle of incidence = the angle of reflection. Therefore this ray is completely internally reflected. The <u>critical angle</u> is the <u>smallest angle</u> such that <u>total internal reflection occurs</u>.

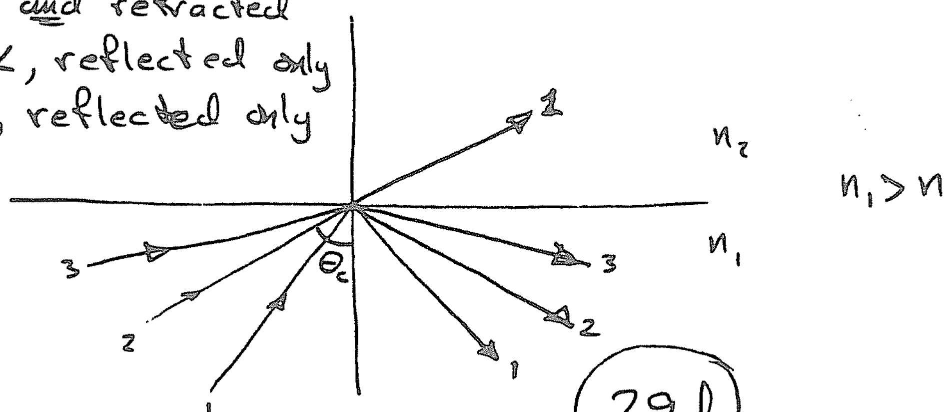
"Light fibers" or "fiber optics" work on this principle. Because the light strikes the surface of the fiber at a grazing angle in each collision, it is totally reflected and hence transmitted along the fiber.



$$\theta_1, \theta_2\ \theta_3 > \theta_c\ !$$

---

1) reflected <u>and</u> refracted
2) critical ∠, reflected only
3) $\theta_i > \theta_c$, reflected only

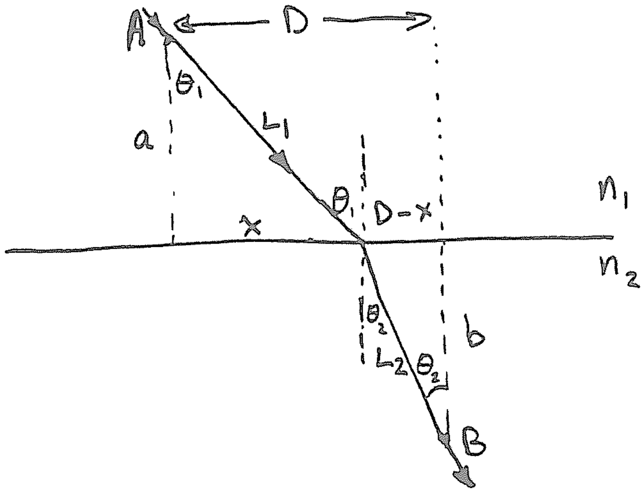


Recall Fermat's Principle: "Light takes the path that minimizes the time of passage from one point to another."

The implication of Fermat's principle is that light will travel in straight lines in a single medium, but when going from one medium to another where its speed changes, it will bend at the surface between them.

We can use Fermat's principle to derive Snell's Law as follows:

Snell's Law



In the picture above, t_1 is the time it takes for light to go from A to x along L_1, t_2 is the time it takes for light to go from x to B. The total time it takes light to go from A to B is thus:

$$t = t_1 + t_2 = \frac{L_1}{v_1} + \frac{L_2}{v_2}$$

where $v_1 = c/n_1$ and $v_2 = c/n_2$ are the speeds of light in medium 1 and medium 2, respectively. Then

$$t = \frac{1}{c}\left[n_1 L_1 + n_2 L_2\right].$$

Now,

$$L_1 = (a^2 + x^2)^{1/2}$$

and

$$L_2 = (b^2 + (D - x)^2)^{1/2},$$

so

$$\frac{dt}{dx} = \frac{1}{c}\left[n_1\frac{dL_1}{dx} + n_2\frac{dL_2}{dx}\right] = 0$$

at the point where the time t is a minimum. Substituting in the derivatives (work it out. . .)

$$\frac{dt}{dx} = \frac{1}{c}\left[n_1\frac{x}{L_1} - n_2\frac{(D - x)}{L_2}\right] = 0$$

or

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

(since $x/L_1 = \sin\theta_1$, etc.) Q. E. D.

Polarization

We derived the wave equation for light by assuming that the electric field vector was parallel to the y-axis. Actually, light emitted from atoms is as likely to have its electric field vector pointing in one direction (perpendicular to the direction of propagation) as another. Light of this sort is called unpolarized light. It is possible (as we shall see) to select out of unpolarized light only that portion that has its electric field vector pointing in a particular direction. When we have done so, the resulting light (with its electric field vector pointing in some systematic, predictable direction is called polarized light.

There are two basic kinds of polarized light.

a) linearly polarized.

b) elliptically (including circularly) polarized.

Linearly polarized light is what we derived the wave equation for. The electric field vector points along a single direction in space (like the y-axis) and has no components at any time in any other direction.

Elliptically polarized light has electric field components in (for example) the y-direction and the z-direction simultaneously, and the sine waves describing the two components are not generally in phase with each other. Consequently the electric field vector (which is the vector sum of these two components) swings around the x-axis, getting longer or shorter in the different directions according to the time. The length and angle of the electric field vector trace out an ellipse if the maximum amplitude of the components is not equal or if the phase difference is not 0 or 90°, and a straight line or circle otherwise. Draw some pictures to be convinced.

Linear polarization. Linearly polarized light can be produced 4 ways (that you are responsible for).

a) absorption (Polaroid sheets)

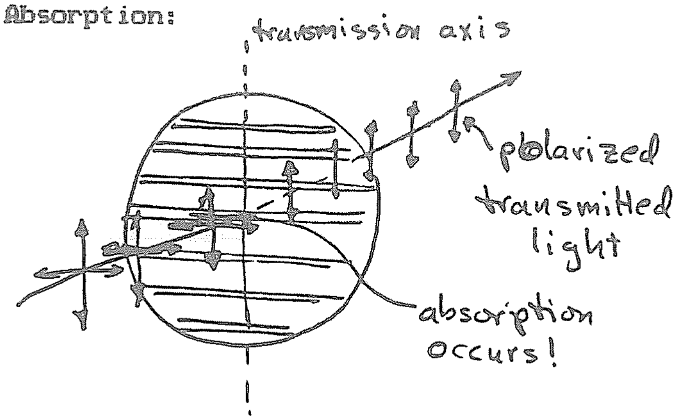
b) reflection (Brewster's law, $\tan\theta_p = \frac{n_2}{n_1}$)

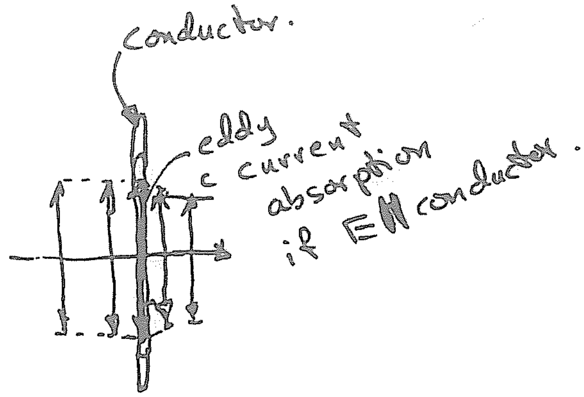
c) scattering (same reason as b)

d) birefringence (because n is different along different planes in a crystal).

You will only be tested in the first three, though you should know at least what is written above about birefringence.

Absorption:





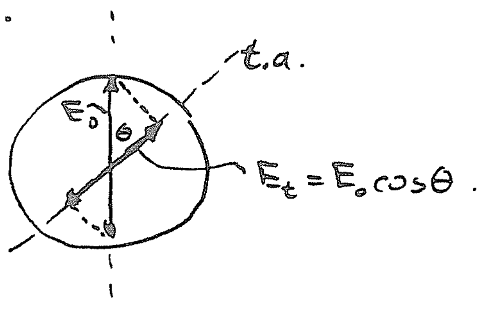
Polaroid sheets consist of plastic with long conducting strands in it lined up along some direction (see above). When light passes through the sheets, the electric field creates a potential difference along the conductors when it lies parallel to the conductors. This causes (eddy) currents to flow, heating the sheet with energy extracted from the light and hence absorbed. But electric field vectors perpendicular to the conducting filaments do not produce eddy currents, and are therefore transmitted instead of being absorbed. As a consequence, the light transmitted is polarized in this direction. The direction perpendicular to the filaments is called the transmission axis, and the transmitted light is polarized in this direction.

True facts:

a) If unpolarized light is incident on a polarizing sheet, the transmitted light is polarized parallel to to the transmission axis and its intensity is 1/2 the indident intensity.

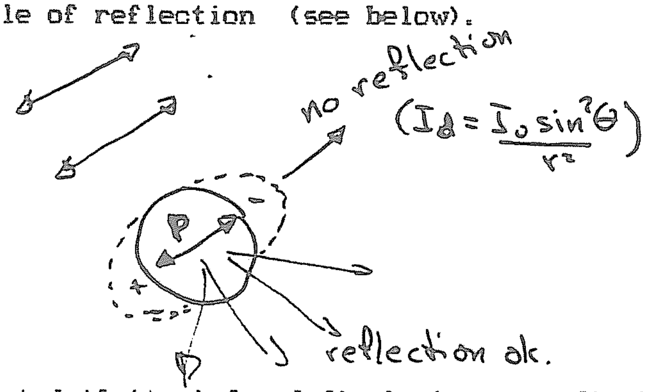
b) If polarized light makes an angle θ as shown with the transmission axis of a polaroid sheet, only that component of E parallel to the t. a. is tranmitted, i.e.-- $E_t = E_0 \cos\theta$. Therefore $I_t = I_0 \cos^2\theta$.

c) Since reflected glare is partially polarized horizontally (see next) polaroid sunglasses have vertical transmission axes.

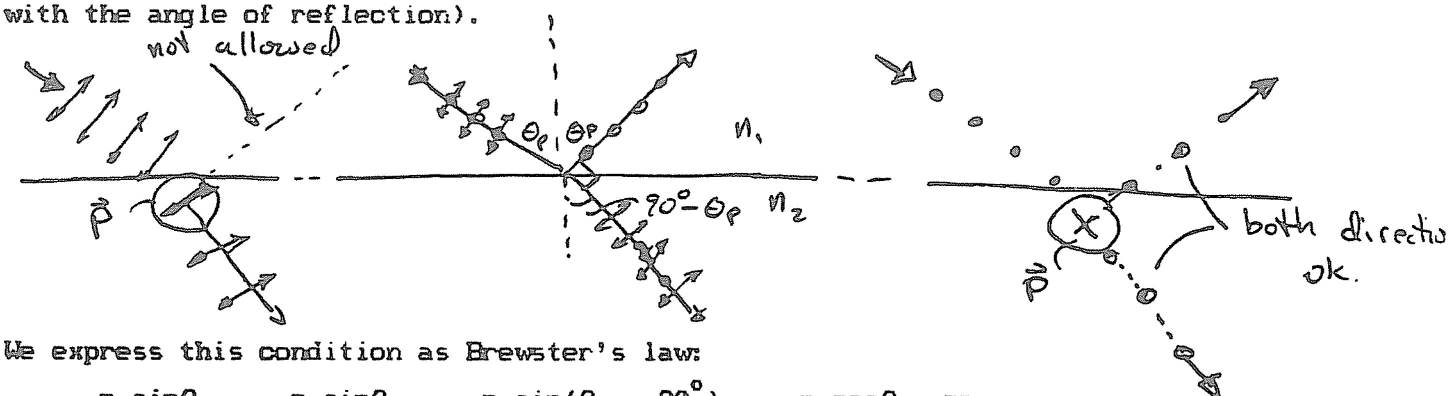


Reflection: Brewster's law.

When light is incident on the surface between two media (see above) it is generally partially reflected and partially refracted. The process of reflection entails that the atoms on the surface absorb the light and reradiate it. (See scattering below) When the atom absorbs, it is "polarized" parallel to the electric field of the light at that moment; when it reradiates it, it does so as if it is an oscillating dipole lined up in that direction. As a consequence, light cannot be reflected if the induced dipole lines up with the angle of reflection (see below).



It can be reflected if its induced dipole is perpendicular to the direction of reflection. The reflected light is thus polarized parallel to the surface and perpendicular to the direction of propagation. In order for the reflected light to be completely polarized, the angle of reflection must thus be perpendicular to the angle of refraction (so the induced dipole perpendicular to the angle of refraction and in the plane of the page lines up exactly with the angle of reflection).



We express this condition as Brewster's law:

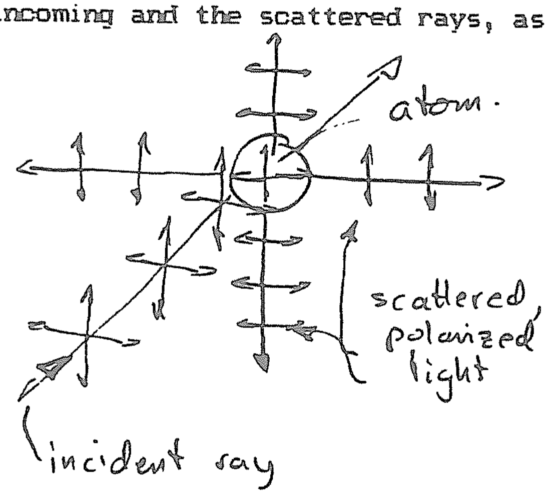
$$n_1\sin\theta_p = n_2\sin\theta_2 = n_2\sin(\theta_2 - 90^\circ) = n_2\cos\theta_p, \text{ or}$$

$$\tan\theta_p = \frac{n_2}{n_1},$$

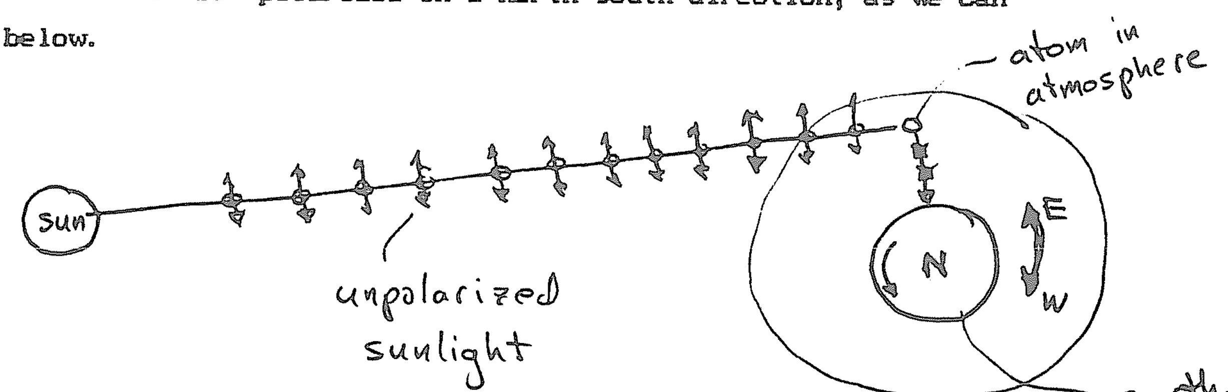
where θ_p is the angle at which the reflected light is completely polarized. Note that the transmitted light is only partially polarized.

Scattering:

As we just saw, when light is absorbed by an atom and reradiated, or scattered, at least one way that can occur is for the electric field of the light to induce an oscillating polarization in the atom, which then reradiates the field like a dipole oscillator. (What actually happens in quantum mechanical and much more complicated, but the essence of this still remains.) A dipole always produces light polarized in the plane of the dipole. Since the induced dipole points in some direction perpendicular to the incoming ray any rays scattered in directions perpendicular to the incoming ray must be polarized perpendicular to both the incoming and the scattered rays, as shown.



As a consequence, light scattered by air in the atmosphere is partially polarized, as can be verified with a polaroid sheet. Viewed at sunrise or sunset, the light scattered from directly overhead will be polarized in a north-south direction, as we can see below.

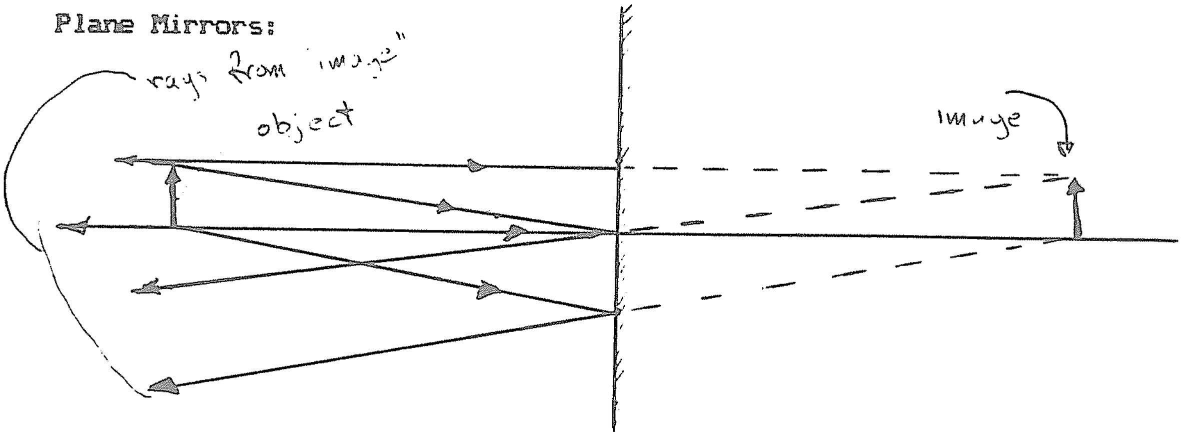


Read about Birefringence on your own, but only True Fact type questions on a test.

Sometimes light behaves like a wave, and sometimes light behaves like a particle. Generally, light behaves like a particle when its wavelength is much smaller than the physical dimensions of the things with which it interacts. In those cases, as we have seen, we can represent the trajectory of light with a straight line (that can bend only at the surface between media with different indices of refraction or at a reflective surface). The approximation (small wavelength ⟶ straight line) that allows us to do this is called the ray approximation, and a diagram that represents those trajectories as straight lines with arrows indicating direction (rays) are called ray diagrams. Geometrical optics is concerned with how to construct a ray diagram representing reflection from and refraction through various curved and flat surfaces. As we shall see, certain mirror surfaces and certain refractive surfaces allow recombination of rays emitted by some source (the object) into a form that appears to originate from some other point in space (the image). The formation of images by lenses and mirrors is an important part of our culture and lives, as it is the imaging process that allows us, among other things, to see. Control of the imaging process allows us to extend the range of our vision to see the very small (with microscopes) and the very distant (with telescopes). It was the extension of the vision of mankind with these devices that was, as much as anything else, responsible for the Rennaissance of Personkind following the discoveries of Gallileo and Gregory (inventors of the telescope, which gave Newton the data upon which to base his theory of Gravitation), and Janssen and Leeuwenhoek (inventor and developer of the compound microscope, which for the first time showed microorganisms, the cellular structure of life, the causes of disease, and the structure of matter on the microscale to Personkind).

Mirrors: Mirrors are smooth surfaces that reflect rays. As long as the geometry of the surface is fairly simple, they tend to form images, as we shall see. We will first study plane mirrors, and learn to draw ray diagrams from it. We will then extend our understanding to include spherical mirrors (concave and convex) as well.

Plane Mirrors:



Rules for ray diagrams. An object emits light rays in all directions from every point on its surface. We generally pick only a few from the top and/or bottom of our "generic" object (↑) and use them to locate the image. In the case of the plane mirror above, any two rays from the top and bottom suffice to locate the image. From the diagram it is obvious that

a) The image is located symmetrically behind the mirror

b) The light rays reflected from the mirror appear to come from an image point behind the mirror. Since no light physically passes through this point (unless the lights are on in your closet) we call this image a virtual image.

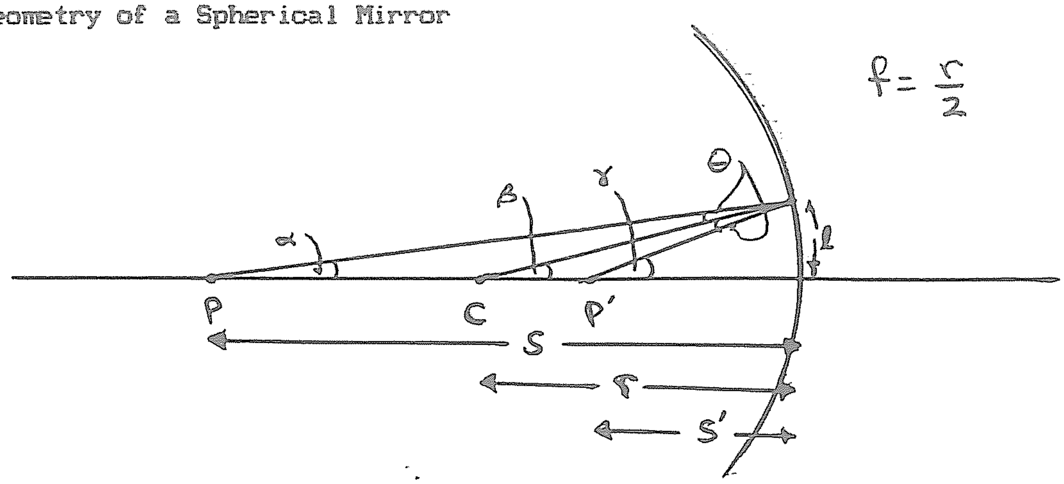
c) The image is erect, and is as big as the object.

Definitions:

A real image is an image where the light rays from a point on the object physically pass through the image. A real image is generally inverted, and can be larger or smaller than the object.

A virtual image is formed when the light rays from a point on the object do not physically pass through the image point. (For example, no light passes through an image behind a mirror.) A virtual image is generally erect, and is usually the same size or smaller than the object.

Geometry of a Spherical Mirror



The object is point P. The center of curvature of the surface is point C. The image formed (determined by angle of incidence = angle of reflection) is located at point P'. α, β, γ are angles as shown, and r is the radius of the sphere. s is the distance from the object to the surface of the mirror, s' is the distance from the image to the surface of the mirror, and ℓ is the arc length. From the triangles drawn, we have

$$\beta = \alpha + \theta$$

and

$$\gamma = \beta + \theta.$$

We eliminate θ, and use the small angle approximation,

$$\alpha \approx \frac{\ell}{s}, \quad \beta \approx \frac{\ell}{r}, \quad \gamma \approx \frac{\ell}{s'}$$

for each of the angles, and one obtains

$$2\beta = \alpha + \gamma$$

or

$$\frac{2}{r} = \frac{1}{s} + \frac{1}{s'} .$$

If the object is at infinity ($s = \infty$) then the light rays from it come in parallel to the mirror axis. Parallel rays are reflected (imaged) through a special point called the <u>focal point</u>. We define the location of f by

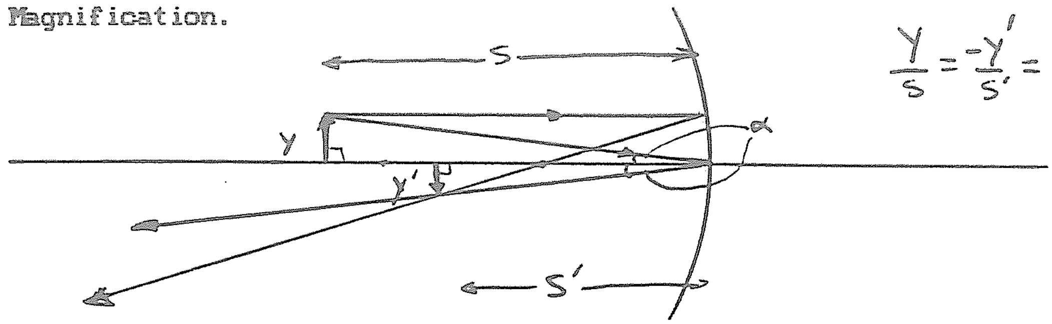
$$\frac{2}{r} = \frac{1}{s'} = \frac{1}{f} \quad (\text{with } \frac{1}{\infty} = 0) \text{ or}$$

$$f = \frac{r}{2}.$$

Then the equation we use for mirrors to locate the image becomes

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

Magnification.



$$\frac{Y}{s} = \frac{-Y'}{s'} = \tan\alpha$$

A little reflection on the ray diagram above will convince one that the image need not be the same size as the object. The ratio of the image size to object size is called the magnification. From the geometry of the diagram above (which is quite general) it is apparent that

$$m = \frac{y'}{y} = -\frac{s'}{s}$$

(from the similar right triangles).

Ray Diagrams. Finally, we need to learn how to draw a ray diagram, with only a few (three) of the many many rays that could be drawn, that will help us locate and understand the images formed by mirrors (and later lenses) for various object distances and focal lengths.

The Rules are: Draw three rays.

1) Draw a ray from the top of the generic object parallel to the axis to the mirror. If the focal length is positive, this ray is reflected through the focal point, by definition. If the focal length is negative, it is reflected so that it appears to come from the focal point.

2) Draw a ray from the top of the g. object that strikes the mirror in the center (on the axis). This ray is reflected symmetrically underneath the object, because the angle of incidence equals the angle of reflection.

3) Dray a ray from the top of the g. object that either goes through or appears to come from the focal point of the mirror. If the focal length is positive, a ray that goes through the focal point is reflected parallel to the axis. If the focal length is negative, a ray that is headed toward the focal point behind the mirror is reflected parallel to the axis. Note that these rays are drawn with a rule exactly opposite to that of ray 1. This is because if you reverse the direction of the light rays, the mirror functions identically, with the role of incident and reflected ray changing places.

Examples:

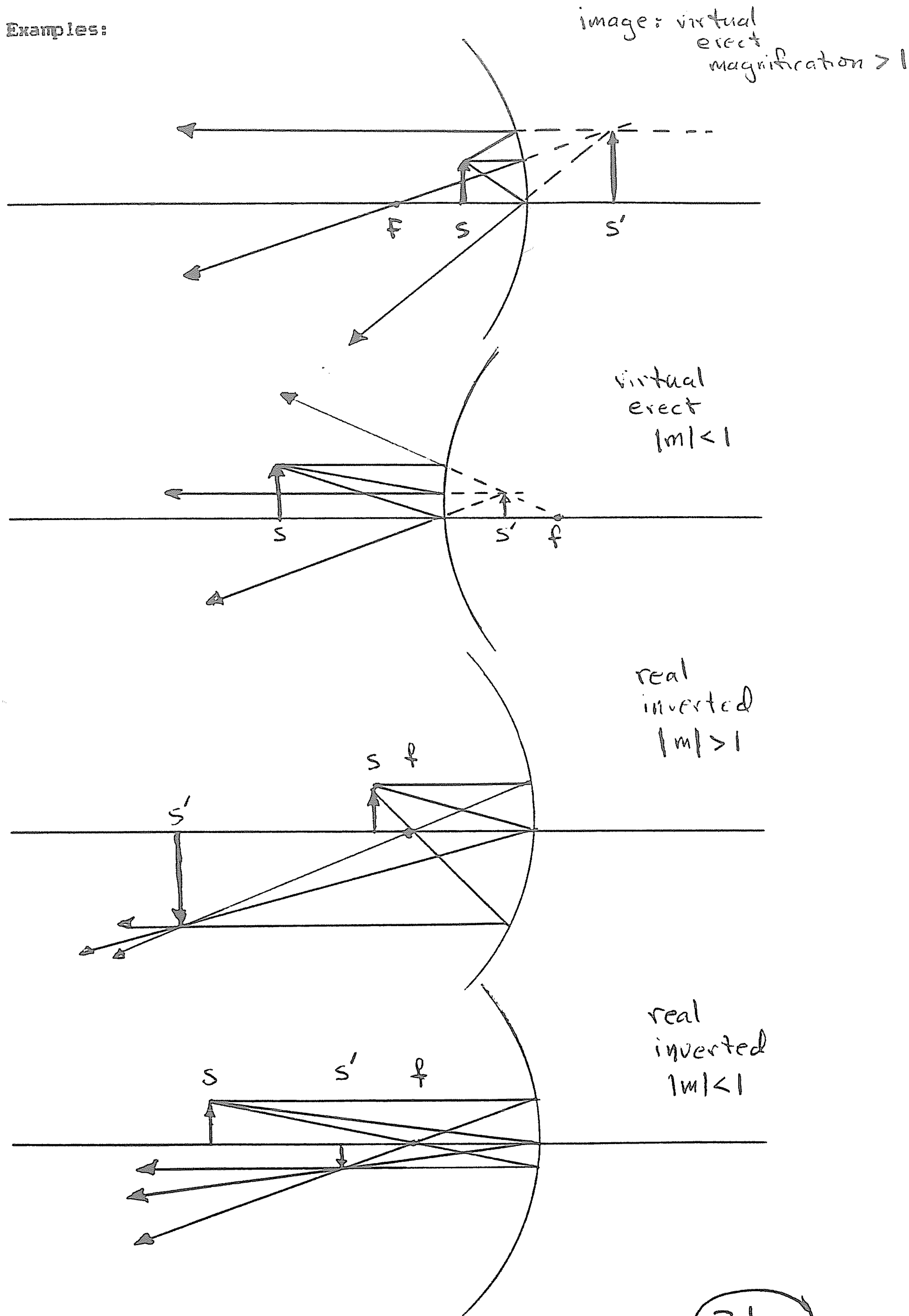
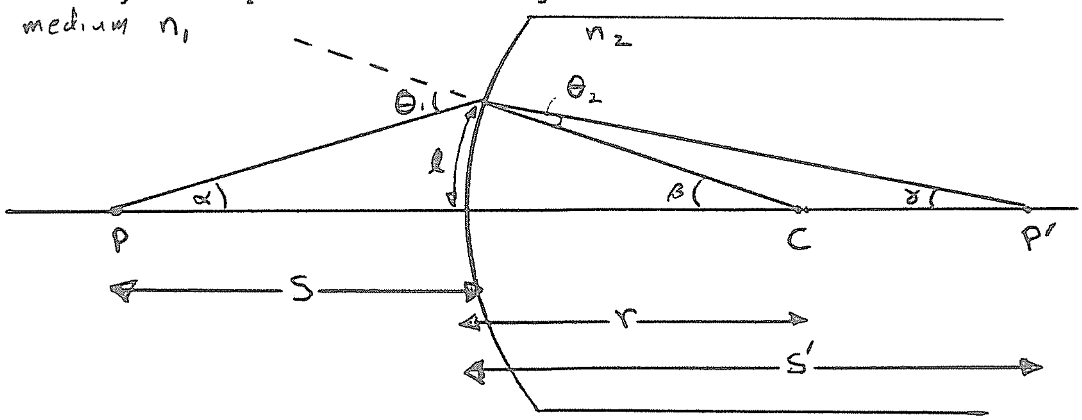


image: virtual
erect
magnification > 1
F
s
s'
virtual
erect
|m| < 1
s
s'
f
real
inverted
|m| > 1
s
f
s'
real
inverted
|m| < 1
s
s'
f

Lenses. Light rays can be focused by anything that can change their trajectory and is suitably shaped. We have just examined one case of that: the mirror, which reflects incident light into a new, but related, trajectory. Another common case is the lens.

The principle behind the lens is that of refraction, which can also alter the trajectory of incident light according to Snell's law. We will understand lenses by first examining refraction off of a spherical surface, and then deducing the lens makers formula, the thin lens equation, and finally by learning to draw ray diagrams for lenses in many useful configurations, e.g.-- the microscope and telescope and the human eye.

Geometry of a Spherical Refracting Surface (lens)



The object is point P. The center of curvature of the surface is point C. The image formed (determined by Snell's law $n_1\sin\theta_1 = n_2\sin\theta_2$) is located at point P'. α, β, γ are angles as shown, and r is the radius of the sphere. s is the distance from the object to the surface of the mirror, s' is the distance from the image to the surface of the mirror, and ℓ is the arc length. For small angles, $\sin\theta_1 \approx \theta_1$ and Snell's law becomes:

$$n_1\theta_1 \simeq n_2\theta_2.$$

From the triangles drawn, we have

$$\beta = \theta_2 + \gamma = \frac{n_1}{n_2}\theta_1 + \gamma$$

and

$$\theta_1 = \alpha + \beta.$$

We eliminate θ_1 to get

$$n_1\alpha + n_2\gamma = (n_2 - n_1)\beta,$$

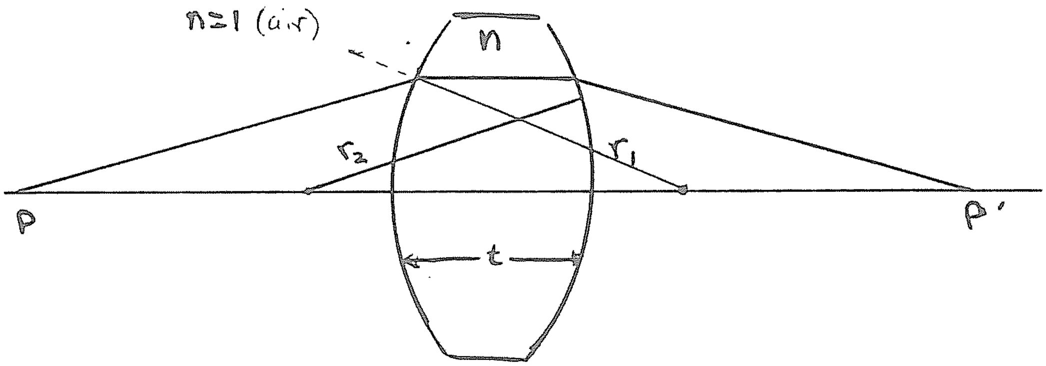
and use the small angle approximation,

$$\alpha \approx \frac{\ell}{s}, \quad \beta \approx \frac{\ell}{r}, \quad \gamma \approx \frac{\ell}{s'}$$

to get

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{(n_2 - n_1)}{r}.$$

Most lenses are not formed out of a solid cylinder of glass, but rather out of a (relatively) thin piece with two surfaces that sits in air ($n_1 = 1$, $n_2 = n_g$). We generalize the results of refracting through two surfaces by making the image from the first surface the object of the second surface. For a lens of thickness t,



(First surface) $$\frac{1}{s} + \frac{n}{s'} = \frac{n - 1}{r_1}$$

and

(Second surface) $$\frac{n}{s''} + \frac{1}{s'''} = \frac{1 - n}{r_2}.$$

Solving the first for s', using the distance t − s' as the object distance (s") for the second surface (Note: s" is negative. This is one of the few cases where "object distance" can be negative, i.e.-- when the "object" for a given lens or mirror is the image of another, and is located on the virtual side.) and substituting it in the second equation, we get:

$$\frac{n}{-s' + t} + \frac{1}{s'''} = \frac{1 - n}{r_2} = -\frac{n - 1}{r_2}.$$

This is too hard to deal with this way (there is actually another whole matrix-based technique for solving this kind of problem that we are not learning now) so we consider only $t \ll s$, $t \ll s'$; the thin lens:

$$\frac{n}{s'} = \frac{n - 1}{r_1} - \frac{1}{s} = \frac{1}{s'} + \frac{n - 1}{r_2}$$

or

$$\frac{1}{s} + \frac{1}{s'} = (n - 1)\left[\frac{1}{r_1} - \frac{1}{r_2}\right].$$

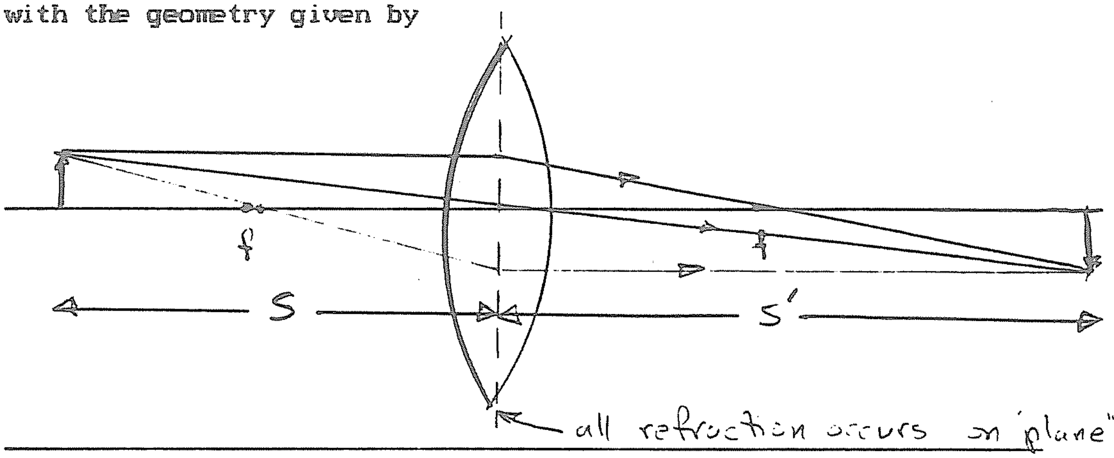
This is known as the lensmakers equation, and you are responsible for it. This equation obtains a simple form that you should recognize when we note that the focal length of the thin lens is given by

$$\frac{1}{f} = (n-1)\left[\frac{1}{r_1} - \frac{1}{r_2}\right].$$

This implies the <u>thin lens equation</u>,

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f},$$

with the geometry given by



True facts about lenses.

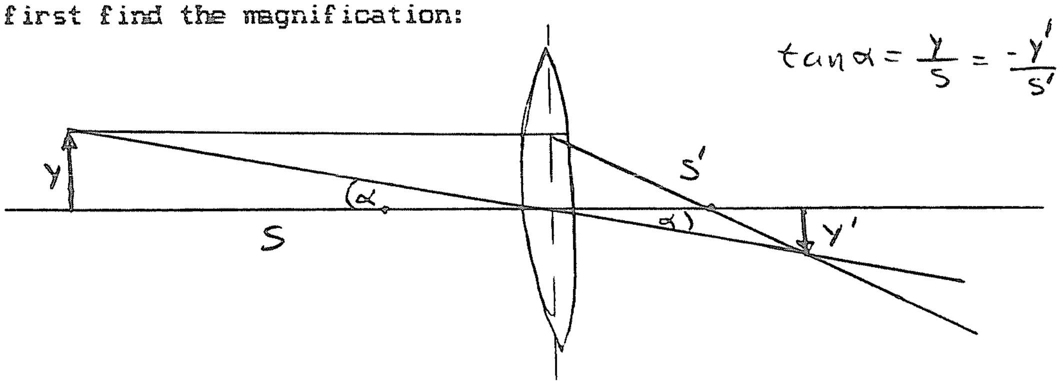
Since the trajectories of light rays are reversible (Snell's law works both ways) the focal length of a lens is the same on both sides of the lens.

 Shaped lenses have positive focal lengths and concentrate light,

 shaped lenses have negative focal lengths and spread light rays apart.

Distances (s', f, r) are positive on the real image side of the lens (opposite the always-positive object distance) and negative on the virtual image side.

Ray diagrams are formed just as they are for mirrors, sort of. We first find the magnification:



As before, $m = \frac{y'}{y} = -\frac{s'}{s}$ from the similar triangles (once we have located the image).

Aberrations. We made a number of approximations in deriving the thin lens, lens makers and mirror equations. Recall that they are: Paraxial rays, n constant for different λ, and rays considered are all close to the axis as well as being paraxial. In reality, non of these things are generally true. The resulting deformation of the images produced by lenses and mirrors due to failures of these approximations are called aberrations.

Below are listed the five kinds of aberration you are responsible for, the cause and the effect.

1) Spherical aberration. This arises because a spherical surface does not focus all rays parallel to the axis through one point (specifically it fails for rays far from the axis). It can be "cured" by using a parabolic surface instead.

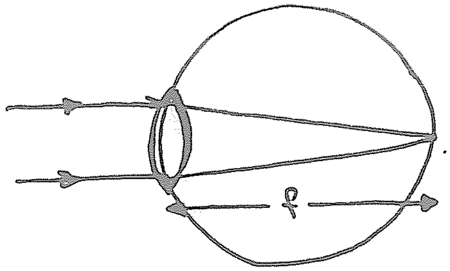
2) Coma. Another spherical aberration, it results when non-paraxial rays are used and the small angle approximation breaks down. It is named coma because it distorts a point into a point with a blurry tail, like that of a comet.

3) Astigmatism. This is caused by a lack of axial symmetry in the lens, i.e. -- the focal lengths in different planes are different. It is a fairly common defect of the human eye.

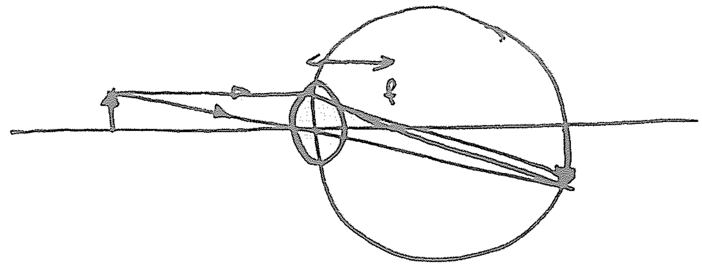
4) Distortion. This is a property of all imaging devices that is caused by the m (the magnification) depending on s (the object distance). For an extended object, different points are different distances from the lens/mirror, and thus they have different magnifications. Instead of seeing a uniformly magnified image, one then sees a distorted image where some points are magnified more than others. This produces, e.g.-- a "big nose" when the camera is too close, etc.

5) Chromatic aberration. This occurs only in lenses (all the others occur in both mirrors and lenses). It results from n (the index of refraction) being a function of λ (the wavelength of the incident light). Thus different wavelengths have different focal lengths, which produces a little "rainbow" around the image.

<u>The Eye.</u> Below is drawn a representation of a "normal" eye.



relaxed



accomodating (f shorter)

A properly functioning eye forms a real image upon the retina. The image is inverted. The "size" of the image (psychologically) is determined by how much of the retina is covered, i.e.-- by how big an angle the image subtends. A normal, relaxed eye is focused on infinity, i.e.-- it focuses parallel rays to a point.

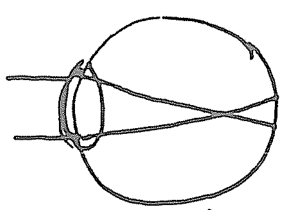
1) <u>Accomodation</u> is the way the eye focuses on objects at different distances. The relaxed eye in focused on infinity. By using muscles in your eye, the lens becomes rounder, shortening the focal length and focusing objects closer than infinity.

2) An the <u>near point</u> the lens is as bulged as it can be and the focal length is as short as it can be. The eye sees objects at this distance (around 10-25 cm) as clearly as it can see anything. This is also called the <u>distance of most distinct vision</u>. We will assume it to be 25 cm, though the lens stiffens with age, and in older or farsighted people it can be a meter or so.

Note: Accomodation can only shorten the focal length of the eye. When the eye is relaxed its focal length is as long as it gets.

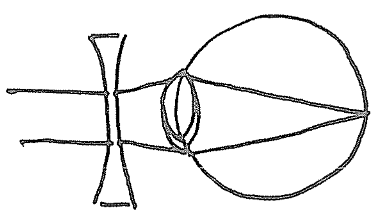
Abnormalities:

Nearsighted Eye



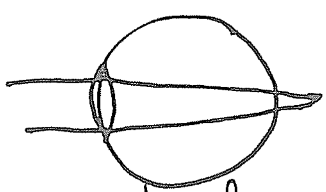
relaxed.
(f too short)

Correction lens



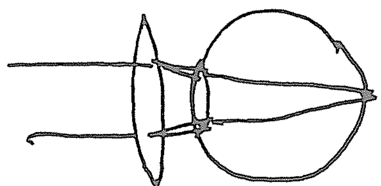
lengthen f w/ diverging lens

Farsighted Eye



relaxed

Correction lens

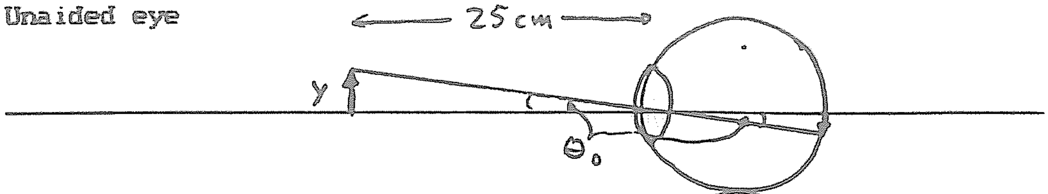


shorten f w/ converging lens

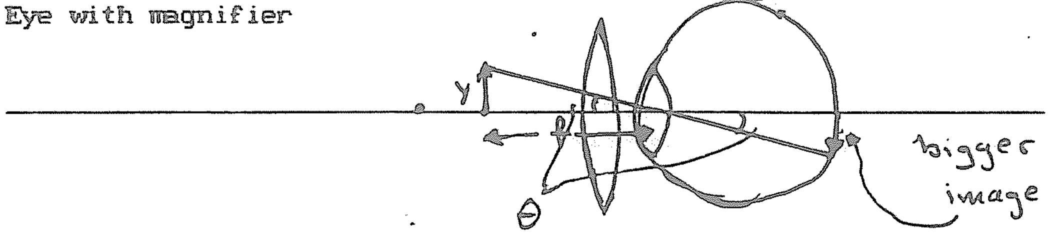
33b

Simple Magnifier

Unaided eye



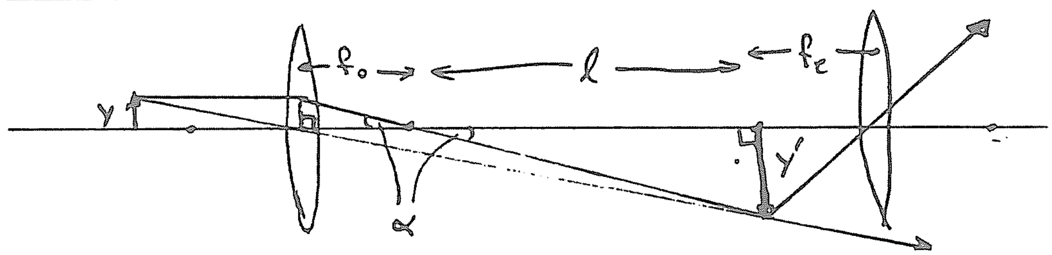
Eye with magnifier



$$M = \frac{\theta}{\theta_0} = \frac{25\text{ cm}}{f}$$

The simple magnifier lets one bring an object closer to the eye and see its image at infinity. One can thus view a nearby object with a relaxed eye. This increases the angle subtended by the image on the retina, i.e.-- magnifies it.

Microscope



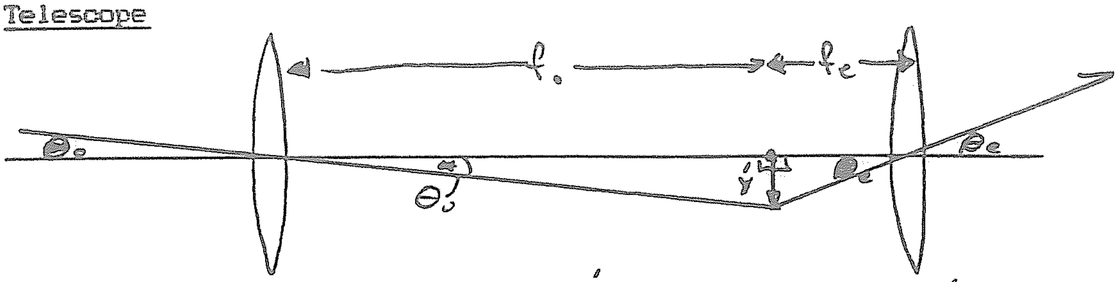
$$\tan\alpha = \frac{y}{f_0} = -\frac{y'}{\ell}$$

2 lenses, called objective and ocular (or eyepiece). The first creates a real image a distance $(\ell + f_o)$ from the lens, at the focal point of the eyepiece, which then acts as a simple magnifier to view the image. ℓ is called the "tube length" of the microscope. Its magnification is $m_o = -y'/y = -\ell/f_o$ for the objective times $M_e = 25/f_e$ for the eyepiece (acting as a simple magnifier). Its total magnification is

$$m = m_o M_e = -\frac{\ell \cdot 25}{f_o f_e}.$$

The image is inverted, virtual, and located at ∞.

Telescope



$$\theta_o \approx \tan\theta_o = \frac{y'}{f_o} \qquad \theta_e \approx \tan\theta_e = \frac{-y'}{f_e}$$

Also two lenses, but now object is close to ∞ ($s \gg f_o$ or f_e). As a result, the first lens forms real image at its focal point. The second lens (as before) acts as a simple magnifier to view the image of the first lens. The total (angular) magnification of the virtual image from the two-lens system is,

$$M = \frac{\theta_e}{\theta_o} = -\frac{f_o}{f_e}.$$

For exam, be able to do any and every lens/mirror ray diagram. Understand the eye, the simple magnifier, the microscope, the telescope, aberrations, the lensmaker formula and magnification. It's all very easy but you <u>must</u> practice to be able to do it correctly and quickly.

EDJ 34. Physical Optics.

Light is, after all, a wave in the electromagnetic field. If light is

a) Monochromatic (one frequency) and

b) Coherent (frequency constant in time over many cycles)

then we can observe the effects of interference and diffraction. Recall that these effects arise from the systematic addition, via the superposition principle for the electric field, of the waves from several coherent "sources".

---

1) Addition of harmonic waves

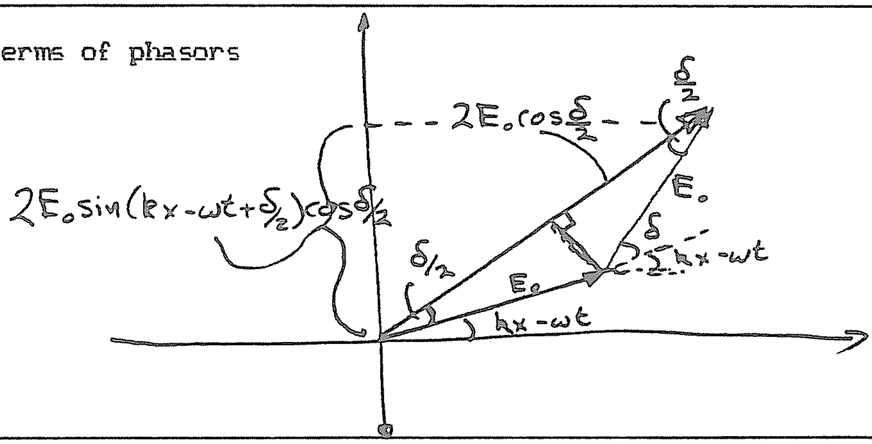
$$E(x,t) = E_0\sin(kx - \omega t) + E_0\sin(kx - \omega t + \delta)$$

$$E(x,t) = 2E_0\sin(kx - \omega t + \delta/2)\cos(\delta/2)$$

so (I_0 proportional to E_0^2)

$$I(x)_{av} = 4I_0\cos^2(\delta/2)$$

In terms of phasors



---

2) Sources of a phase shift.

a) Path difference (thin film, two, three, N slits)

b) phase shift of π on reflection off of a medium with larger n.

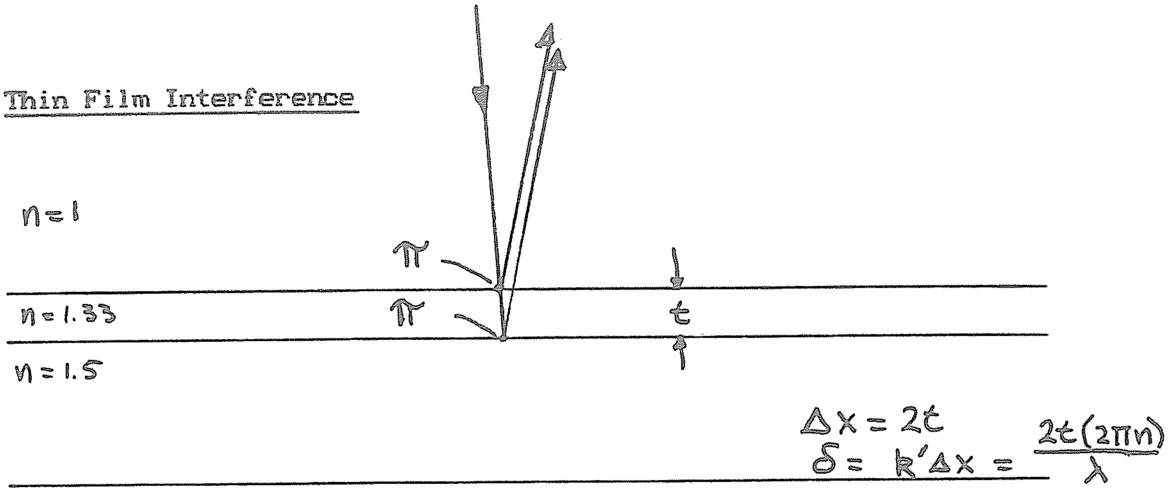
We must learn to discover when constructive and destructive interference occur. Constructive interference occurs when the waves from the two sources are in phase as they reach the point of observation. Destructive interference occurs when the waves are out of phase as the reach the point of observation. In terms of δ,

$\delta = 0, 2\pi, 4\pi, 6\pi$ (or $-2\pi, -4\pi,$). . . for constructive

$\delta = \pi, 3\pi, 5\pi$ (or $-\pi, -3\pi$). . . for destructive.

This is for the interference of two sources.

Thin Film Interference



There are two sources of phase shift in thin film interference. The FIRST is, every time light reflects from a denser (higher n) medium, its phase is shifted by π (the wave flips over). The SECOND is that the light that is reflected off of the second surface above travels a distance 2t farther. If x is the distance of the observer from the first surface, we are thus adding

$$E(x,t) = E_1\sin(kx - \omega t + \pi) + E_2\sin(kx + k'(2t) - \omega t + \pi).$$

We will assume that E_1 and E_2 are approximately equal. The phase shifts of π cancel. Thus

$$\delta = k'\Delta x = k'(2t),$$

which arises from the extra distance travelled in the second medium. (NOTE WELL! $k' = nk = \frac{2\pi n}{\lambda}$ is the wavenumber of the wave <u>in the medium</u>). We thus expect constructive interference (maxima in reflected intensity) when

$$\delta = 2t\frac{2\pi n}{\lambda} = 0, 2\pi, 4\pi, \ldots$$

or

$$2t = m\frac{\lambda}{n} \text{ (for } m = \ldots -2,-1,0,1,2 \ldots).$$

We would (similarly) get destructive interference when

$$\delta = 2t\frac{2\pi n}{\lambda} = \pi, 3\pi, 5\pi, \ldots$$

or

$$2t = (m + \frac{1}{2})\frac{\lambda}{n}.$$

These results can easily be understood. When the path difference (2t) contains an integral number of wavelengths ($m\lambda'$, in the medium) then the waves recombine in phase and add. When the path difference (2t) contains an odd-half-integral number of wavelengths, the waves recombine exactly out of phase and cancel.

Soap Bubble.

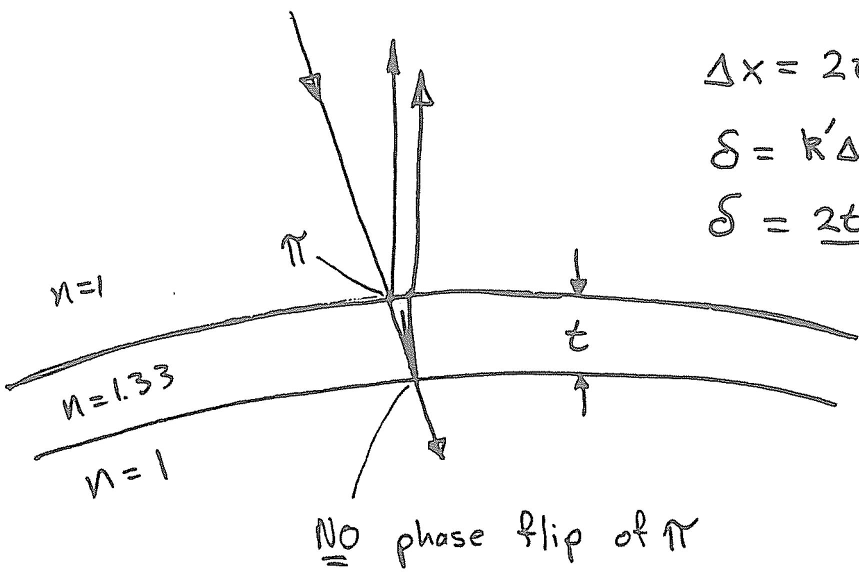
In the case of a soap bubble, there is a phase shift of π at the first surface but not at the second surface, so there is an overall phase difference of π added to the phase difference due to path difference. We then get

$$\delta = 2t\frac{2\pi\ n}{\lambda} + \pi, \text{ and}$$

$$2t = (m - \frac{1}{2})\frac{\lambda}{n} \quad \text{for constructive interference,}$$

$$2t = m\frac{\lambda}{n} \text{ for destructive interference.}$$

As a consequence, when the soap bubble gets very thin due to evaporation ($t \longrightarrow 0$) we get a reflection minimum in all wavelengths due to the phase shift of π at one surface, followed by a negligible path difference. Soap bubbles (as you have seen) get very clear right before they pop (light that isn't reflected is transmitted).



$$\Delta x = 2t$$

$$\delta = k'\Delta x + \pi$$

$$\delta = \frac{2t(2\pi n)}{\lambda} + \pi$$

$\delta = 0, \pm 2\pi, \pm 4\pi$ for constructive interference, or

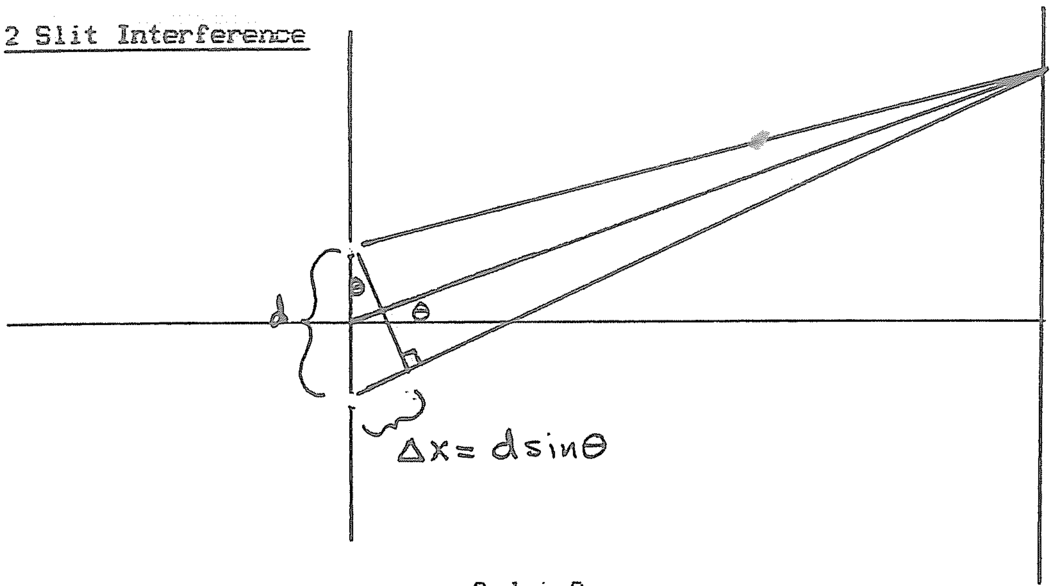
$$\frac{2t(2\pi n)}{\lambda} + \pi = 0, \pm 2\pi \ldots \text{etc.} \quad \text{or}$$

$$2t = \frac{\lambda}{n} \times \left[-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \cdots\right] \quad \text{or}$$

$$2t = (m - \tfrac{1}{2})\frac{\lambda}{n} \quad \text{for } m = 0, \pm 1, \pm 2 \ldots$$

EDJ 35. Physical Optics.

2 Slit Interference



$\Delta x = d\sin\theta$, so $\delta = k\Delta x = \dfrac{2\pi d\sin\theta}{\lambda} = m(2\pi)$ at maximum, so

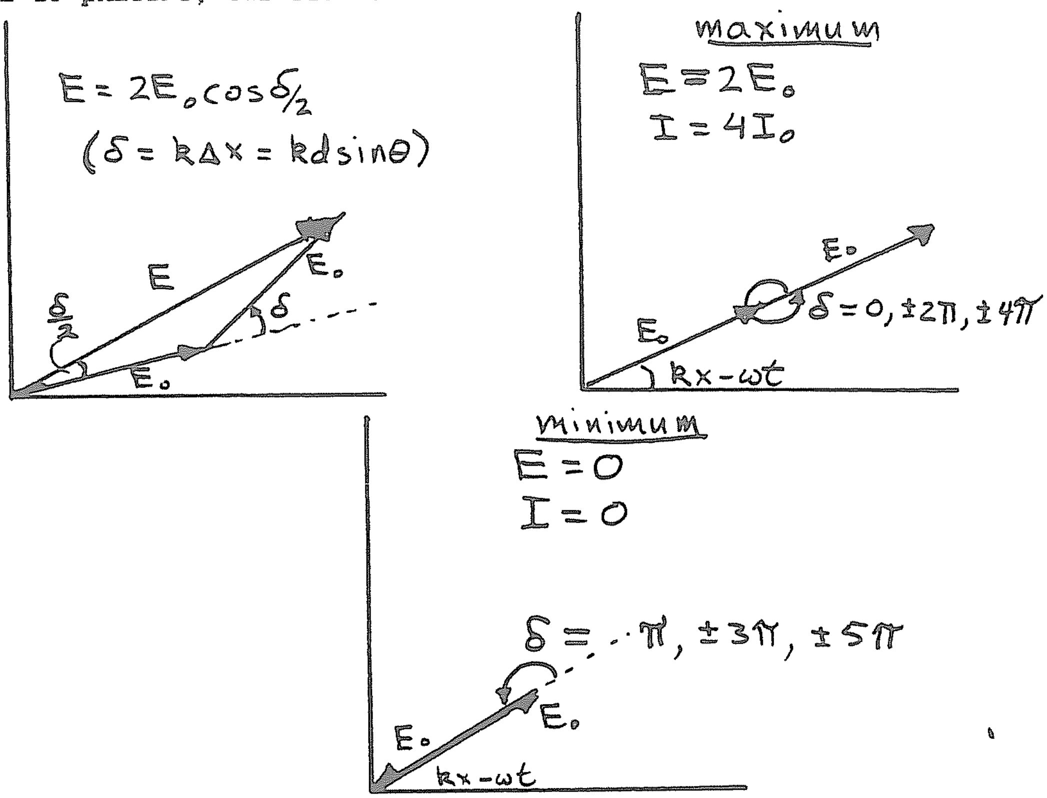
$$d\sin\theta = m\lambda \quad \text{(maximum)}$$

$$d\sin\theta = (m + \tfrac{1}{2})\lambda \quad \text{(minimum), and}$$

$$I = 4I_0\cos^2(\delta/2)$$

(from addition of two harmonic waves and I proportional to E^2).

In terms of phasors, two slit interference looks like:



Example Problem: Suppose $\lambda = 600$ nm, $d = 1800$ nm $= 3\lambda$. Then

$$d \sin\theta = m\lambda \quad \text{(maxima) or}$$

$$\theta_{max} = \sin^{-1}\left[\frac{m\lambda}{d}\right] = \sin^{-1}\left[\frac{m}{3}\right], \quad \text{for } m = 0, 1, 2, 3.$$

Note that m cannot be larger than 3 in this case because θ cannot be larger than 90°! Similarly,

$$d \sin\theta = (m + \frac{1}{2})\lambda \quad \text{(minima) or}$$

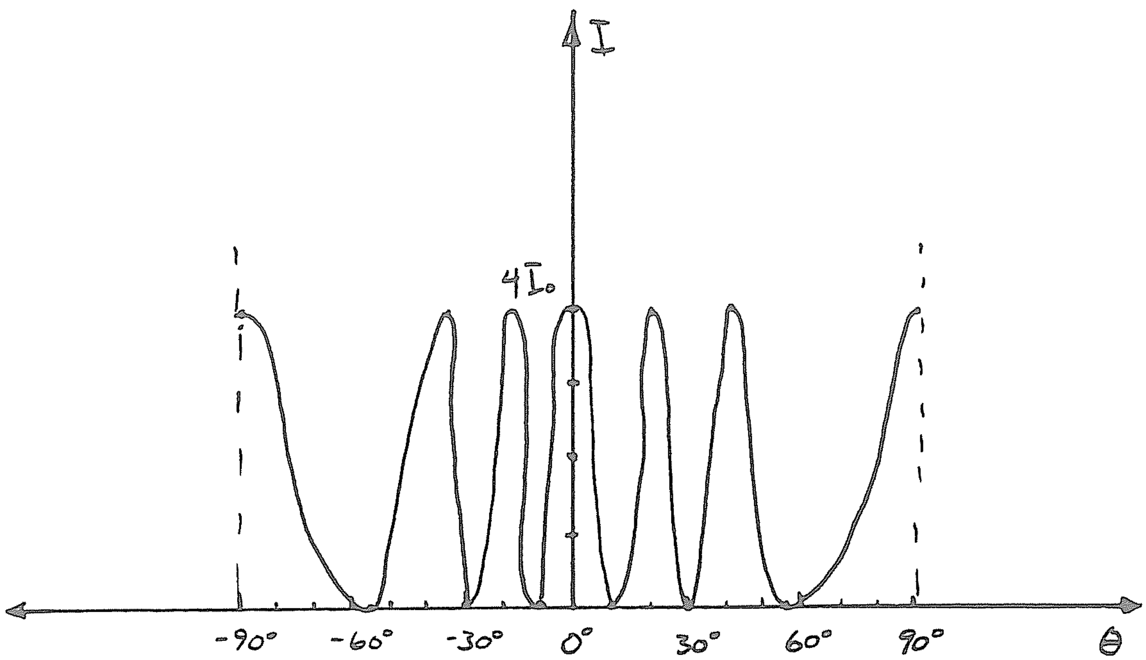
$$\theta_{min} = \sin^{-1}\left[\frac{m + \frac{1}{2}}{3}\right] \quad \text{for } m = 0, 1, 2.$$

We explicitly get

$$\theta_{max} = 0^\circ, 19.47^\circ, 41.81^\circ, 90^\circ$$

and $$\theta_{min} = 9.59^\circ, 30^\circ, 56.44^\circ.$$

The interference pattern is semi-quantitatively sketched below. Recall that the "exact" solution is $I = 4I_0 \cos^2(\delta/2)$



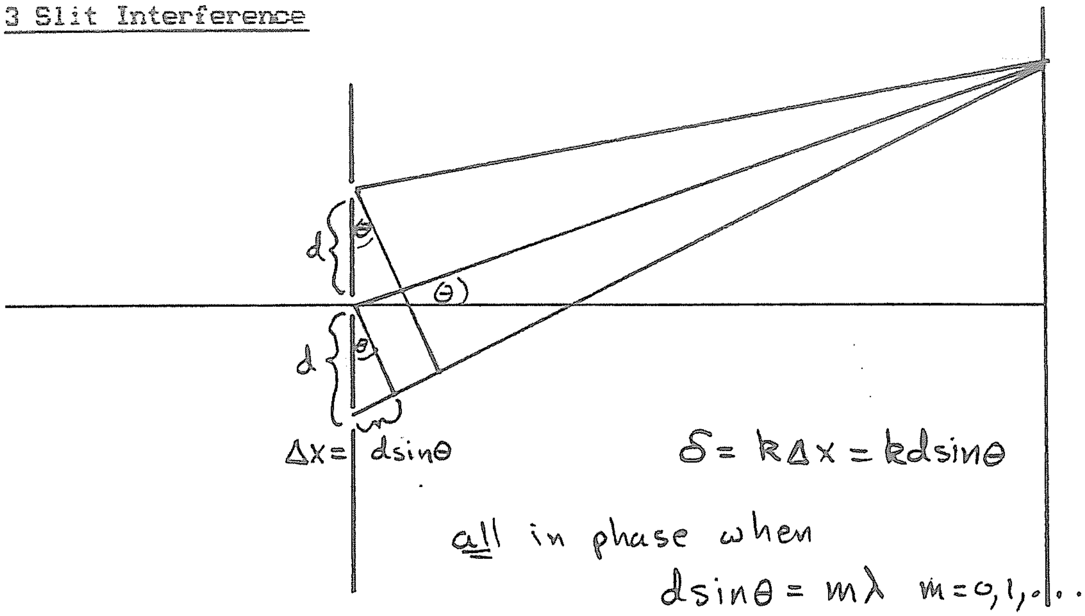
6 minima

7 maxima

$I_{max} = 4I_0$

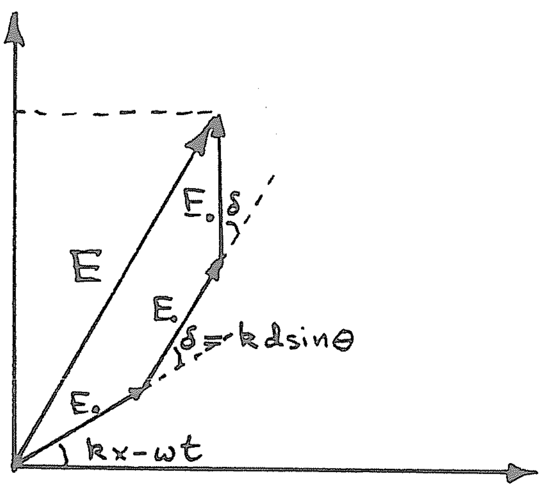
$I_{min} = 0$

3 Slit Interference



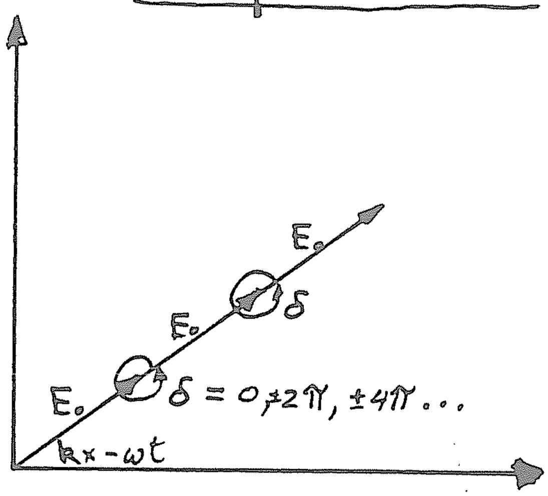
Now we can't (easily) use trig identities to obtain the desired answers, which are: Where do interference maxima occur? How bright are they (relative to the intensity of a single slit)? Where do the minima occur? What does the interference pattern look like, qualitatively? But we can use the phasors to answer these questions. Thats why we used them for two slits, where we didn't really need them.

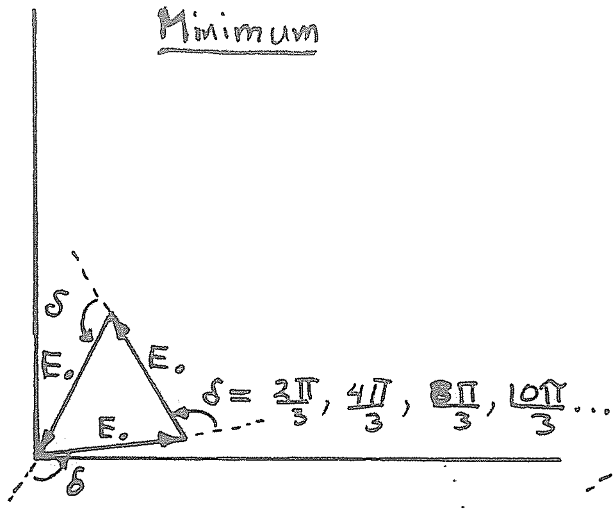
Phasor Diagrams for Three Slits

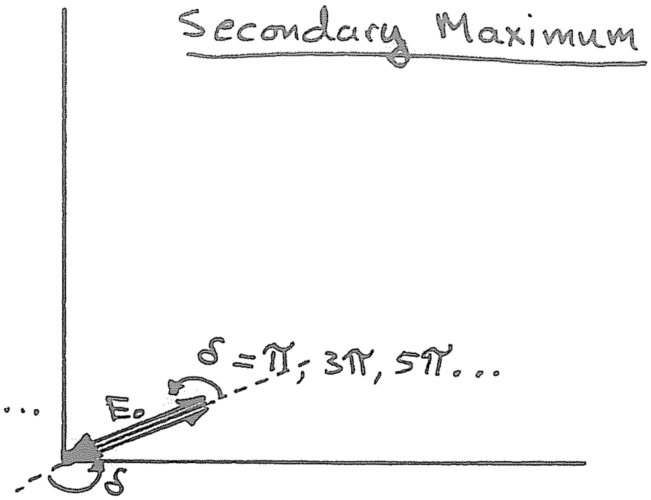


General δ

Principal Maximum







We can summarize the results of these diagrams as follows: We get a (principal) maximum when

$$d\sin\theta = m\lambda \quad (m = 0, 1, 2, \ldots).$$

We get a secondary maximum when

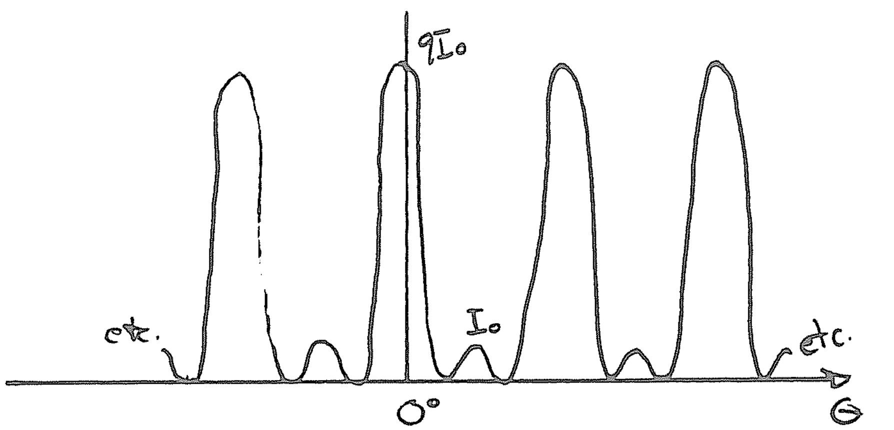
$$d\sin\theta = \frac{m}{2}\lambda \quad (m \text{ not multiple of } 2).$$

We get a minimum when

$$d\sin\theta = \frac{m}{3}\lambda \quad (m \text{ not multiple of } 3).$$

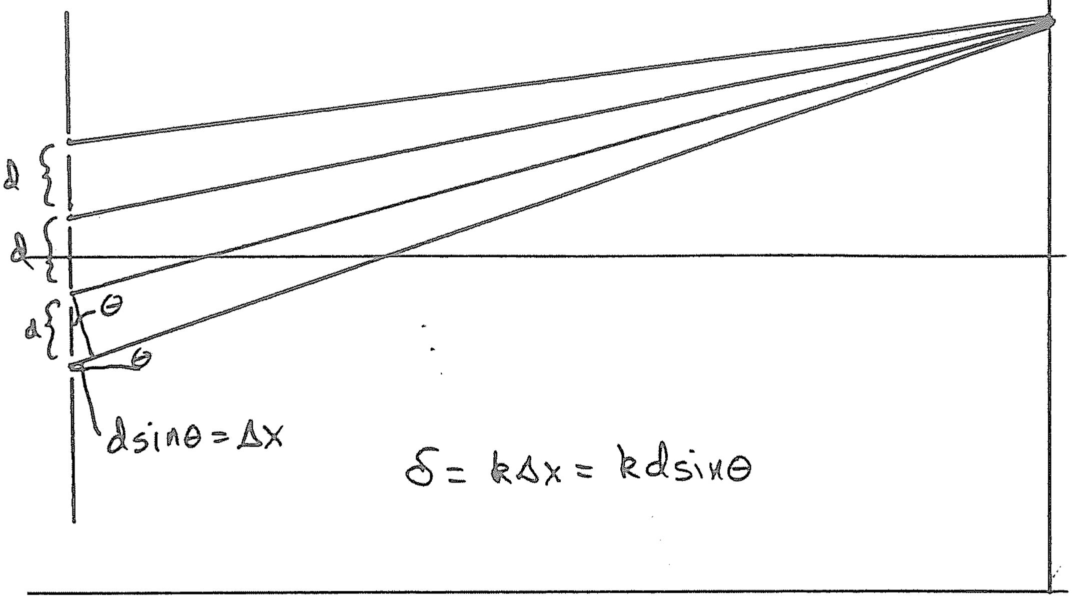
The Interference pattern (as a function of θ) looks like:

$I_{max} = \underline{9} I_0$

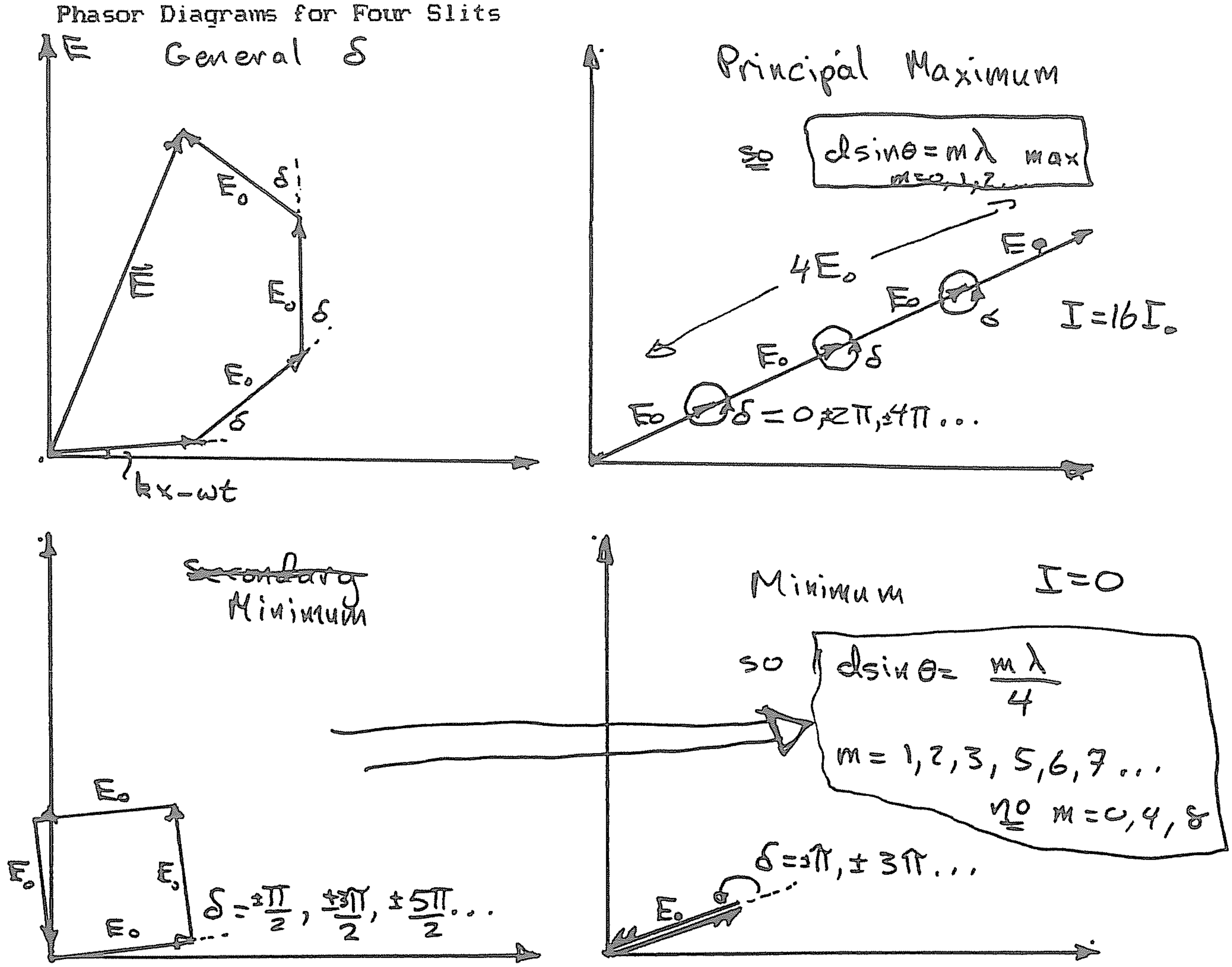


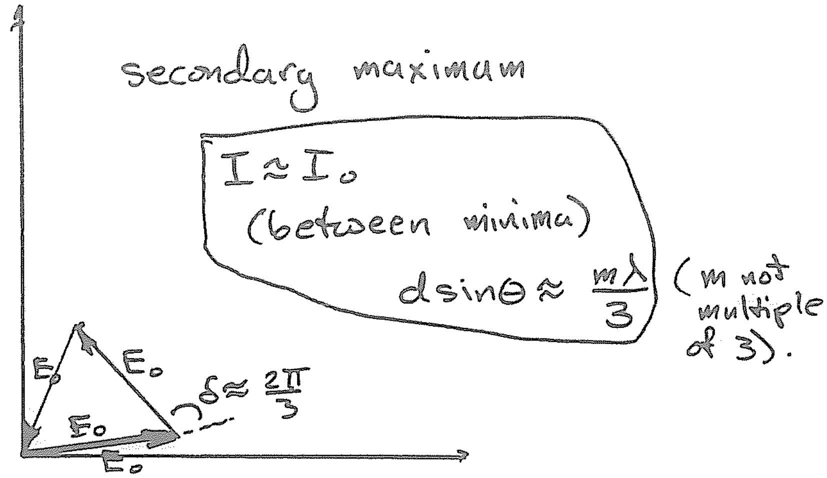
secondary Maximum height I_0

4 Slit Interference



Phasor Diagrams for Four Slits





We can summarize the results of these diagrams as follows: We get a (principal) maximum when

$$d\sin\theta = m\lambda \quad (m = 0, 1, 2, \ldots).$$

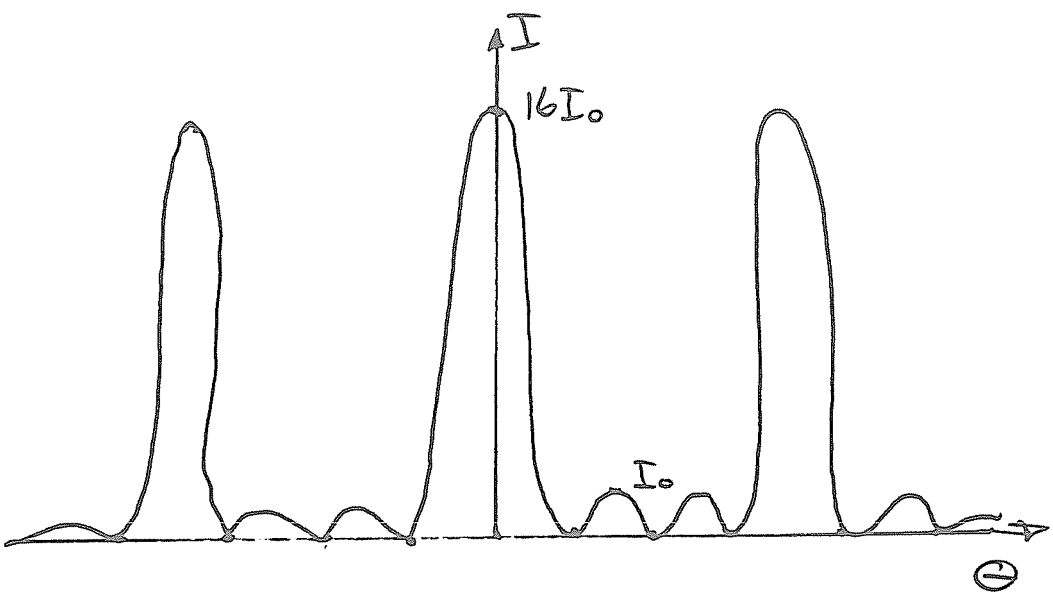
We get a secondary maximum when

$$d\sin\theta \approx \frac{m}{3}\lambda \quad (m \text{ not multiple of } 3).$$

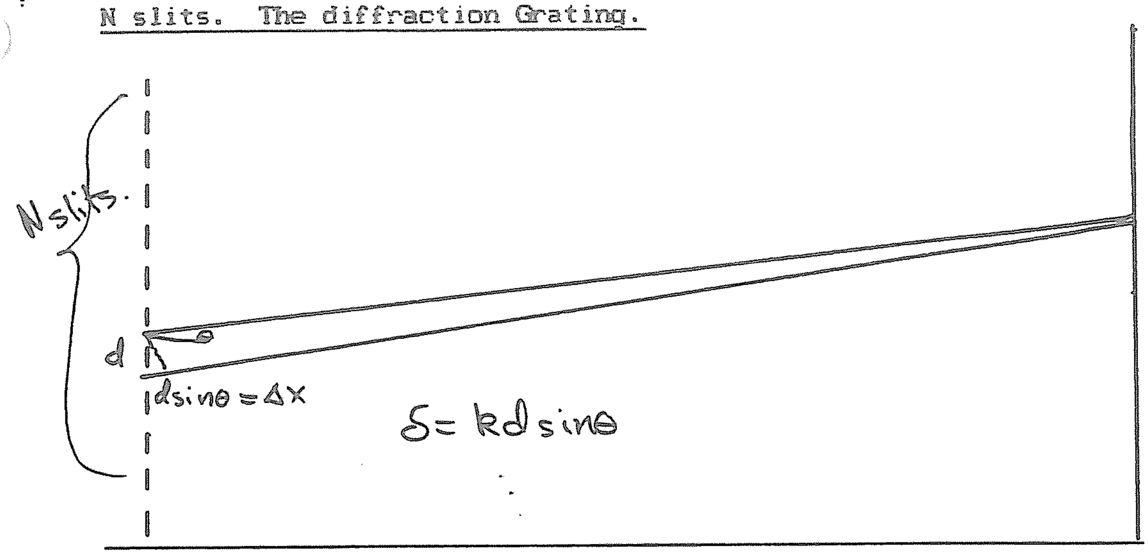
We get a minimum when

$$d\sin\theta = \frac{m}{4}\lambda \quad (m \text{ not multiple of } 4).$$

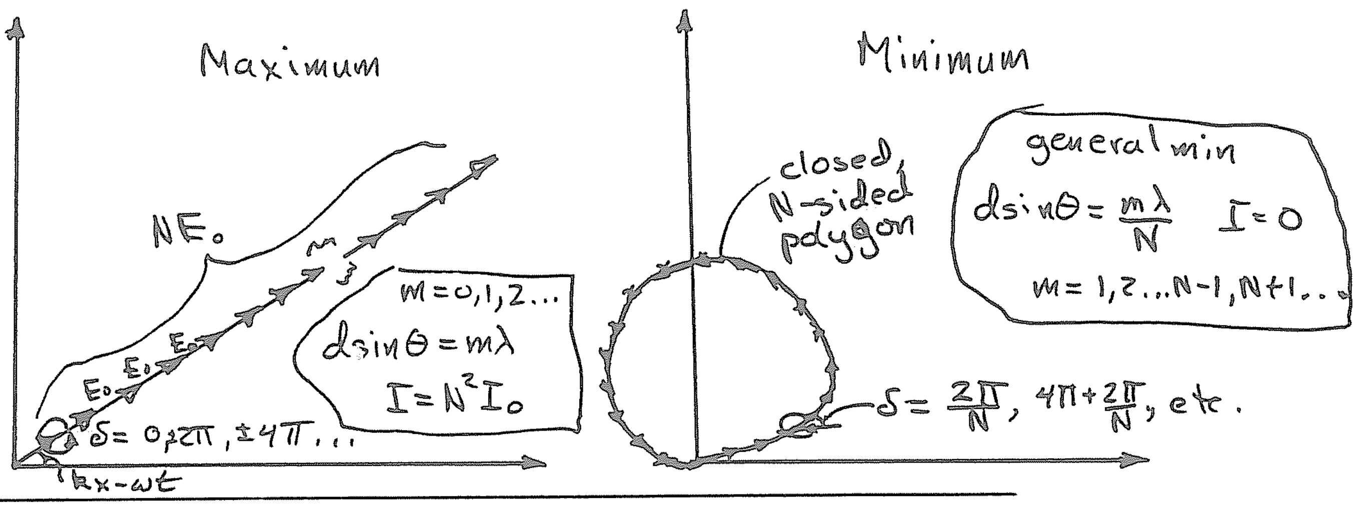
The Interference pattern (as a function of θ) looks like:



N slits. The diffraction Grating.



Phasor Diagram for Pricipal Maximum and First Minimum



From this we see that we get (principal) maxima when

$$d \sin\theta = m\lambda \quad (m = 0, 1, 2 . . .)$$

as usual. We get the first minimum when

$$d \sin\theta = \frac{\lambda}{N}, \text{ or}$$

$$\Delta\theta \sim \sin\theta = \frac{1}{N}\frac{\lambda}{d}.$$

This is the "angular width" of the central maximum. In general the width of a maximum is about 1/N times the separation of the orders, which goes like λ/d. We can thus resolve two closely separated wavelengths in the mth order if

$$\theta_1 = m\frac{\lambda_1}{d}$$

$$\theta_2 = m\frac{\lambda_2}{d}, \quad \text{so}$$

$$\Delta\theta = \theta_2 - \theta_1 = m\frac{\Delta\lambda}{d}$$

is greater than the angular separation of the first minimum past the mth order:

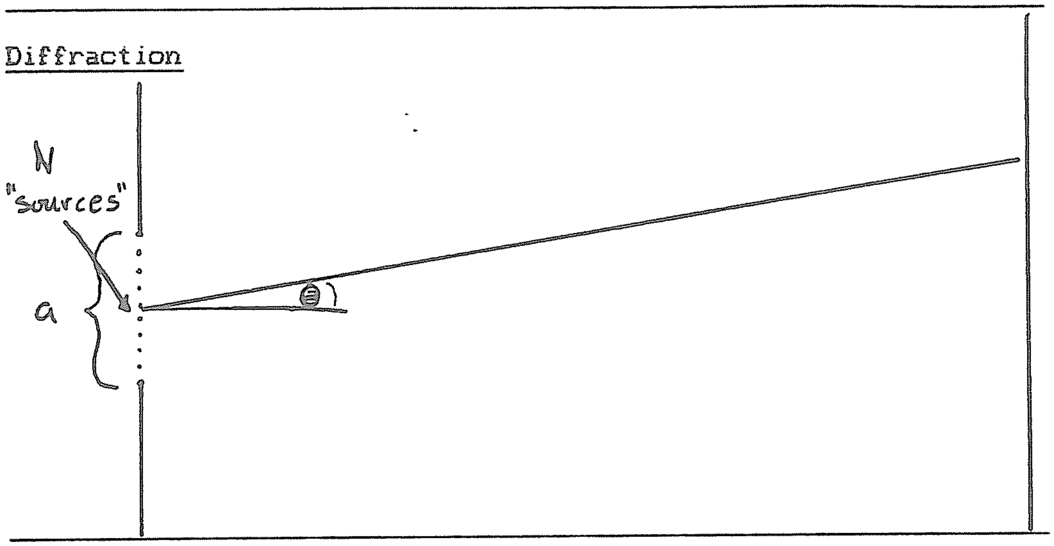
$$m\frac{\Delta\lambda}{d} = \frac{1}{N}\frac{\lambda}{d}.$$

The resolving power of a diffraction grating is a measure of its ability to resolve closely separated lines. We define it to be

$$R = \frac{\lambda}{|\Delta\lambda|} = mN.$$

(Recall the definition of Q in driven oscillators, which serves a similar function).

Diffraction



We assume that the slit is made up of N "point sources" of the field. In this respect the problem greatly resembles a diffraction grating. BUT, while we clearly get a maximum when $\theta = 0$, the conditions for minima and secondary maxima are quite different.

Consider the wave originating from the top of the slit and interfering only with the wave originating from the middle. If we treat these as a two slit problem, we expect cancellation to occur when

$$\frac{a}{2}\sin\theta = (m + \frac{1}{2})\lambda \quad \text{or}$$

$$a\sin\theta = (1, 3, 5, \ldots)\lambda.$$

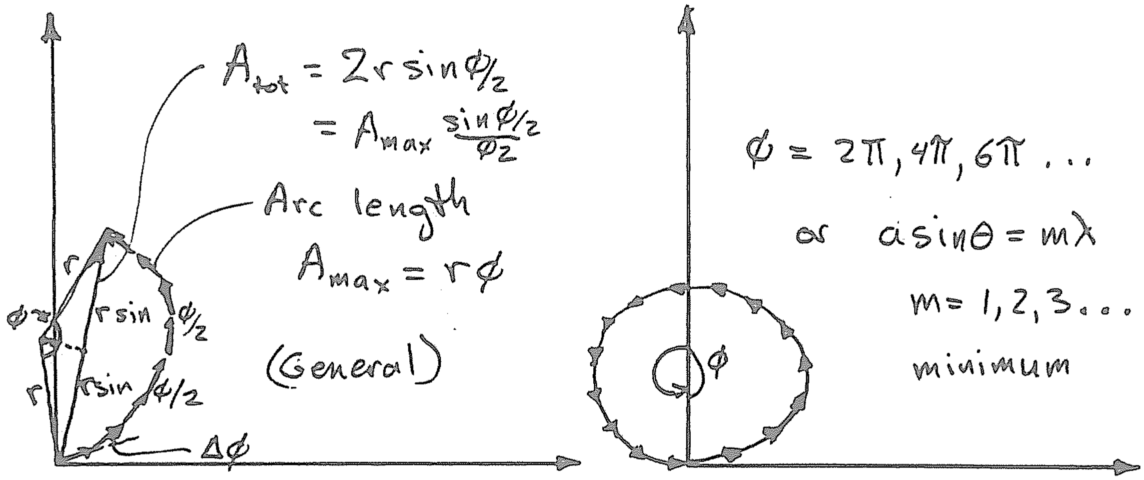
Similarly, the interference of waves from the top and a/4 result in

$$a\sin\theta = (2, 6, 10 \ldots)\lambda.$$

Proceeding in this way, we expect to get diffraction minima when

$$a\sin\theta = m\lambda, \quad m = 1,2,3,\ldots \text{ (NOT } m = 0!).$$

This is not very rigorous, and does not predict where maxima should occur. To do that, we have to resort to a Phasor diagram

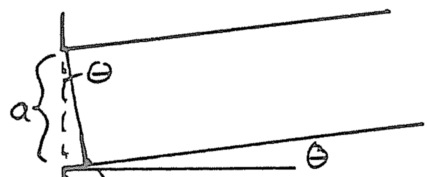


From the first one, if we break the slit up into N pieces, the phase difference between any two adjacent sources is

$$\Delta\phi = k\frac{a}{N}\sin\theta$$

so that the phase difference between the top and the bottom of the slit is

$$\phi = k\ a\ \sin\theta.$$



$$\Delta x = a\sin\theta \Rightarrow \phi = k\Delta x = N\Delta\phi = ka\sin\theta$$

Adding up N sources like we do above, and letting N go to ∞ while we keep a constant, the polygon drawn above goes smoothly to a circular arc and $\Delta\phi$ goes to $d\phi$. The geometry is unaffected and

$$\sin\frac{\phi}{2} = \frac{A}{2}\frac{1}{r}, \text{ or}$$

$$A = 2r\ \sin(\phi/2).$$

But the length of the arc is conserved (and equal to A_{max}) so that

$$\phi = \frac{A_{max}}{r} \text{ or}$$

$$r = \frac{A_{max}}{\phi}.$$

Substituting this in the equation for A above, we get

$$A = A_{max} \frac{\sin(\phi/2)}{\phi/2}.$$

Since I_0 is proportional to A^2_{max},

$$I = I_0\left[\frac{\sin(\phi/2)}{\phi/2}\right]^2 \text{ in terms of}$$

$$\phi = \frac{2\pi}{\lambda} a \sin\theta.$$

Clearly $\sin(\phi/2)$ will equal 0 and a minimum will obtain when

$$\frac{\phi}{2} = \pi, 2\pi, 3\pi. . . \text{ or}$$

$$a \sin\theta = m\lambda \text{ (as before).}$$

But, a maximum occurs at ϕ such that $\frac{dI}{d\phi} = 0$ (and not a minimum). This expression requires the solution of a trancendental equation and hence you are not responsible for it, though I might ask you to find the trancedental equation to be solved as part of an optional problem on the final.

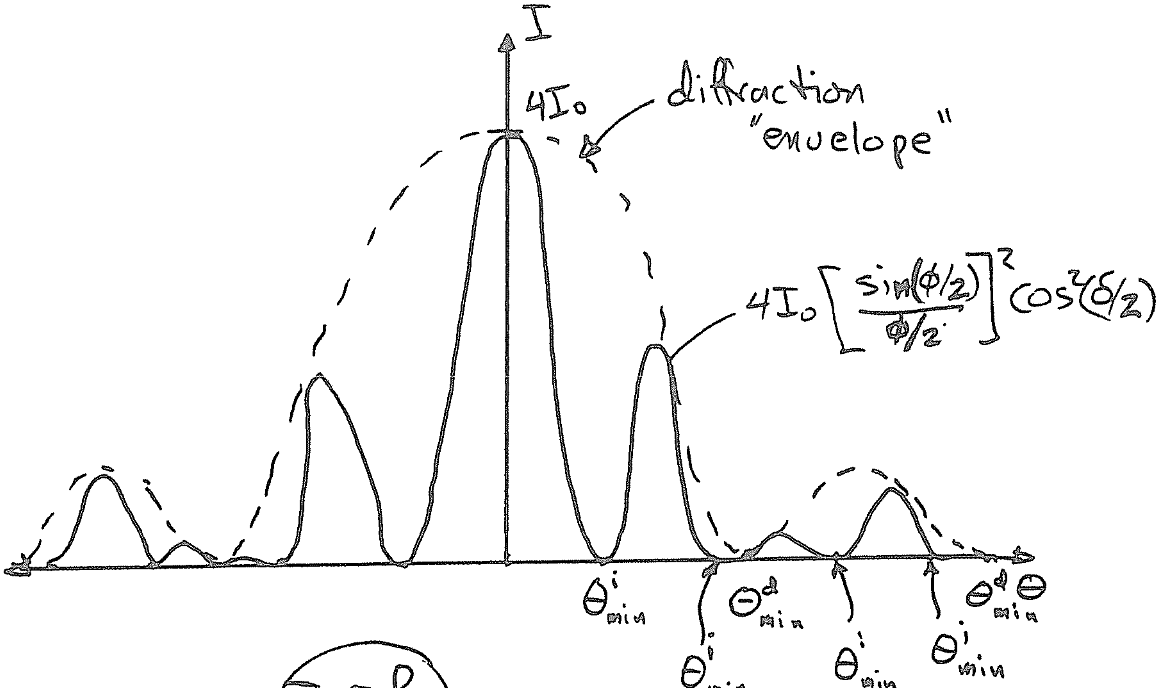
Interference and Diffraction from two slits of width a, separation d. We <u>combine</u> the expressions for the intensities by multiplying them,

$$I = 4I_0\left[\frac{\sin(\phi/2)}{\phi/2}\right]^2 \cos^2(\delta/2),$$

with $\delta = kd\sin\theta$

and $\phi = ka\sin\theta$.

The result looks qualitatively like:



Resolution

We define two sources that diffract through the same slit to be resolved if the central maximum from one source is at least at the minimum of the other. This is the Rayleigh criterion for resolution. For circular apertures, the first minimum occurs when

$$\sin \theta = 1.22\frac{\lambda}{D},$$

where D is the diameter of the opening. Thus if

$$\alpha \geq 1.22\frac{\lambda}{D}$$

(with α the angular separation of the two sources) then they are resolved.

Summary: We now are in a position to understand when (and why!) waves behave like particles and do not "bend" around corners and cast shadows. Diffraction occurs on a spatial scale commensurate with the wavelength of the light used. When the scales we are interested in are on the order of cm or m, the diffraction and interference effects of visible light are negligible. But, when we are looking at objects within a cell that are nanometers or less in size, visible light (after passing through the circular aperture of the lens) cannot resolve the details. Similarly, radar waves ($\lambda \sim 1$cm) are fine for detecting objects several meters in size and up, but are useless for detecting and resolving, say, bullets or coins or other objects around a centimeter or less in size. X-rays could be used to look at cellular structure, but unfortunately we can't make useful lenses for X-rays and hence build microscopes (they also are too energetic and can heat up the objects being studied enough to alter their structure).

Real matter doesn't really behave like particles either. It turns out that it actually behaves like a wave just like light (sort of) but its wavelength tend to be so short that we can treat it like a particle. Newtons laws and classical mechanics in general is actually the "geometrical optics" of quantum wave mechanics just like geometrical optics is the small wavelength approximation to physical wave optics.

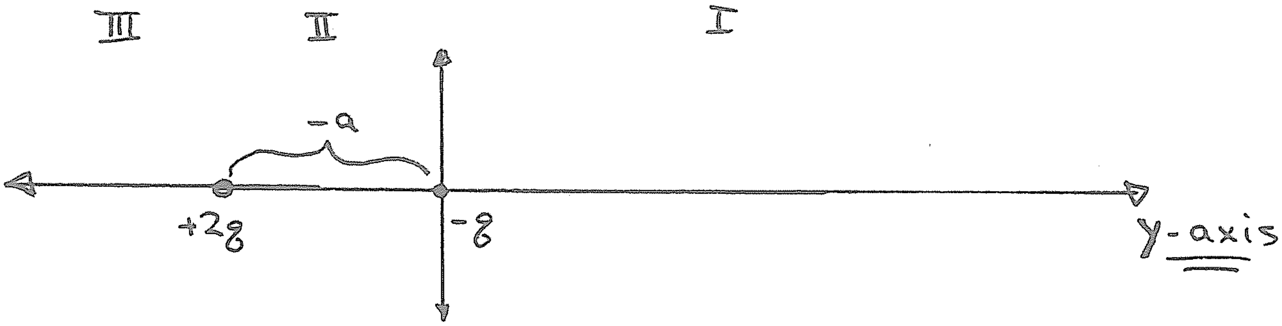
For that reason we use electron microscopes to examine small (~1 Å) objects, because electrons have very short wavelengths and can be focussed by electrostatic "lenses". At this time, however, the only probes we have into structures on the order of the size of an atom or a molecule tend to be destructive, and so one tends to destroy or alter that which one wishes to measure. This is the manifestation of an important physical principle, called the Heisenberg Uncertainty Principle, that expresses some of the fundamental limitations on what we can expect to learn from nature.

Quiz 1

Physics 52.1 & 52.2 — 9/(2-3)/85

The Electric Field — R. G. Brown

(10 points) In the picture below, a charge of +2q is located at y = -a, and a charge of -q is located at y = 0.



a) (6 points) Find the electric field (magnitude and direction) at an arbitrary point on the y-axis. (You will need to do three regions: y>0, 0>y>-a, -a>y.)

b) (3 points) Find a spot on the y-axis where the force on a charge placed there would be zero. (Hint: Where is the field zero?) Extra credit: is this position a stable equilibrium point? Why or why not.

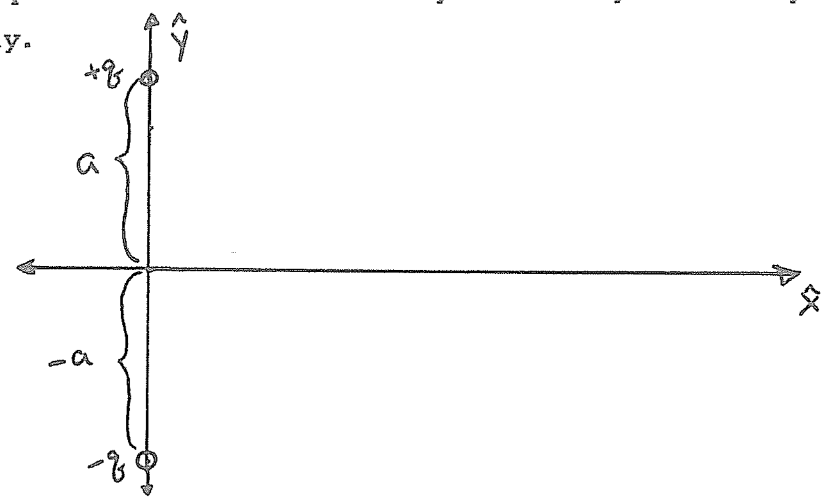
c) (1 point) What is the magnitude and direction of the field when q = 1μC, y = 10 cm, x = 0 cm, and a = 10 cm.

Quiz 1

Physics 52.1 & 52.2 9/(2-3)/85

The Electric Field R. G. Brown

(10 points) In the picture below, two equal and opposite charges (+q and −q) are located on the y-axis at y = +a and y = −a, respectively.



a) (6 points) Find the electric field at an arbitrary point on the x-axis (magnitude and direction).

b) (3 points) Show that the electric field on the x-axis has the form $\vec{E} = \frac{-k\vec{p}}{x^3}$ when x>>a. (recall $\vec{p} = q\vec{\ell} = q(2a\hat{y})$.)

c) (1 point) What is the magnitude of the electric field if q = 1 μC, a = 30 cm, x = 40 cm, y = 0?

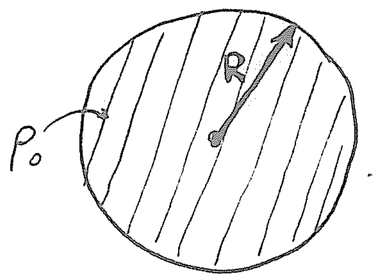
Quiz 1

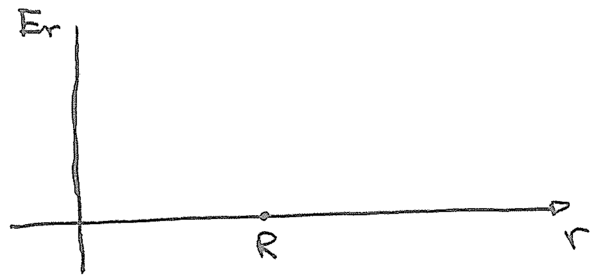
Physics 52.1 & 52.2 — 9/(16-17)/85

Gauss' Law and Potential — R. G. Brown

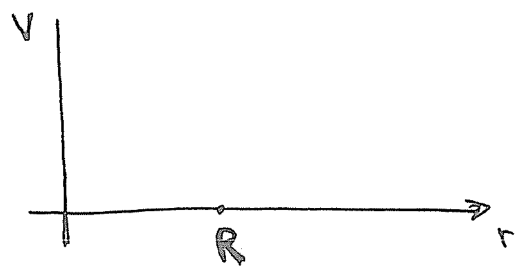
---

a) (10 points) Find the electric field for a solid sphere of radius R and with a uniform charge density ρ_0 C/m^3 at all points in space. Graph (roughly) your answer.





b) (10 points) Using your answer(s) to part a), find the electric potential (relative to ∞) at all points in space. Graph (roughly) your answer.



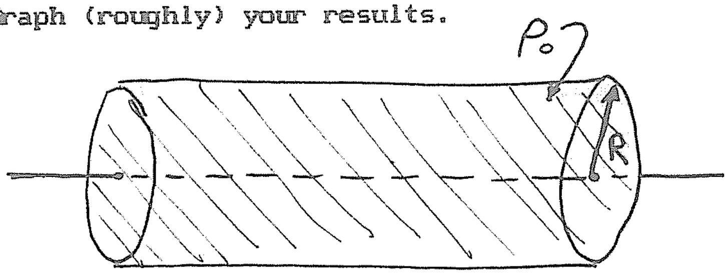
Quiz 1

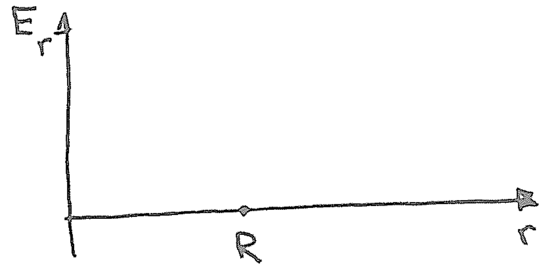
Physics 52.1 & 52.2 9/(16-17)/85

Gauss' Law and Electric Potential R. G. Brown

---

a) (10 points) A solid cylindrical shell of "infinite" length and radius R has a uniform charge density of ρ_0 C/m^3 distributed throughout it. Find the electric field at all points in space. Graph (roughly) your results.





b) (10 points) Find the potential difference between the surface of the cylinder (r = R) and all other points in space (r inside and outside the cylinder).

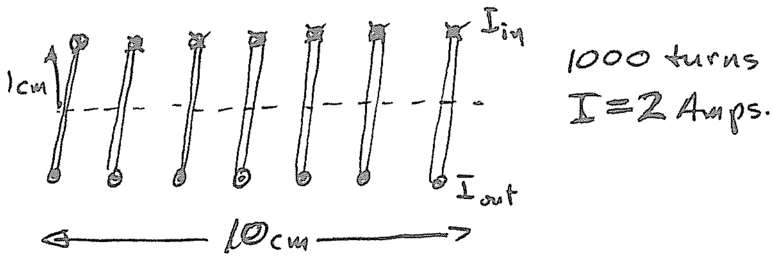
Quiz 6

Physics 52.1 & 52.2 — 10/(21-22)/85

DC-Circuits — R. G. Brown

---

a) (5 points) Use Ampere's law and find the magnetic field inside a circular solenoid 10 cm. long and 1 cm in radius, containing 1000 turns and carrying a current of 2 Amps. Express your units in Gauss (recall 1 Tesla = 10^4 Gauss). And you thought I'd never give you numbers, didn't you.



b) (5 points) Find the total magnetic flux through the solenoid and its self-inductance (in Henries).

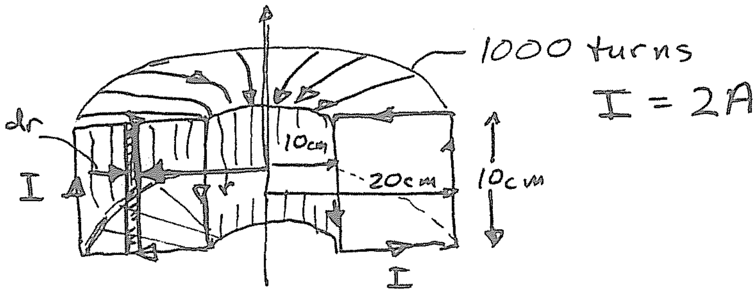
Quiz 6

Physics 52.1 & 52.2 10/(21-22)/85

DC-Circuits R. G. Brown

---

a) (5 points) Use Ampere's law and find the magnetic field as a function of r inside a toroidal solenoid with a rectangular cross-section. Its inner radius is 10 cm., its outer radius is 20 cm., and it is 10 cm. high. It has 1000 turns and is carrying a current of 2 Amps. Express your units in Gauss (recall 1 Tesla = 10^4 Gauss). And you thought I'd never give you numbers, didn't you.



cut-away view.

b) (5 points) Find the total magnetic flux through the solenoid and its self-inductance (in Henries). (hint: consider flux through strip of width dr drawn)

Hour Exam 1
Physics 52.1/52.2

Name_______________
R. G. Brown, Inst.

There are five 2 point "True Facts" questions that can be answered either with a formula or a short written answer. Following that are three 10 point short problems. There is a 35 point question exploring your understanding of the electric field, electric potential, capacitance, etc. as discussed extensively in class and recitation. Finally, choose one of two 25 point questions for

10 + 30 + 35 + 25 = 100 points.

Use good testpersonship. Relax. Leave no question blank; sometimes even a picture is worth a few points. Do the ones you are best prepared for first.

==

SHOW YOUR WORK!

1) True Facts. (2 pts. each)

i) What is Coulomb's Law?

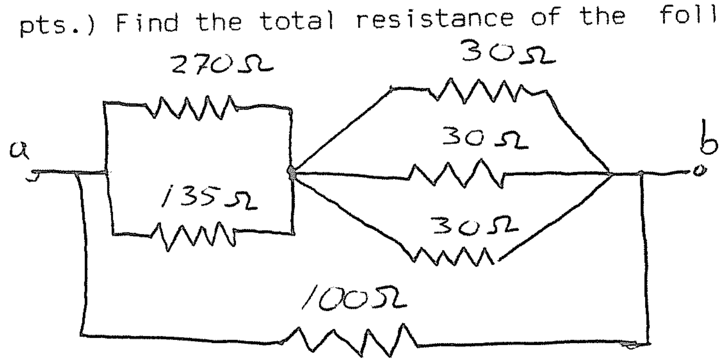
ii) What is Gauss' Law?

iii) What are Kirchoff's Rules?
a)

iv) b)

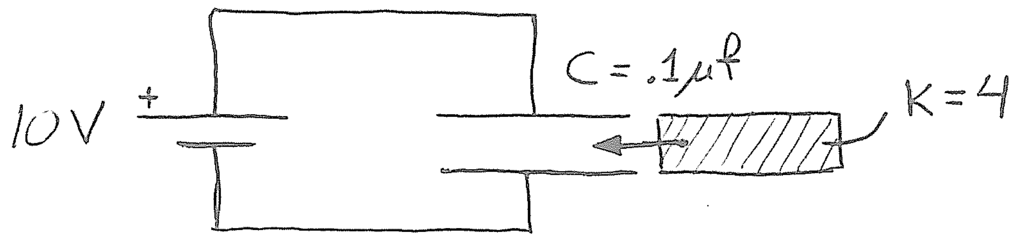
v) Current in a metal wire flows (circle one) a) in the same direction the electrons move. b) in the opposite direction the electrons move. c) perpindicular to the direction the electrons move.

2) (10 pts.) Find the total resistance of the following resistance network.



$R_{ab} =$ ______

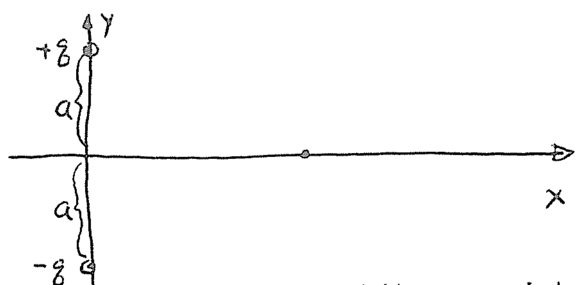
3) (10 pts.) Below is a parallel plate capacitor with C = 0.1 microfarads when the space between the plates is empty. It is connected to a 10V battery as shown and then a dielectric slab with dielectric constant K=4 is inserted that completely fills the space between the plates. The battery is NOT disconnected while this occurs.



a) How much additional charge flows onto (or off of) the capacitor as the dielectric is inserted? Note that to answer this, one must know how much was there to start with.

b) What is the final energy stored in the electric field of the capacitor and where did it come from (i.e.--what did the work of moving charge around)?

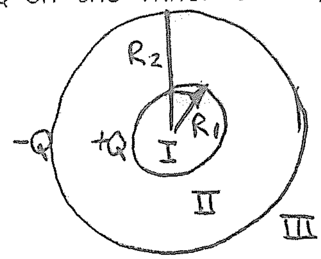
4) (10 points)



a) Find the Electric field at an arbitrary point on the x-axis. (Magnitude and Direction).

b) Find the first two terms in the binomial expansion of the field when x>>a, in terms of $\vec{p}$, the dipole moment.

5) (35 points!) Everything you ever learned about Electric field in one painless question. Below is an (easy) spherical capacitor with a charge +Q on the inner shell, -Q on the outer shell.



a) Use Gauss' law to find the field at all points in space. (regions I, II, III)

b) Use the definition of electric potential to find the potential at all points in space, relative to zero potential at infinity.

c) From the potential difference of the shells, find the capacitance of the spheres.

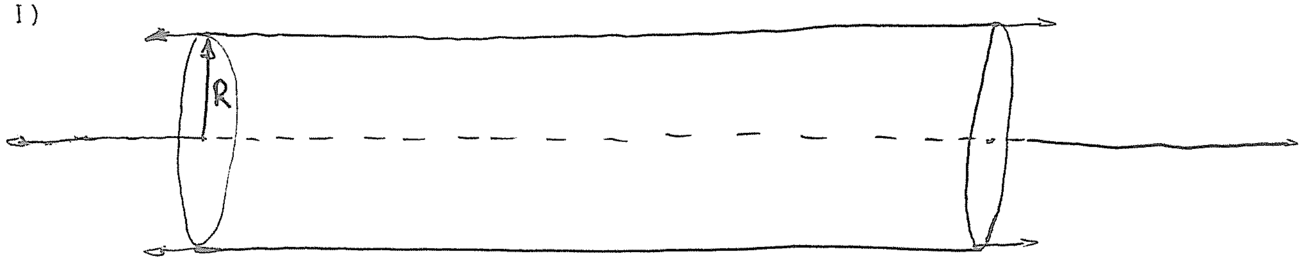
d) Now tell me the energy density, as a function of r, throughout the region between the spheres. (II)

e) And finally, integrate your answer to d) to find the total energy stored in the capacitor. If you can't do that, at <u>least</u> tell me what the energy stored on the capacitor is.

6) (25 points) Choose one of the following two questions and answer it completely. Yes, we'll look at the other one for partial credit, but only if your total score (on the rest of the test) is below 50. Trying both of them may keep you from failing, but otherwise don't waste your time.

==

I)



Above is a solid insulating cylinder with a charge distribution of $\rho(r) = 3r$ (charge/unit volume).

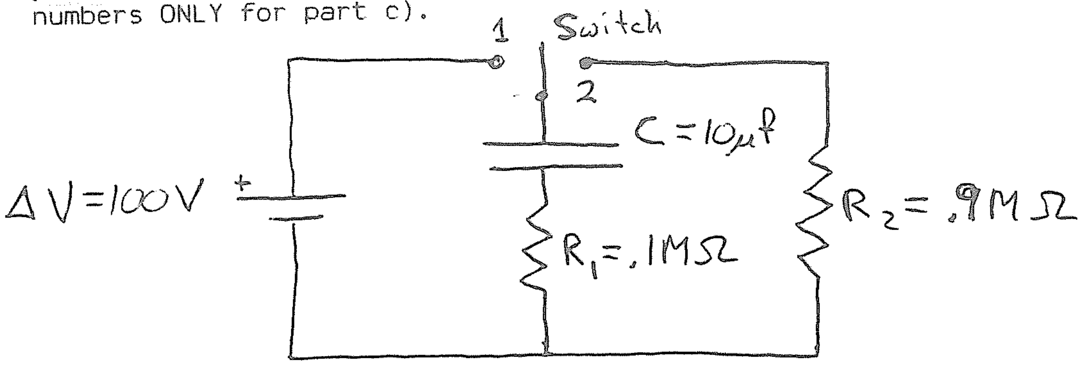
a) Use Gauss' law to find the Electric field at all points in space.
(inside cylinder)

(outside cylinder)

b) Find the electric potential at all points in space, relative to zero potential on the axis of the sphere. Note that in this case the potential on the axis is well defined.
(inside cylinder)

(outside cylinder)

II) Below is an RC circuit. At time t=0 the switch is moved to postition 1 and current flows, charging the capacitor. $C = 10\mu f$ (microfarads), $V = 100$ volts, $R_1 = .1\ M\Omega$, $R_2 = .9\ M\Omega$. (Use these numbers ONLY for part c).



a) Use Kirchoff's rules to find the charge on the capacitor as a function of time. (Find the differential equation and show that your answer is a solution. Hint: Find I(t), Q(t) and substitute them into the D. E. you obtain.)

b) The switch is moved to position 2 when the charge on the capacitor is $Q_0 = 1\mu C$. Find the energy stored on the capacitor as a function of time as it discharges. Where does the energy go? (Don't derive it, just tell me answer.)

c) How long does it take for the charge on the capacitor to reduce to 10% of its value at the time the switch is moved?

Good Luck. . .

Hour Exam 2
Physics 52.1 & 52.2
R. Brown, Instructor

Name:__________________
November 9, 1984

Instructions: Below are seven questions. You MUST answer the first five (worth 5,10,20,20,20 points respectively). Then answer ONE of the two optional questions (worth 25 points).

As usual, use good testpersonship. Answer the familiar ones and the "easy" ones first, to warm up, and gradually tackle the harder ones. Budget your time and make sure you at least get something down for each problem. Remember, partial credit is (generously) awarded!

1. (5 points)
Please write down Maxwell's equations, in integral form. Below each one, state in a sentence or two (or even a phrase!) what the equation means. No essays, Please!

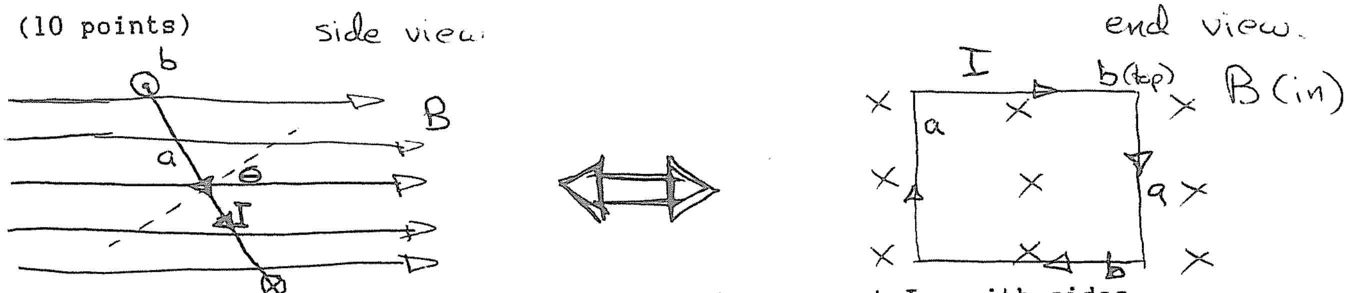
a)

b)

c)

d)

2. (10 points)



Above is a rectangular current loop carrying current I, with sides a and b, in a uniform magnetic field B to the right as shown. Find:
a) the net force on the loop. Don't work too hard at it.

b) the magnetic moment of the loop. (show direction on diagram above).

c) the net torque on the loop (magnitude and direction).

This page is extra workspace. Be sure to clearly indicate which problem any work here belongs to, and note it at the problem also for Dennis to find.

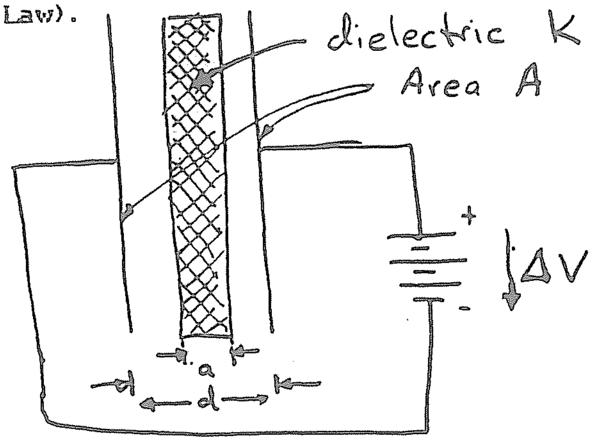
Good Luck, and RELAX! It's ONLY a PHYSICS TEST. . . Rob.

Quiz 4

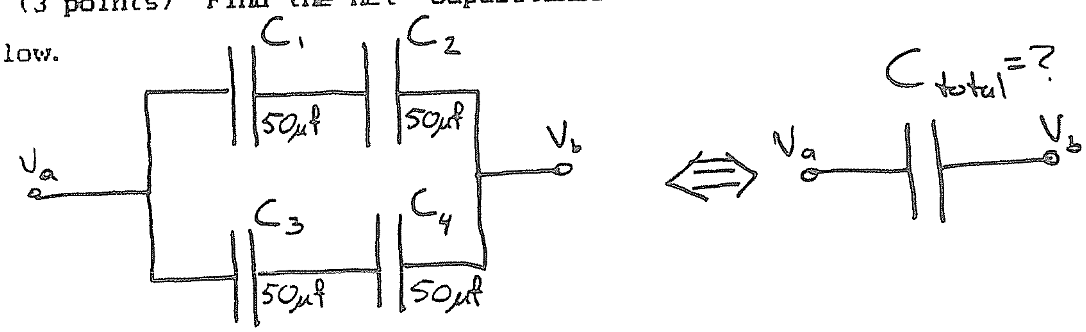
Physics 52.1 & 52.2 — 9/(23-24)/85

Capacitance — R. G. Brown

(7 points) a) Below is pictured a parallel plate capacitor with cross sectional area A and total separation d. A dielectric slab of width a<d and dielectric constant K has been inserted as shown. Find the capacitance of the arrangement. (show all work. Start from Gauss' Law).



b) (3 points) Find the net capacitance of the the arrangement below.



C_{total} = ____________

$C_1 = 50\mu f$ $C_2 = 50\mu f$ $C_3 = 50\mu f$ $C_4 = 50\mu f$

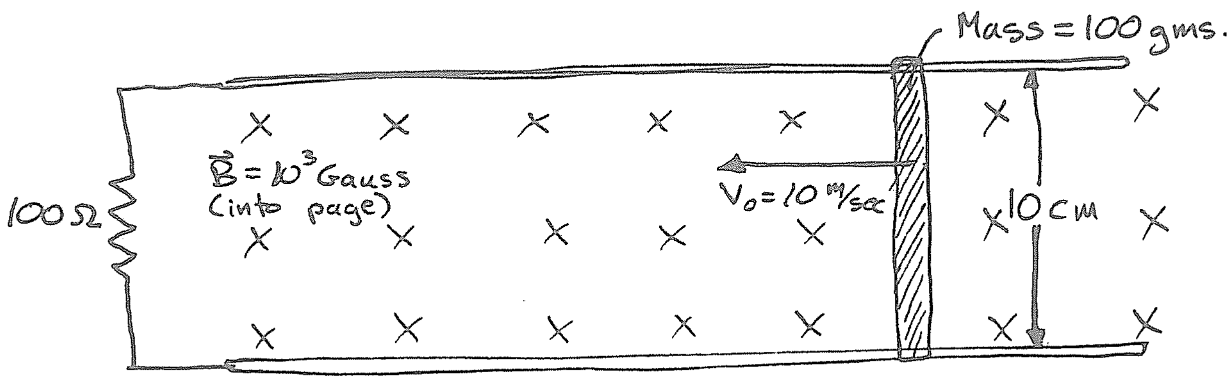
Quiz 6

Physics 52.1 & 52.2

DC-Circuits

10/(21-22)/85

R. G. Brown



Above is pictured a 10 cm. conducting rod sliding on frictionless, conducting rails, through a magnetic field of 10^3 Gauss (not Tesla, 1 Tesla = 10^4 Gauss). The rod is initially (at t = 0) moving to the left with a velocity of 10 m/sec. The rails are connected by a 100 Ω resistor. The rod has a mass of 100 gms.

a) (6 points) Find the force on the rod as a function of velocity. To do this, find the EMF, the current, and the force in that order. Do this part algebraically. (show all directions on diagram).

b) (4 points) Using Newton's second law, show that the velocity should decay exponentially (just like the RC circuit or the LC circuit). What is the time constant for the decay of the velocity (in numbers)? (Hint: For exponential decay, $\frac{dv}{dt}$ must be proportional to $-v$, right?)

Quiz 5

Physics 52.1 & 52.2

DC-Circuits

10/(7-8)/85

R. G. Brown

(7 points) You are given a galvanometer that reads (at full scale deflection) a current of 1mA. Its internal resistance is 10 Ω. You wish to construct an Ammeter out of the galvanometer and whatever else you require (resistors, capacitors, etc.) that reads currents of 20 A at full scale deflection. Draw the correct circuit configuration below, and solve for the values of any extra parts you may need (i.e.-- the values of the resistors, capacitors, that you use in the circuit). Be sure to show all your work and label the various currents and voltages that are relevant.

(3 points) Draw below the correct placement of the ammeter (as a symbol "Ⓐ") in a circuit with a voltage V = 100 Volts, R = 5 Ω. What does the ammeter read?

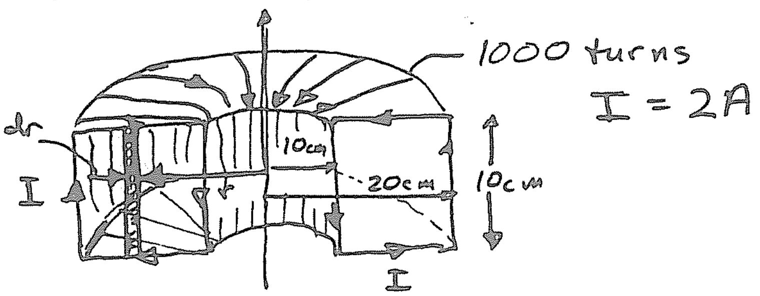
Quiz 6

Physics 52.1 & 52.2 10/(21-22)/85

DC-Circuits R. G. Brown

---

a) (5 points) Use Ampere's law and find the magnetic field as a function of r inside a toroidal solenoid with a rectangular cross-section. Its inner radius is 10 cm., its outer radius is 20 cm., and it is 10 cm. high. It has 1000 turns and is carrying a current of 2 Amps. Express your units in Gauss (recall 1 Tesla = 10^4 Gauss). And you thought I'd never give you numbers, didn't you.



cut-away view.

b) (5 points) Find the total magnetic flux through the solenoid and its self-inductance (in Henries). (hint: consider flux through strip of width dr drawn)

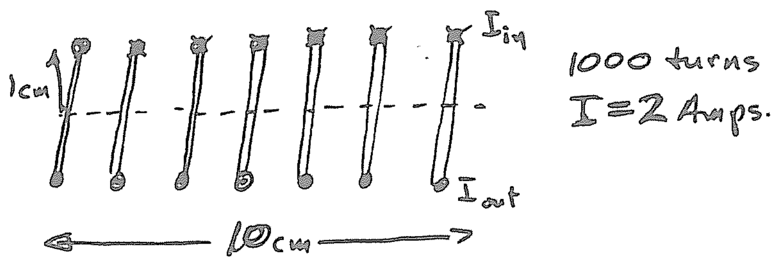
Quiz 6

Physics 52.1 & 52.2 — 10/(21-22)/85

DC-Circuits — R. G. Brown

---

a) (5 points) Use Ampere's law and find the magnetic field inside a circular solenoid 10 cm. long and 1 cm in radius, containing 1000 turns and carrying a current of 2 Amps. Express your units in Gauss (recall 1 Tesla = 10^4 Gauss). And you thought I'd never give you numbers, didn't you.

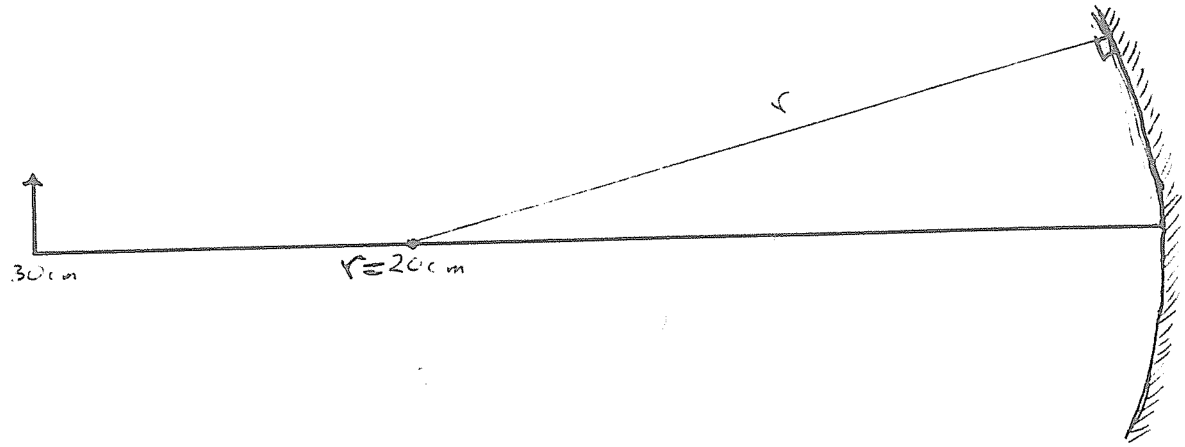


b) (5 points) Find the total magnetic flux through the solenoid and its self-inductance (in Henries).

Physics 52.1
12/3/84

Name:________________

Quiz 7 Score:________



Above is a concave mirror with radius of curvature r=20 cm. An object 1 cm high is placed at s=30 cm as shown and its reflection is viewed. Find:

a) (2 pts.) The focal length of the mirror.

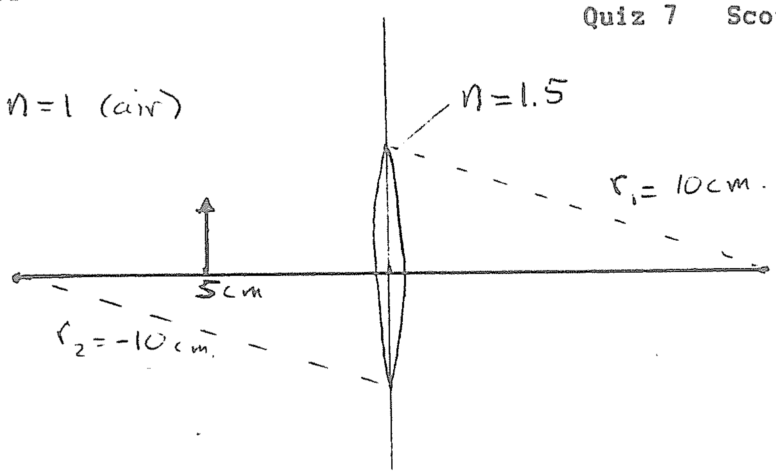
b) (2 pts.) The distance from the mirror to the image. (Is it positive or negative?)

c) (2 pts.) The magnification of the image. Please indicate whether the image is virtual or real, inverted or erect, and bigger than or smaller than the original object.

d) (4 pts.) Draw the ray diagram in on the picture above. Draw at least three rays that can be used to locate the image and pictorially indicate its relative magnification. No sloppy drawings please! Use a straight edge or draw very carefully.

Physics 52.2
12/4/84

Name:____________________
Quiz 7 Score:_____



Above is a converging lens with radii of curvature r_1 = 10 cm, r_2 = -10 cm, n=1.5. An object 1 cm high is placed at s=5 cm as shown and its image is viewed. Find:

a) (2 pts.) The focal length of the lens. (Use the lensmaker's formula)

b) (2 pts.) The distance from the plane of the lens to the image. (Is it positive or negative?)

c) (2 pts.) The magnification of the image. Please indicate whether the image is virtual or real, inverted or erect, and bigger than or smaller than the original object.

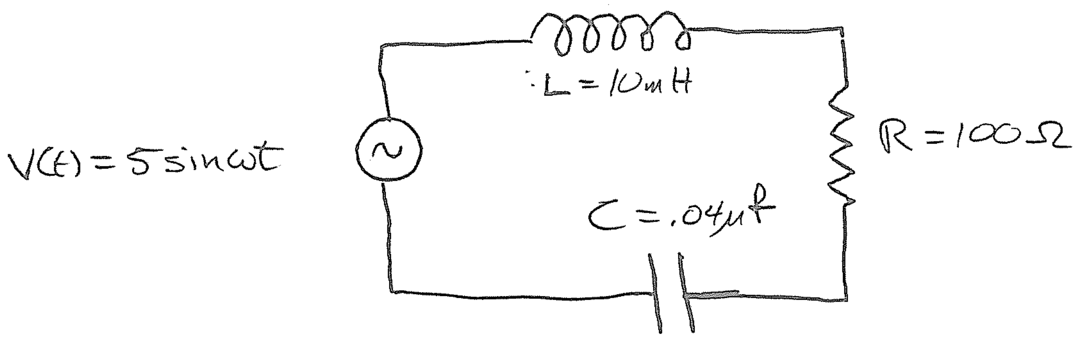
d) (4 pts.) Draw the ray diagram in on the picture above. Draw at least three rays that can be used to locate the image and pictorially indicate its relative magnification. No sloppy drawings please! Use a straight edge or draw very carefully. (If you couldn't do part a, use f = 15 cm.)

Quiz 7

Physics 52.1 & 52.2 — 11/(4-5)/85

DC-Circuits — R. G. Brown

Below is pictured a series AC-LRC circuit. The potential $V(t) = 5\sin\omega t$, $R = 100\Omega$, $L = 10mH$, $C = .04\mu f$. Justify your answers below with at least an equation or two. Remember, partial credit can be awarded only if we can see what you're trying to do!



a) (3 points) Find the impedance Z of the circuit drawn above, as a function of ω.

b) (3 points) What is the Q-value of the circuit? Draw a rough picture of the average power delived to the circuit as a function of frequency, based on the value of Q!

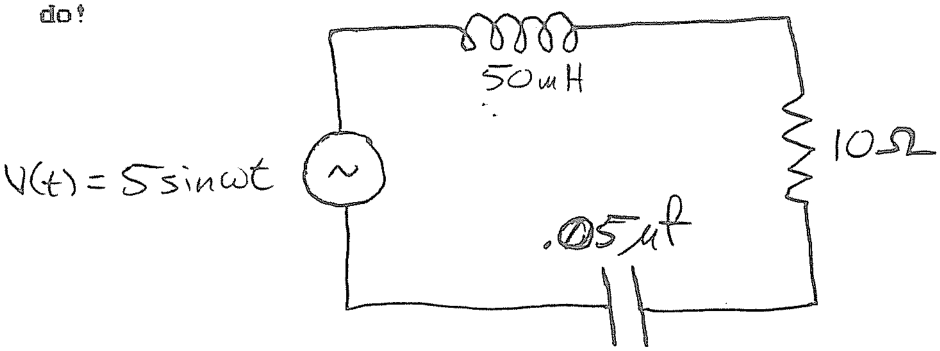
c) (4 points) Find the phase angle ϕ and the instantaneous power delivered to the capacitor as a function of time. What is the average power used by the capacitor?

Quiz 7

Physics 52.1 & 52.2 — 11/(4-5)/85

AC-Circuits — R. G. Brown

Below is pictured a series AC-LRC circuit. The potential $V(t) = 5\sin\omega t$ Volts, $R = 10\Omega$, $L = 50$ mH, and $C = .05\mu f$. Justify your answers below with at least an equation or two. Remember, partial credit can be awarded if we can see what you're trying to do!



a) (3 points) What is the resonant frequency for the circuit?

b) (3 points) Find the Q-value of the circuit and draw a reasonable picture for the average power delivered to the circuit as a function of frequency, <u>based on your answer for Q!</u>

c) (4 points) Find the phase angle ϕ and the instantaneous power delivered to the inductor as a function of time. What is the average power used by the inductor?

Quiz 8

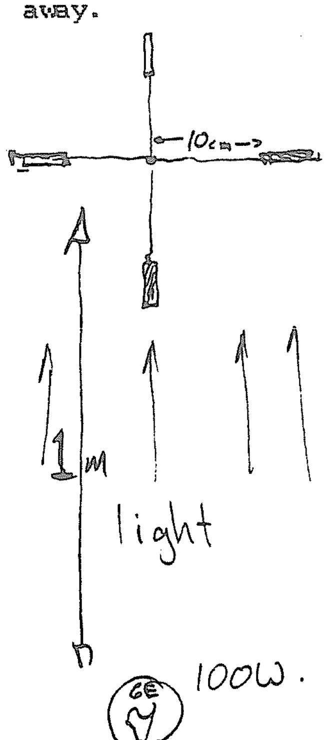
Physics 52.1 & 52.2 11/(11-12)/85

Light R. G. Brown

Aha! I fooled you! (It was easy. . .) Due to the large number of quiz scores people will have by the end of the semester, there will be NO QUIZ TODAY! Don't tell the Tuesday lab people, though. We want them to be as surprised as you are. If I WERE to give you a quiz today, here is what I'd ask.

a) (5 points) Find the intensity of light 1 meter away from a 100 Watt light bulb. Assume that all the energy appears as light.

b) (5 points) A Radiometer is formed of four paddles each a centimeter square, mounted on arms 10 centimeters long as shown. One side of each paddle is black (perfectly absorbing), the other is white (perfectly reflecting). Find the maximum torque exerted by the 100 watt bulb above if the radiometer is placed a meter away.



Quiz 8

Physics 52.1 & 52.2 11/(11-12)/85

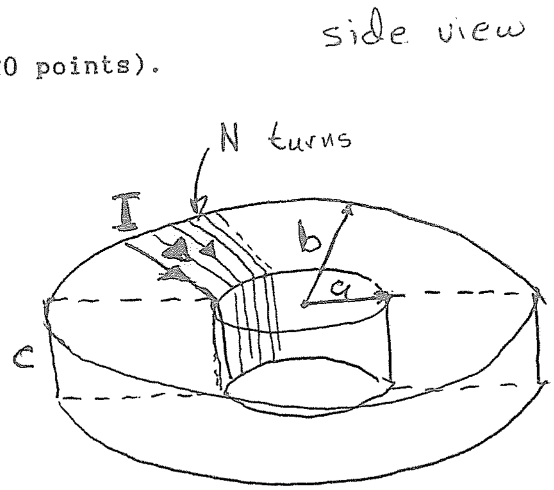
Light R. G. Brown

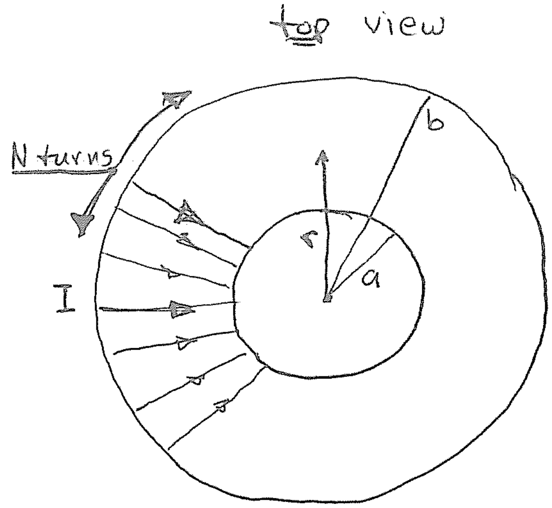
Aha! I fooled you! (Unless people from yesterday's section told you. . .) Due to the large number of quiz scores people will have by the end of the semester, there will be NO QUIZ TODAY! If I WERE to give you a quiz today, here is what I'd ask.

a) (5 points) The index of refraction of glass is approximately 1.5 ($\frac{3}{2}$). The index of refraction of water is 1.33 ($\frac{4}{3}$). A beam of light is passing from the glass to the water. Find the critical angle at which total internal reflection occurs. Explain (briefly) why total internal reflection occurs at this point.

b) (5 points) The light in the problem above just happened to be violet light (λ = 400 nm in air). What is the wavelength of the light in glass? In water? What is its frequency? After it has passed through the water and the glass, what color do we see? Why? (That's five one point questions. . .)

3 (20 points).



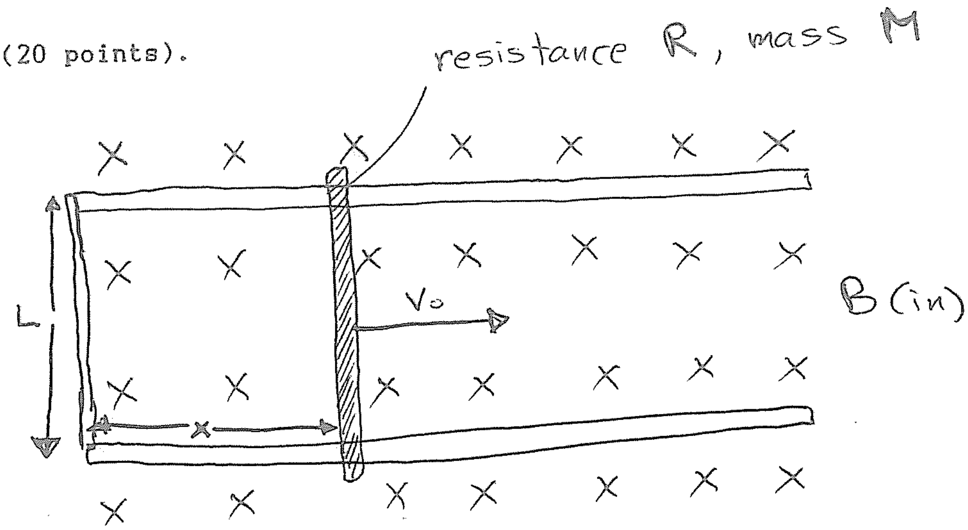


Above is pictured a rectangular toroidal solenoid with N turns. Each turn carries a current I. Its inner radius is a, its outer radius is b, and its height is c (as drawn). Find:

a) The magnetic field inside the solenoid as a function of r (the distance from the axis of the solenoid). <u>Use Ampere's Law!</u> Do **NOT** just write the answer down. Draw curves into the picture that show that you know what you are doing.

b) The self-inductance (L) of the toroidal solenoid.

4 (20 points).



Above is pictured a "rod on rails". The rod has mass M, resistance R, and length L. It is given an initial velocity v_0 to the right. Find:

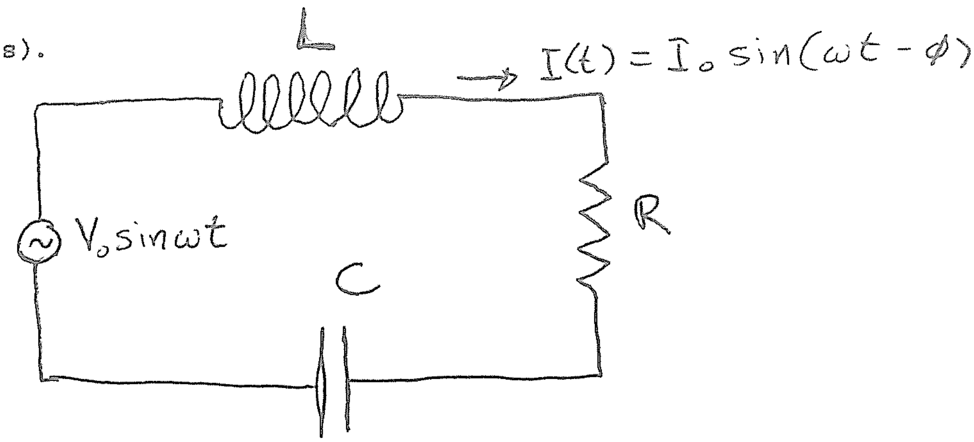
a) the induced emf in the rod as a function of v. Use Faraday's Law.

b) the current in the rod as a function of v. (also draw direction in on diagram)

c) the Force on the rod as a function of v.

d) and finally the velocity of the rod as a function of time. To find this you must integrate Newton's 2nd Law F = M dv/dt. But you should know how to do this by now. . .

5 (20 points).



And another easy one. Above is the same old boring LRC circuit that we have grown to know and love. $V(t) = V_o\sin(wt)$. $I(t) = I_o\sin(wt - \phi)$.

a) Draw the phasor diagram for the voltage gain/drop across each element of the circuit. Be sure to define each quantity in the diagram in terms of R, L, C and w.

b) Express the phase angle ϕ in terms of the given quantities. Draw the triangle from which we find the impedance, Z and show ϕ on the triangle. (Also, what is Z?)

c) What is the instantaneous power into the capacitor? What is the average power?

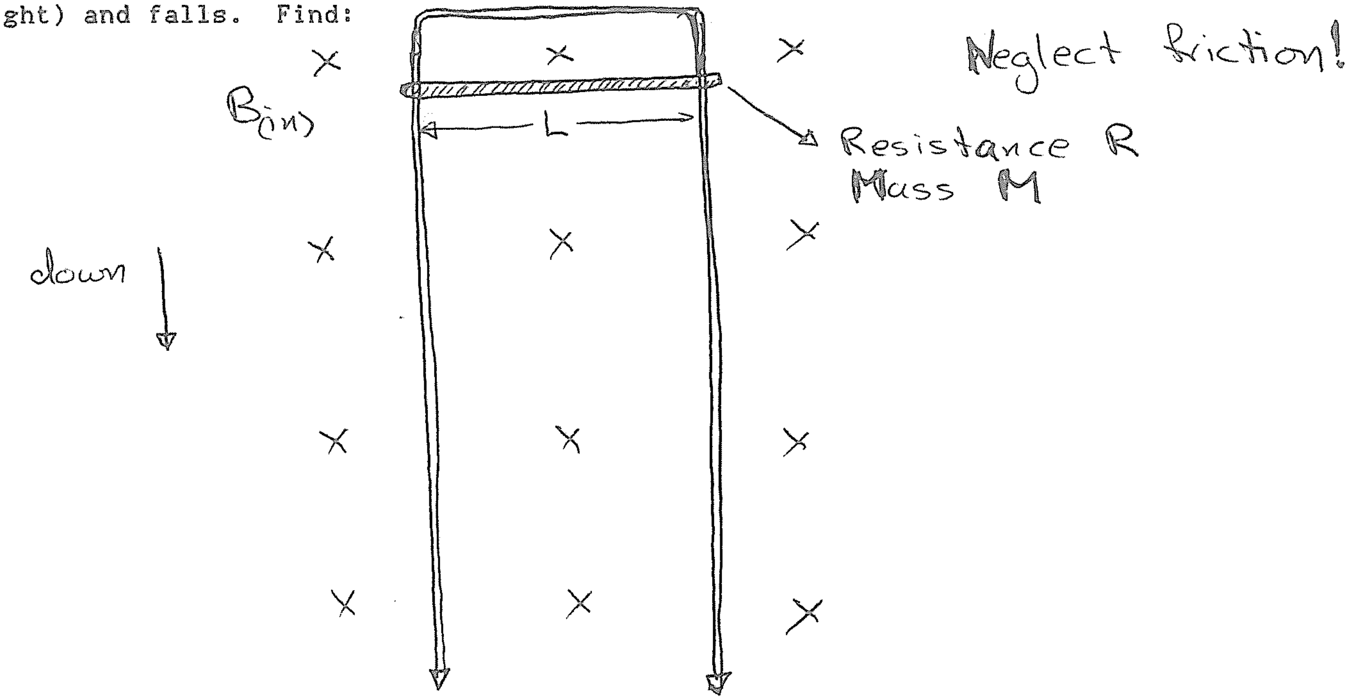
Choose <u>one</u> of the next two problems. They are each worth 25 points.

OPTION 1 (25 points).

Derive the wave equation for the electric field from Maxwell's equations. Such a derivation should include: Simple diagrams that indicate the paths of integration; a brief statement of the assumptions (if any) involved; the algebra. DO NOT solve the wave equation once you've got it, unless you want 10 extra points and have extra time.

Instead, show how you understand the solutions for the E and B fields. What are the solutions? Are they in phase? How are the magnitudes related? What is the energy density of the electromagnetic wave (in terms of E and/or B)? What is the intensity (in terms of E and B or the energy density)?

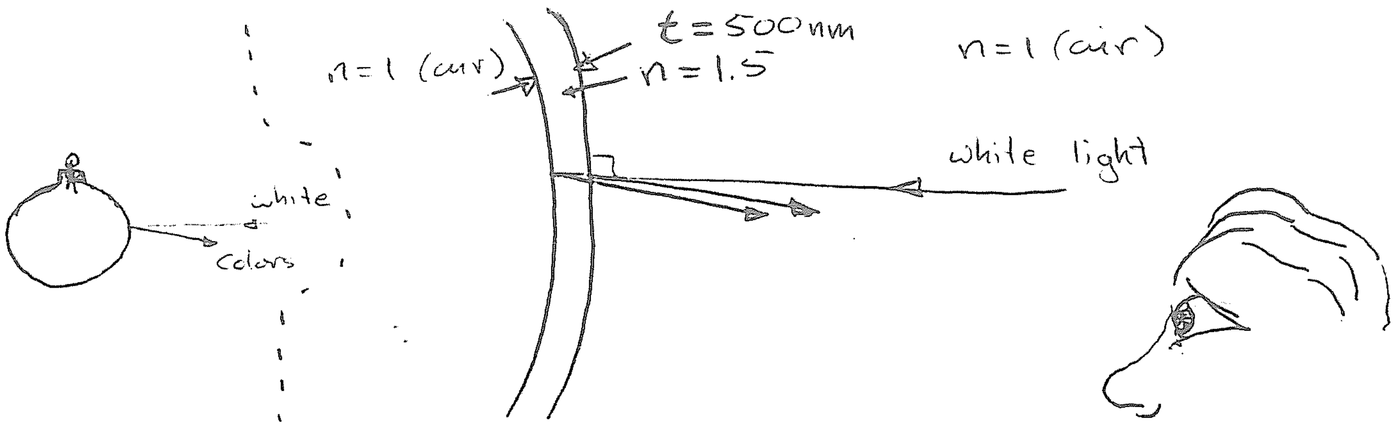
OPTION 2 (25 points).
Below is pictured another "Rod on Rails, but this time the (frictionless) rails are vertical. The resistance of the rod is R, its length is L, and its mass is M. The magnetic field, B, is into the paper as drawn. At time t=0, the rod is released from rest (at a great height) and falls. Find:



a) The equation for its velocity as a function of time. As a hint, what is the terminal velocity of the rod? (Use Newton's 2nd, Faraday's, Ohm's Law)

b) Show that, at terminal velocity, the power input by gravity into the system equals the power dissipated by joule heating of the rod. There are a variety of ways to do this. Some are easier than others, and any valid argument will be accepted.

Practice Quiz on Physical Optics.
Please keep and do at your liesure before the hour exam.
No credit, do not hand in.



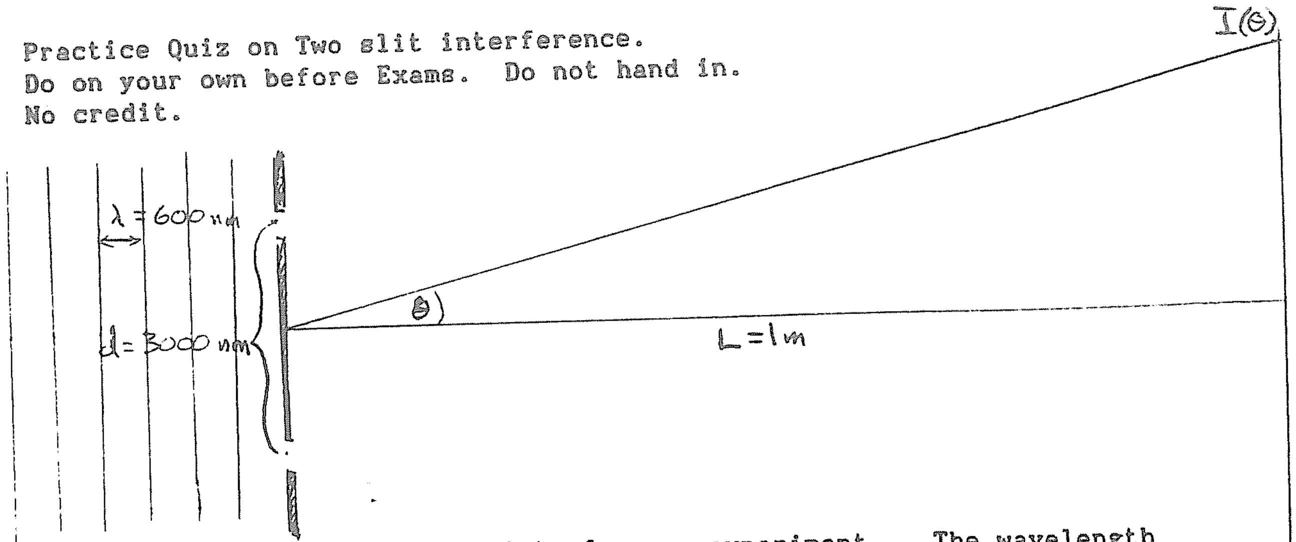
Above is drawn a very thin clear glass (n=1.5) christmas tree ornament. When viewed with white light, the ornament is irridescent and many rainbow colors are seen playing across its surface. The glass is known to be 500 nanometers thick.

1) What two wavelengths of visible light are brightest when seen reflected at normal incidence as drawn above?

2) What wave length of visible light is <u>not</u> seen in light reflected at normal incidence as drawn above?

Hint for both parts of the problem. First work out which equation you want for constructive vs. destructive interference, including phase shifts at surfaces etc. Solve these equations for λ_o. Then work your way through m (the order) until the results lie in the visible spectrum. You do know where the visible spectrum lies, don't you?

Practice Quiz on Two slit interference.
Do on your own before Exams. Do not hand in.
No credit.

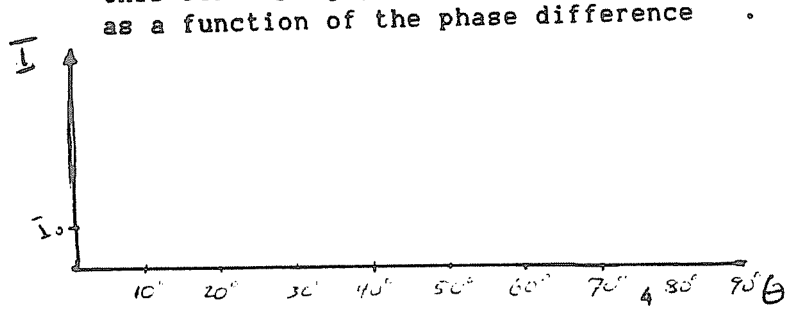


Above is pictured a two slit interference experiment. The wavelength of the incident light is $\lambda = 600$ nm. The slit separation d = 3000 nm. The distance L is 1 m >> d. Find:

a) The angles θ at which constructive interference occurs. (All of them).

b) The angles at which destructive interference occurs. (All of them).

c) Draw a picture of the interference pattern produced I(θ). Do this for θ =[0,90] degrees. Write the expression for the intensity as a function of the phase difference .

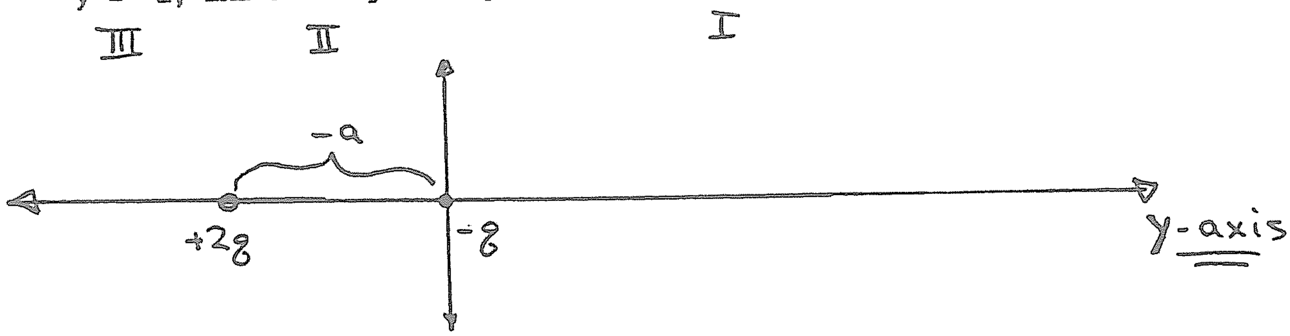


Quiz 1

Physics 52.1 & 52.2 — 9/(2-3)/85

The Electric Field — R. G. Brown

(10 points) In the picture below, a charge of +2q is located at y = -a, and a charge of -q is located at y = 0.



a) (6 points) Find the electric field (magnitude and direction) at an arbitrary point on the y-axis. (You will need to do three regions: y>0 (I), 0>y>-a (II), -a>y (III).)

b) (3 points) Find a spot on the y-axis where the force on a charge placed there would be zero. (Hint: Where is the field zero?) Extra credit: is this position a stable equilibrium point? Why or why not.

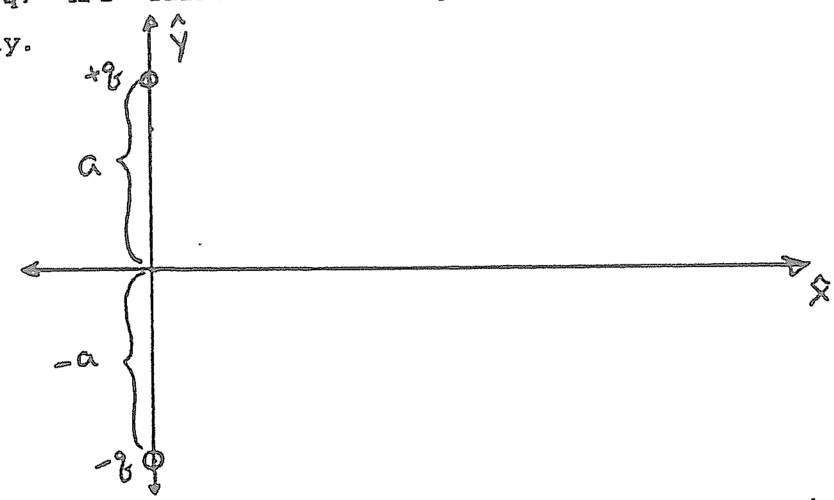
c) (1 point) What is the magnitude and direction of the field when q = 1μC, y = 10 cm, x = 0 cm, and a = 10 cm.

Quiz 1

Physics 52.1 & 52.2 — 9/(2-3)/85

The Electric Field — R. G. Brown

(10 points) In the picture below, two equal and opposite charges (+q and −q) are located on the y-axis at y = +a and y = −a, respectively.



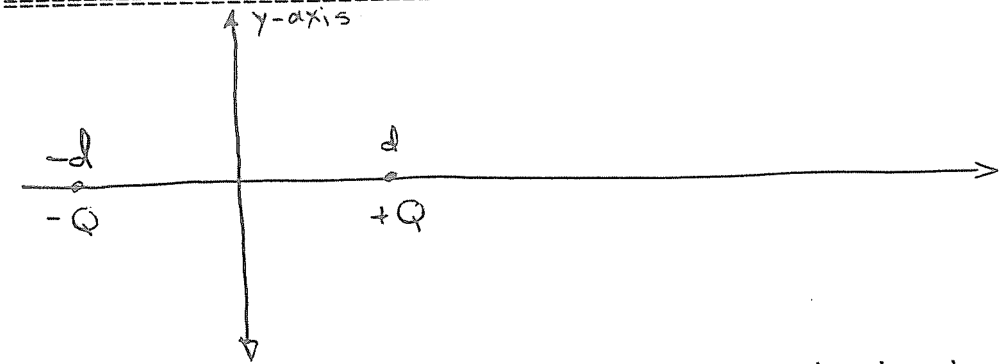
a) (6 points) Find the electric field at an arbitrary point on the x-axis (magnitude and direction).

b) (3 points) Show that the electric field on the x-axis has the form $\vec{E} = \frac{-k\vec{p}}{x^3}$ when x>>a. (recall $\vec{p} = q\vec{l} = q(2a\hat{y})$.)

c) (1 point) What is the magnitude of the electric field if q = 1 μC, a = 30 cm, x = 40 cm, y = 0?

Physics 52.2
Quiz 1

9/4/84
Name:



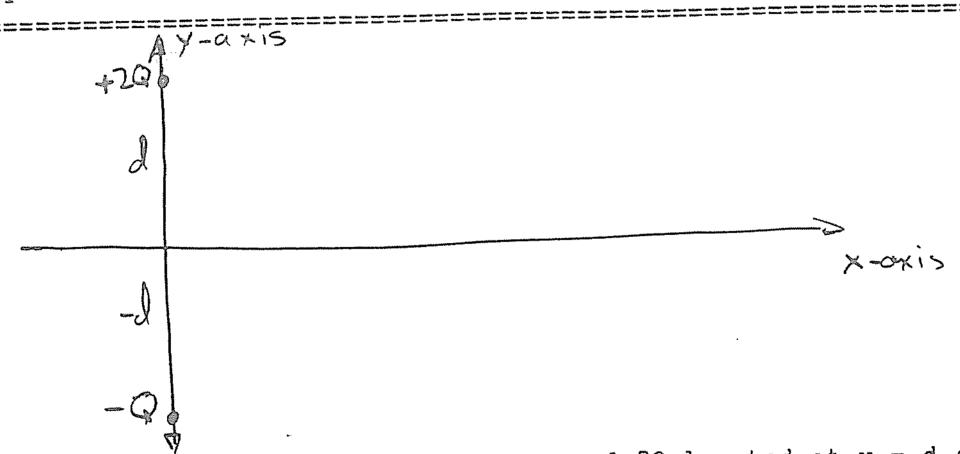
Above is a diagram showing a charge of +Q at x = d and a charge of -Q located at x = -d.

a) (5 pts) Find the electric field on the x-axis for $x > d$ in terms of k, Q, x and d.

b) (4 pts) Find the electric field to lowest order if $x >> d$. How does the field vary? (This is the dipole field).

c) (1 pt) Suppose Q = 20 μC, d = 3m. Find the force (magnitude and direction, remember) on a test charge of q = -20 μC placed at the origin.

Physics 52.1 9/3/84
Quiz 1 Name:



1) Above is a diagram showing a charge of 2Q located at y = d and a charge of -Q located at y = -d.

a)(5 pts) Find the electric field at an arbitrary point on the x-axis, in terms of k, Q, x and d. Do it algebraically.

b)(4 pts) Find the electric field (to lowest order in x) if x>>d. In which direction does it point? Does your answer make sense?

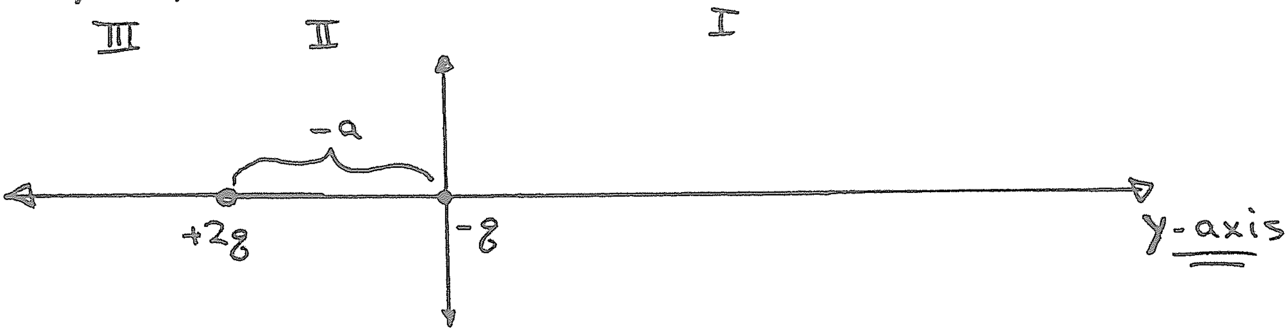
c)(1 pt) Let Q = 30μC, d = 3m. Find the force (magnitude and direction) that would be exerted on a test charge of 10μC placed at the origin.

Quiz 1

Physics 52.1 & 52.2 9/(2-3)/85

The Electric Field R. G. Brown

(10 points) In the picture below, a charge of +2q is located at y = -a, and a charge of -q is located at y = 0.



a) (6 points) Find the electric field (magnitude and direction) at an arbitrary point on the y-axis. (You will need to do three regions: y>0, 0>y>-a, -a>y.)

b) (3 points) Find a spot on the y-axis where the force on a charge placed there would be zero. (Hint: Where is the field zero?) Extra credit: is this position a stable equilibrium point? Why or why not.

c) (1 point) What is the magnitude and direction of the field when q = 1μC, y = 10 cm, x = 0 cm, and a = 10 cm.

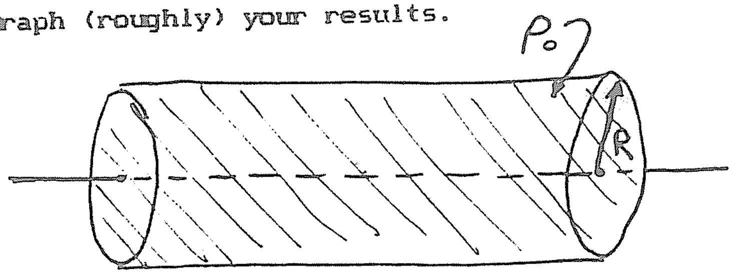
Quiz 1

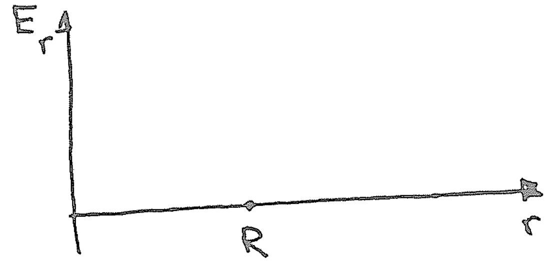
Physics 52.1 & 52.2 9/(16-17)/85

Gauss' Law and Electric Potential R. G. Brown

---

a) (10 points) A solid cylindrical shell of "infinite" length and radius R has a uniform charge density of ρ_0 C/m^3 distributed throughout it. Find the electric field at all points in space. Graph (roughly) your results.





b) (10 points) Find the potential difference between the surface of the cylinder (r = R) and all other points in space (r inside and outside the cylinder).

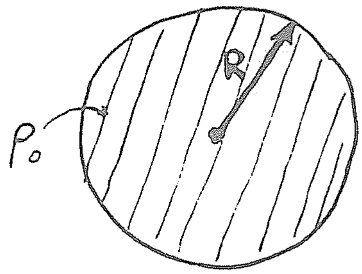
Quiz 1

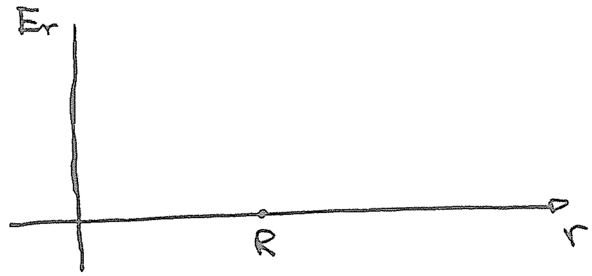
Physics 52.1 & 52.2 9/(16-17)/85

Gauss' Law and Potential R. G. Brown

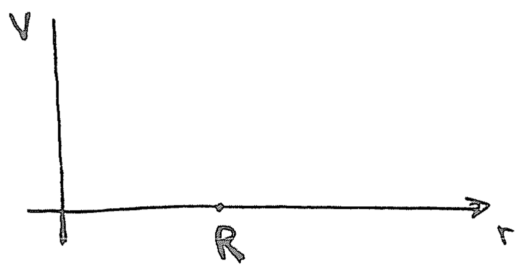
---

a) (10 points) Find the electric field for a solid sphere of radius R and with a uniform charge density ρ_0 C/m^3 at all points in space. Graph (roughly) your answer.





b) (10 points) Using your answer(s) to part a), find the electric potential (relative to ∞) at all points in space. Graph (roughly) your answer.



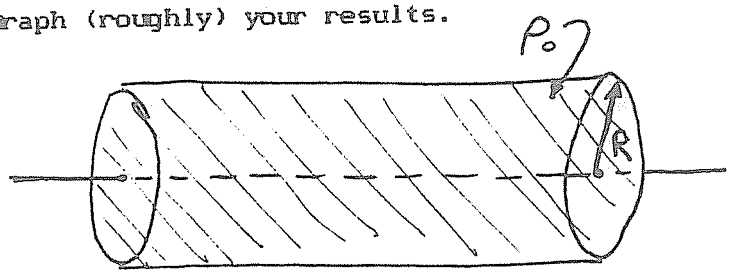
Quiz 1

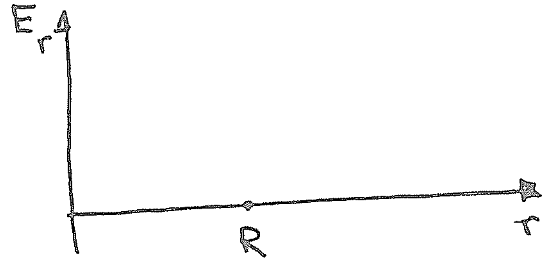
Physics 52.1 & 52.2 9/(16-17)/85

Gauss' Law and Electric Potential R. G. Brown

---

a) (10 points) A solid cylindrical shell of "infinite" length and radius R has a uniform charge density of ρ_0 C/m^3 distributed throughout it. Find the electric field at all points in space. Graph (roughly) your results.

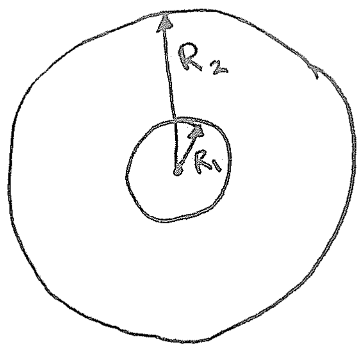




b) (10 points) Find the potential difference between the surface of the cylinder (r = R) and all other points in space (r inside and outside the cylinder).

Physics 52.1 or 52.2
Quiz 3

Name:____________________
9/25/84



1) (4 pts.) Find the capacitance of the spherical shells drawn above in terms of R_1, R_2, and ϵ_0. Use whatever laws you need to and show all work, indicating how you would solve for $\vec{E}$ and V rather than actually solving for them if you like.

2) (4 pts.) What is the total energy stored in the electric field of the capacitor above? What is the energy density (as a function of r)? Show all work, again, starting from the known field or potential if you wish.

3) (2 pts.) If the capacitor above were filled with a dielectric material with dielectric constant K=2.0, how would your answers to 1) and 2) change? (i.e.-- find the modified field/potential/capacitance and energy)

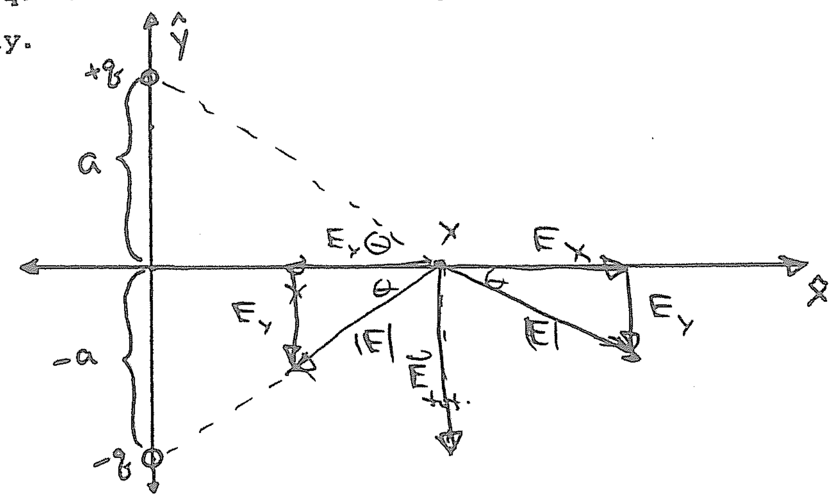
Quiz 1

Key.

Physics 52.1 & 52.2 — 9/(2-3)/85

The Electric Field — R. G. Brown

(10 points) In the picture below, two equal and opposite charges (+q and -q) are located on the y-axis at y = +a and y = -a, respectively.



a) (6 points) Find the electric field at an arbitrary point on the x-axis (magnitude and direction).

$$|E| = \frac{kq}{(x^2+a^2)} \qquad E_x = \pm \frac{kq\,x}{(x^2+a^2)^{3/2}} \quad (|E|\cos\theta)$$

$$E_y = \frac{-kq\,a}{(x^2+a^2)^{3/2}} \quad (|E|\sin\theta)$$

$$\text{so } E_x^{tot} = 0 \qquad E_y = -2kqa/(x^2+a^2)^{3/2}$$

b) (3 points) Show that the electric field on the x-axis has the form $\vec{E} = \frac{-k\vec{p}}{x^3}$ when x>>a. (recall $\vec{p} = q\vec{\ell} = q(2a\hat{y})$.)

$$E_y = \frac{-2kqa}{(x^2+a^2)^{3/2}} = \frac{-2kqa}{x^3}\left(1+\frac{a^2}{x^2}\right)^{-3/2} \approx \frac{-2kqa}{x^3}\left(1 - \frac{3a^2}{2x^2} + \dots\right)$$

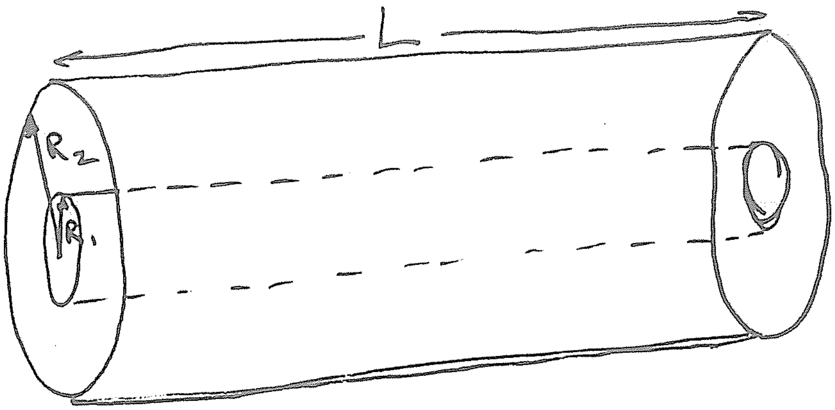
$$= -\frac{k|p|}{x^3} \quad \text{so} \quad \vec{E} = -\frac{k\vec{p}}{x^3} \quad (\vec{p} = 2aq\hat{y})$$

c) (1 point) What is the magnitude of the electric field if q = 1 μC, a = 30 cm, x = 40 cm, y = 0?

$$E = \frac{2(9\times10^9)(1\times10^{-6})(.30)}{(.25)^{3/2}} = \frac{9\times10^3(2)^3}{(.25)} = 9\times8\times10^3 = 72\times10^3(\times2)\times(.30)$$

Physics 52.1 & 52.2
Quiz 3

Name:____________
9/24/84



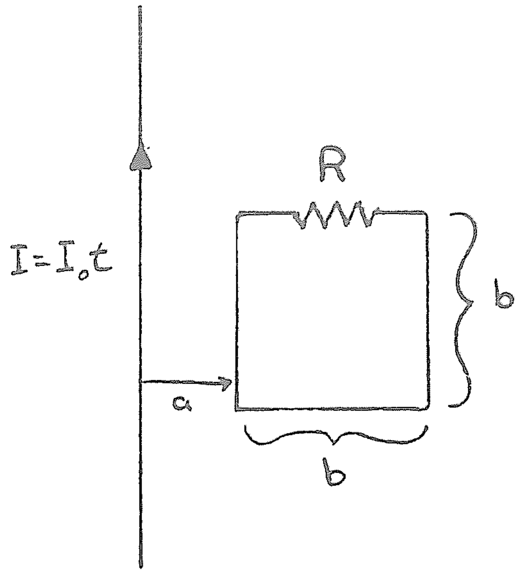
1) (4 pts.) Find the capacitance of the cylindrical shells drawn above in terms of L, R_1 , R_2, and ϵ_0. Use whatever laws you need to and show all work, indicating how you would solve for $\vec{E}$ and V rather than actually solving for them if you like.

2) (4 pts.) What is the total energy stored in the electric field of the capacitor above? What is the energy density (as a function of r)? Show all work, again, starting from the known field or potential if you wish.

3) (2 pts.) If the capacitor above were filled with a dielectric material with dielectric constant K=4.0, how would your answers to 1) and 2) change? (i.e.-- find the modified field/potential/capacitance and energy)

Quiz 4
Physics 52.1

Name:____________
10/22/84



The current in the long straight wire above is $I(t) = I_0 t$ Amps. Express all answers in terms of μ_0, I_0, a, b, and R.

a) Find the magnetic field as a function of r due to the long wire using Ampere's law. Indicate the direction on the diagram.

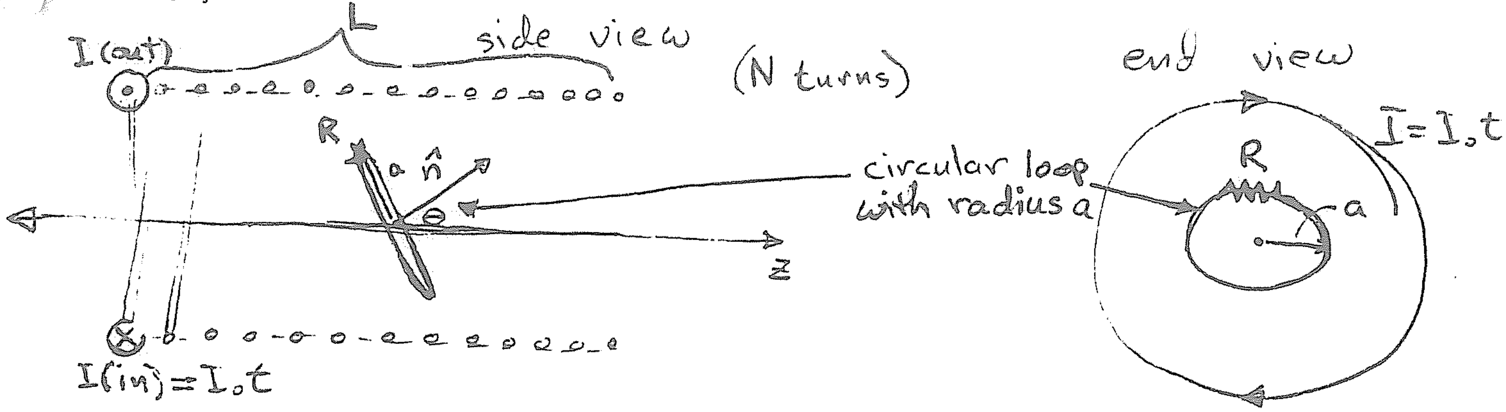
b) Find the flux through the square loop due to the current in the long wire. From this calculate M, the mutual inductance.

c) Find the EMF in the loop, using Faraday's law. Indicate the direction of current flow (found from Lenz's law).

d) Find the net force on the loop.

Quiz 4
Physics 52.2

Name:____________________
10/23/84



Above is a picture of a solenoid with length L and N turns, each carrying a current $I = I_o t$. Express all answers in terms of μ_o, I_o, a, R and θ. (R is resistance of circular loop)

a) Find the field inside the solenoid (using Ampere's law). Draw it's direction on the picture above.

b) Find the flux through the circular loop as a function of I. Note that the loop is angled with respect to the field. What is the mutual inductance of the loop?

c) Find the EMF induced in the circular loop from Faraday's law. Indicate the direction of current flow (from Lenz's law) on the diagram.

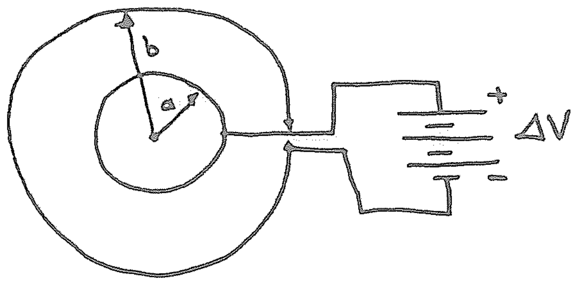
d) What is the torque on the loop? (Hint: Find the current in the loop, and thus the magnetic dipole moment.)

Quiz 4

Physics 52.1 & 52.2 — 9/(23-24)/85

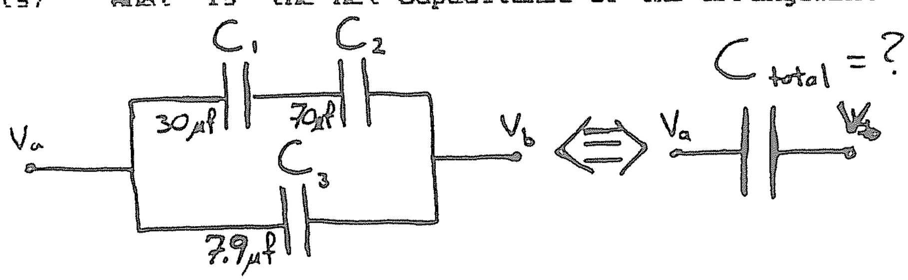
Capacitance — R. G. Brown

(7 points) Below you see drawn a spherical shell capacitor made up of two concentric conducting shells of radii a and b. Find the capacitance. (Show all of your work. Start with Gauss' Law)



---

b) (3 points) What is the net capacitance of the arrangement below.



C_{total} = ___________

$C_1 = 30\mu f$ $C_2 = 70\mu f$ $C_3 = 7.90\mu f$

Quiz 5
Physics 52.1

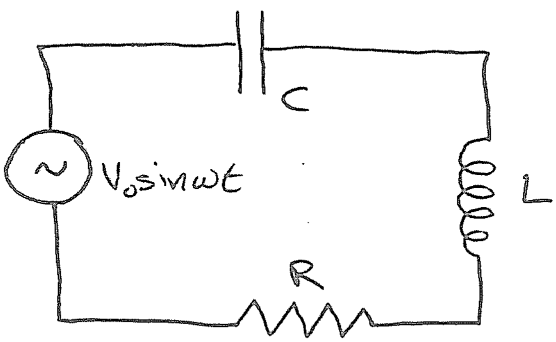
Name:________________
10/29/84

As promised, for one point each, for each material described below, tell whether the material is paramagnetic, diamagnetic, or ferromagnetic.

a) At low temperature the material develops a magnetization that is positive (parallel to the applied field) and inversely proportional to the temperature.
a:________________________

b) The material seems to develop a strong, "permanent" magnetic moment after being heated and then cooled in a strong magnetic field.
b:________________________

c) The material repels the poles of a magnet brought near the substance. It seems to repel either pole equally at any temperature.
c:________________________



$V_0 = 100$ Volts
$C = 1 \times 10^{-6}$ farads.
$L = 40 \times 10^{-3}$ henries
$R = 400\ \Omega$
$\omega = 10000$ rad/sec.
recall $f = \frac{\omega}{2\pi}$!

2) (7 pts.) Above is pictured an LRC circuit. R = 400 ohms, C = 1 microfarad, and L = 40 millihenries. Find:

a) the resonant frequency of the circuit.

b) the impedance (Z) of the circuit, if the angular frequency of the the applied voltage, ω, is 10000 rad/sec.

c) the average power drawn by the circuit if ω = 10000 rad/sec and V(max) = 100 volts.