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Ultra-high-frequency piecewise-linear chaos using delayed feedback loops

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We report on an ultra-high-frequency (>1 GHz), piecewise-linear chaotic system designed from low-cost, commercially available electronic components. The system is composed of two electronic time-delayed feedback loops: A primary analog loop with a variable gain that produces multi-mode oscillations centered around 2 GHz and a secondary loop that switches the variable gain between two different values by means of a digital-like signal. We demonstrate experimentally and numerically that such an approach allows for the simultaneous generation of analog and digital chaos, where the digital chaos can be used to partition the system’s attractor, forming the foundation for a symbolic dynamics with potential applications in noise-resilient communications and radar.

Recently, Corron et al. realized a piecewise-linear (PWL) electronic circuit that simultaneously generates an analog chaotic waveform and an associated digital waveform that represents the system’s symbolic dynamics. Their technique is data-efficient for chaos-based applications in that the symbolic dynamics is sufficient for specifying the underlying deterministic features of the chaotic carrier signal, but their implementation was limited to the audio frequency range (20 Hz–20 kHz). Here, we propose a new circuit architecture that operates at ultra-high-frequencies (UHFs) (0.3–3 GHz). Our approach is based on two time-delayed electronic feedback loops, one analog and the other digital. Together, they ensure an increase in complexity of the analog signal and enable fast control over its dynamics, so that the overall system still behaves in a piecewise-linear fashion. Our architecture represents a potential breakthrough in the design of inexpensive and robust electric circuits that generate piecewise-linear chaos.

I. INTRODUCTION

A PWL operator with two-states is defined as

\[ f(x) = \begin{cases} f_1(x), & \text{if } x < I \\ f_2(x), & \text{if } x \geq I \end{cases} \]

where \( f_1(x) \) and \( f_2(x) \) are linear operators with respect to \( x \) and \( I \) is a threshold. Every time the value of \( x \) crosses \( I, f(x) \) switches between \( f_1(x) \) and \( f_2(x) \). This definition extends naturally to multiple states and thresholds.

Piecewise-linear operators have been used with discrete maps,\(^1\) continuous systems,\(^2,3\) and to describe certain classes of electronic circuits\(^4\) and regulatory systems.\(^5\) They introduce a nonlinear component in the dynamics of systems (called PWL systems), a minimum requirement for potentially leading to chaos. For example, when a PWL operator controls the parameters of discrete maps and linear differential equations, chaos can exist in the temporal evolution of the systems’ state variables.\(^6,7\) Recent studies also demonstrate that a two-state PWL operator can generate low-dimensional chaos if the system has hysteresis\(^8\) or uses multiple thresholds.\(^9,10\) Such theoretical evolutions, referred to as PWL chaos, are also observed in experimental systems.

Creating an experimental embodiment of a PWL chaotic system resides in the ability to design multiple linear systems (or a single system with variable parameters) and a physical switch. Examples of linear systems used to produce experimental PWL chaotic dynamics are LRC (inductance, resistance, capacitance) audio-frequency circuits.\(^11\) They are popular because their parameters can be easily switched with conditions based on their voltage or current. This allows the LRC circuits to operate in two different linear states.\(^12\) Physical switches implemented in PWL systems use various components such as digital logic gates and operational amplifiers.\(^13\)

Though the switching times between the linear states are not infinitely fast, they are usually considered negligible for LRC circuits (being orders of magnitude faster than the system’s dynamics) and thus make LRC systems good for PWL experiments with chaos.\(^14\)

A novel LRC-based implementation of a PWL chaotic system has been designed recently by Corron et al.\(^15\) In their circuit, negative damping (gain greater than one) induces oscillations about a piecewise-constant voltage with a growing amplitude that is bounded by a piecewise constant driving voltage. The driving voltage switches its sign in response to a guard condition on the system’s present dynamics, and leads to the generation of audio-frequency PWL chaos. Such chaos is fully characterized by the temporal evolution of the piecewise-constant voltage that serves as an easily accessible, real-time symbolic dynamics. This approach, with its simultaneous generation of analog and digital chaos, demonstrates advantages over traditional chaos generators from an application point-of-view. One of them is the existence of a matched filter for robust chaos-based communications and radars\(^16\) in noisy environments.

The range of applications for PWL chaos, as envisioned by Corron et al., motivates an ultra-high-speed version of their circuit. However, non-ideal behaviors of LRC circuits...
make it difficult to realize an ultra-high-speed PWL mode-of-operation. Above a certain frequency, the propagation
times of signals through LRC circuits are no longer negligi-
ble, and the dynamics evolves on a time scale that is com-
parable to the switching times of the system’s electronic
PWL operators. Furthermore, nonlinearities tends to arise in
LRC systems at UHF and introduces undesired
effects into the PWL system. We acknowledge that optical
and opto-electronic systems have been shown to generate
UHF chaos without such non-ideal behaviors. However,
in order to overcome these obstacles in an all-electronic
system, a new architecture must addresses these challenges that
are at the forefront of nonlinear dynamics and electronic
design.

In this paper, we present the design and operating char-
acteristics of a novel chaotic system, shown in Fig. 1, that
uses feedback loops to realized UHF chaos. In Secs. II–IV,
we first describe the two main components of our electronic sys-
tem: (i) an analog time-delayed feedback-loop, which gener-
ates a multi-mode oscillations; and (ii) a digital time-delayed
control loop, which bounds the amplitude of the oscillations
using a threshold condition and generates a digital-like
waveform that follows the threshold crossings. Finally, in
Secs. V–VII we present a PWL physical model and a time-
delayed discrete map that capture the dynamics in our system
and provide estimates for its maximal Lyapunov exponent.

II. ANALOG MULTI-MODE FEEDBACK LOOP

The time-delayed feedback loop in Fig. 1 is used to gen-
erate growing-amplitude oscillations, similar to the negative
damping in designs of Corron et al.13,22 It comprises a variable
gain amplifier (VGA) that is self feedback. The VGA produces
a voltage $v_{out} = v(t)$ that is delayed by time $\tau_f$ by means of
coaxial cables and connected to its input [$v_{in} = v(t-\tau_f)$].

The presence of a time-delayed feedback loop has impor-
tant implications for the spectral properties of our system
compared to that of Corron et al. For example, it generates
multi-mode oscillations instead of mono-mode oscilla-
tions.13,15 Within the bandwidth $B$ of the VGA, resonances
that lead to multi-mode oscillations occur approximately at
integer multiples of $1/\tau_f$. If the pass-band of $B$ is larger than
the (approximate) inter-mode spacing, the system becomes
multi-mode.

In addition, the multi-mode resonances and $B$ are de-
pendent on the value of $\tau_{ctl}$. To characterize these spectral
changes, we measure the feedback-loop transfer function for
two different values of the voltage control port $\tau_{ctl}$, where
the setup is shown in Fig. 2(a). It comprises a sinusoidal
waveform generator with automatic frequency sweep, which
drives the feedback loop, and a power spectrum analyzer
(PSA) that records the output power. The experiment is first
realized with $\tau_{ctl} = \tau_{f} < 0.5$ V, shown in Fig. 2(b). It con-
firms the existence of multiple modes within $B \sim 0.4$ GHz
centered at $f_c \sim 2.1$ GHz. For $\tau_{ctl} = \tau_{ctl} > 0.5$ V (Fig. 2(c)),
we observe resonances with relatively constant amplitude,
and we approximate $B \sim 0.6$ GHz with a shift in the central
frequency $f_c \sim 2.5$ GHz (see Appendix).

![DIAGRAM](image_url)

FIG. 1. Piecewise-linear chaotic waveform generator. An analog feedback
loop connects the output voltage ($v_{out}$) of a VGA (Hittite HMC287MS8) and
its input voltage ($v_{in}$). The VGA output also feeds a digital control loop with
a LGA (Analog Devices AD8319) and a TTL (Texas Instruments
SN74AUC1G04) inverter gate in series (a second TTL gate is driven in par-
allel in order to monitor $s(t)$ with correct impedance matching). We monitor
the signals $s(t)$ and $\xi(t)$ using an oscilloscope (osc).

![DIAGRAM](image_url)

FIG. 2. Multi-mode characteristics of the feedback loop. (a) Experimental
setup to determine the open-loop transfer function of the feedback loop. The
input of a VGA is fed by a sinusoidal waveform generator and a delayed ver-
sion of its output $v(t-\tau_f)$, where the signal $v(t)$ is monitored using a PSA.
For this measurement, an attenuator of gain $g_{ctl}$ in the feedback loop to main-
tain the net-gain $g_{ctl}(t)$ below unity, avoiding self-oscillation and saturation at
the resonant frequencies. In our experiment, we observe multi-mode charac-
teristics with (b) $\tau_{ctl} = \tau_f$, the high-gain state, and (c) $\tau_{ctl} = \tau_{ctl}$, the low-gain
state, where both states allow for the existence of up to ~20 modes (mode
spacing ~1/\tau_f ~ 25 MHz).
We maintain the saturated behavior for approximately
erned solely by the values of its net gain (saturation, the feedback loop operates linearly and is gov-
tem can generate multi-mode UHF oscillations. Away from
oscillations in the feedback loop decay until they reach the
uations (see Fig. 3(b)). When oscillations grows
such that the dynamics becomes a stable periodic oscillation
fies and band-pass filters electrical noise after propagating
saturates. Figure 3 illustrates how the feedback loop ampli-
(see Appendix for characterization of these components). As a whole, the
control-loop generates a digital signal and is designed to pre-
vant saturation in the feedback loop.
As shown in Fig. 1, the feedback and control loops are
combined by means of a voltage divider, so that \( v(t) \) drives the
VGA, the LGA, and the TTL gate. The output of the TTL
gate generates a digital signal \( s(t) \) that is fed into a fixed volt-
age attenuator connected directly to \( v_{ctl} \). We use a fixed voltage
attenuator to map the low and high state from \( s(t) \) (\( s(t) = 0 \text{~V} \),
and 2 V, respectively) onto high and low values desired for \( g(t) \)
\( (v_{ctl} = v_{f} \text{ and } v_{tl}) \), respectively). This design is adequate to re-
izle a switching condition for PWL dynamics, except that it
requires additional time-delay to guarantee proper control on the
fast dynamics existing in the feedback loop.

With low-speed systems, the processing time of the
ponents in a control loop, called the control-latency-time
(CL2T), is usually orders of magnitude faster than the time-
cales of the dynamics to be controlled and therefore can be
neglected.\(^8,13,25\) In our UHF system, however, this is not the
case. The average period in \( v(t) \) is 0.5 ns, which should be
compared to the summed CLT of the LGA and TTL gate of
approximately 9 ns. As a result, it is not possible for the
switching condition to affect instantaneously the feedback-
loop oscillations as they are generated.

However, we can control the envelope of the oscillations
(evolving on a slower time-scale) if \( s(t) \) is applied to the
VGA at the proper time with respect to feedback-loop delay
time. This is obtained by adjusting the time delays of
both the control and feedback loops to be approximately
equal. Such an approach has been already used successfully
to stabilize a very-high-frequency optoelectronic chaotic system.\(^{26}\)

The time delay \( \tau_c \) of the feedback loop is tunable with
respect to the length of coaxial cable used in the design of
our circuit. First, we guarantee that \( \tau_c \) is larger than the CLT
and choose \( \tau_c \sim 41 \text{~ns} \). Then, we use coaxial cables of time
delay \( \tau_{coax} \) in the control loop to approximately match the
propagation time of the control signal with \( \tau_c \). We denote
this net control-loop delay time by \( \tau_c = \text{CLT} + \tau_{coax} \) (see
Fig. 1) and adjust its value to be \( \tau_c \sim 40 \text{~ns} \) (see Appendix
for a detailed description of the total control loop CLT).

In Sec. IV, we analyze the dynamics in \( v(t) \) from the full
system (shown in Fig. 1) and its relation to the digital output
of the control loop \( s(t) \).

III. DIGITAL CONTROL LOOP

We design our control scheme to vary \( v_{ctl} \) in response to a
signal that monitors the amplitude of the multi-mode oscillations in \( v(t) \). When the amplitude or envelope of \( v(t) \) grows
beyond (decays below) a set threshold, the control loop
changes the value of \( v_{ctl} \) and switches the system to its decay
(growth) regime. In our design, we use a logarithmic amplifier
(LGA) to detect the envelope of \( v(t) \) and a transistor-transistor
logic (TTL) inverter gate as a digital switch (see Appendix
for a detailed description of the total control loop CLT).

However, if the VGA gain is switched fast enough and
with proper timing, it can avoid saturation and remain in its
linear mode-of-operation. For this, in Sec. III, we introduce
an additional time-delayed feedback loop that bounds the
amplitude of the multi-mode oscillations. This digital control
loop is specifically designed to parallel aspects of the guard
condition from Ref. 13, where it uses a switching condi-
tion\(^{23,24}\) that produces a digital-like waveform \( s(t) \).
40-GS/s oscilloscope (DSO80804A) and analyze separately the time series and spectral content of both signals. Along with our analyses, we explain the observed dynamics based on the system design.

A. Experimental time series

After initial transients, the system dynamics shows oscillations in $v(t)$ with a time-varying amplitude and digital-like switchings in $s(t)$. As shown in Fig. 4(a), the evolution of $v(t)$ is aperiodic and its amplitude remains approximately between $\pm 50 \text{ mV}$, within the VGA’s linear mode-of-operation. The amplitude modulation in $v(t)$ leads to spectral broadening of its power spectral density (PSD) with a peak at $\sim f_c$ and multi-mode power symmetrically distributed, as shown in Fig. 4(b). The small peaks in the PSD are spaced by $\sim 1/\tau_f$. From the peak at $f_c$, the PSD shows a 10-dB-bandwidth of $\sim 100 \text{ MHz}$ (consistent with the bandwidth of the LGA). Hence, the dynamics of $v(t)$ shows a UHF broadband spectrum due to aperiodic amplitude modulations.

The complexity of the dynamics in $v(t)$, which lies only in the modulation of its amplitude, has two different origins: (i) multi-mode feedback and (ii) long memory. First, noise seeds the transient oscillations of the system, which are amplified and filtered based on the system’s multi-mode characteristics (see the Appendix for the initial, transient dynamics). Then, these transient amplitude modulations are stored in the system’s memory as they propagate around the feedback loop and continuously affect the future dynamics at time $t + \tau_f$, causing the dynamics to reside in an infinite-dimensional phase space and allowing for the existence of chaos.27

Simultaneously, the control loop generates $s(t)$, a digital-like, asynchronous (unclocked) voltage. It switches aperiodically between its low and high states with minimum pulse widths of approximately 10 ns (see Fig. 4(c)). The finite bandwidth of $s(t)$ is visible on the experimental PSD spanning frequencies from dc to $\sim 100 \text{ MHz}$, the cutoff frequency at $-10 \text{ dB}$ (see Fig. 4(d)). The signal $s(t)$ also shows short-pulse rejection, a mechanism by which the TTL gate generates intermediate voltages in response to fast fluctuations at its input. As a result, our system shows slight deviations from ideal PWL behavior, discussed in Subsection IV B.

We finish this section by comparing the time series of $v(t)$ and $s(t)$. Unlike the audio-frequency PWL system presented in Ref. 13, there is no direct correlation between the analog and digital dynamics. This feature is unique to our system and is caused by the multi-mode feedback loop. In Subsection IV B, we construct two and three-dimensional (2D and 3D) projections of the system’s attractor. We demonstrate a relationship existing between $v(t)$ and $s(t)$, and we also examine the probability distribution of the feedback-loop gain to highlight the non-ideal behaviors of the circuit with respect to a perfectly PWL system.

B. Projection of experimental attractor

We examine projections of the system’s attractor using time-delay embeddings of the dynamics. In our system shown in Fig. 1, we choose the embedding time based on the relation

$$v(t) \sim g(t)v(t - \tau_f).$$

The approximate relation results from the filtering characteristics of the VGA. Similar to Ref. 13, we also discretize Eq. (2) and use the local maxima in $v(t)$, labeled as $v_n$, where $n$ is an integer incremented over each maxima that are spaced by $\sim 1/f_c \sim 0.5 \text{ ns}$. These local maxima contain the information about the system’s amplitude modulations. Within the interval $[t, t + \tau_f]$, there are on average $T_f = 87$ local maxima, and therefore, using the discrete embedding time $T_f$, we can construct attractor projections.

A 2D projection of the system’s attractor is represented graphically by examining the pairs $(v_{n-T_f}, v_n)$. Figure 5(a) shows the projection with two different clusters of points with finite widths and slopes that coincide approximately with the gains $g_{\text{Hi}}$ and $g_{\text{Lo}}$ (labeled accordingly in the figure). The linearity of these clusters is expected because of the relation $g_n = g(t_n)$ is the gain at discrete

FIG. 4. Experimental time series and PSDs. (a) Time series of $v(t)$ showing UHF oscillations with complex amplitude modulations. A zoom of $v(t)$ is shown for $t = 0.5 - 0.525 \mu\text{s}$. (b) PSD of $v(t)$. (c) Time series and (d) PSD of the switching state $s(t)$. The PSDs in (b) and (d) have a resolution bandwidth of 80 kHz and are smoothed over a window of 10 MHz. We note that the dynamics for this system is stable for extended periods of time, even when no thermal stabilization is used to prevent parameter drifts.
times $t_n$. The finite widths of the clusters relate to small over and under shoots in the values of the gain (due to filtering) and intermediate $s(t)$ values applied to $v_{c1}$.

In addition, depending on the state of $s(t)$, the map’s dynamics resides on one of its two clusters, respectively. Using $s_n = s(t_n)$, we construct a 3D projection of $(s_n-T_n, v_{n-T_n}, v_n)$ in Fig. 5(b). Every time $s_n$ switches, the dynamical state $(v_{n-T_n}, v_n)$ transitions from one cluster to the other, demonstrating that $s_n$ partitions the system’s attractor (See the Appendix for a preliminary analysis of the alphabet in the symbolic dynamics and its underlying grammar).

We also analyze the attractor by looking at the statistics of $g_n \sim v_n/v_{n-T_n}$. In Fig. 5(c), the probability density of the gain $P(g_n)$, is given as a function of $g_n$ and shows two primary peaks that occur at approximately $g_{L,H}$, the most probable values of $g_n$. There is also a higher probability to have $P(g_n) > 1$ because of the larger area under the $g_{L,H}$ peak, which indicates on average more growth than decay in the dynamics. The spread of the peaks is due to the finite bandwidth of the VGA, where $v_n$ cannot rise or fall infinitely fast. Finite bandwidth is a limiting property of all high-speed electronics, and is the main cause of non-ideal PWL behaviors in our system. We now verify these non-ideal behaviors using a continuous model of the dynamics.

V. PIECEWISE-LINEAR PHYSICAL MODEL

In this section, we model both the feedback and control loops using delay differential equations (DDEs). All of the model parameters are obtained from regression analysis using experimental data as detailed in Appendix.

A. Multi-mode feedback loop model

We model the feedback loop as a band-pass filter with time-delayed feedback and gain

$$\frac{\dot{v}(t)}{\Delta(t)} + v(t) + \frac{(\omega(0) L(t))^2}{\Delta(t)} \int v(t') \, dt' = g(t) v(t)$$

where $\Delta(t) = f(t)$ is the feedback-loop time delay and $\Delta(t)$, $\omega(0)$, and $g(t)$ are approximately piecewise-constant (APC) parameters that continuously switch. We note that Eq. (3) is completely linear for fixed $\Delta(t)$, $\omega(0)$, and $g(t)$, as we have defined the feedback loop without a nonlinear saturation term. Therefore, for $g(t) > 1$, Eq. (3) becomes unstable and $v(t)$ diverges. As we will discuss in Subsection V B, the control loop, which continuously switches the values of the APC parameters, is the only mechanism that keeps $v(t)$ from diverging (experimental saturation).

The first two APC parameters are modeled as $\omega(0)(t) = 2\pi \sqrt{f(t)(f(t) - f_{L}^{(+)})}$ and $\Delta(t) = 2\pi \Delta(t) = 2\pi (f(t) - f_{L}^{(-)})$, where $f(t)$ and $f_{L}^{(+)}, f_{L}^{(-)}$ are the upper (+) and lower (-) cutoff frequencies of the multi-mode band-pass filter ($-3$ dBm drop off), respectively. In the model, these cutoff frequencies shift depending on the switching state $s(t)$ as

$$f(t) = s(t) = \frac{\Delta(t)}{A}$$

where $\Delta(t) = g_{L,H}, \Delta(t) = \Delta(t) = 2\pi(f_{L}^{(+)}) - f_{L}^{(-)}$.

The last APC parameter is the gain $g(t)$, which is also dependent on $s(t)/A$ through the relation

$$g(t) = \frac{\Delta(t)}{A} (g_{L} - g_{H})$$

where $g_{H} = 1.5$ ($g_{L} = 0.2$) is the high (low) gain state. In our model, all of the APC constants have the same rise and fall times (instead of infinitely fast switches) that follow the physical switching state $s(t)$. This approximation captures some of the non-ideal behaviors of the experiment while maintaining approximate PWL dynamics.

B. Control loop model

The control loop is modeled using two separate nonlinear operators to approximate the LGA and TTL gate. The LGA first rectifies a delayed version of $v(t)/A$, provided by the feedback loop, with a logarithmic response in amplitude and saturation for high inputs using the nonlinear function

$$F_{LGA}(v(t) = C_{1} - C_{2} \left( \tanh \left( \log_{10} \left( \frac{v}{C_{3}} \right) \right) \right) + 1$$

where $C_{1}, C_{2}, C_{3}$ are parameters.
Experimental results. The output of this function drives a first order low-pass filter

\[ \zeta(t) = -2\pi f_L \zeta(t) + 2\pi f_L F_{LG}(v(t - \tau_c)), \]

(7)

where \( f_L = 0.02\pm 0.01 \) GHz is the low-pass cut-off frequency. The output \( \zeta(t) \) is smoothed to recover the envelope of \( v(t) \). This completes the model of the LGA.

The LGA output \( \zeta(t) \) then drives the TTL gate, which we approximate as a continuous nonlinear switching function

\[ s(t) = F_s(\zeta(t)) = \frac{A}{2}(1 + \tanh[m(\zeta(t) - \theta)]), \]

(8)

where \( A = 2.00\pm 0.04 \) V, \( I = 0.96\pm 0.01 \) V, and \( m = 51.00 \pm 0.01 \) V\(^{-1} \). Equation (8) switches with rise and fall times proportional to \( m \) and asymptotically approaches \( s(t) = 0 \) and \( s(t) = A \) for inputs \( \zeta(t) \) above and below the threshold \( I \), respectively (Recall that the value of \( s(t)/A \) is also used in Eqs. (4) and (5)).

C. Simulation results

1. Numerical time series

We integrate Eqs. (3)–(8) using a third-order Adams-Bashforth algorithm. The simulated time series and frequency spectra for \( v(t) \) and \( s(t) \) are plotted in Fig. 6, which mirrors the format of Fig. 4 for easy comparison with the experimental results.

As shown in Figs. 6(a) and 6(b), the central frequency of \( v(t) \) is \( \sim f_c \), and the complex amplitude modulations of \( v(t) \) are bounded between \( \pm 80 \) mV. Similar to the experiment, the PSD of \( v(t) \) shows a multi-mode power spectrum. The dynamics of the model is thus a good approximation to the experiment, producing a qualitatively similar UHF, multi-mode \( v(t) \).

Recall that the model does not include any saturation, and thus, in order for such dynamics in \( v(t) \) to remain bounded, the control loop must produce \( s(t) \) to switch the gain. In Fig. 6(c), \( s(t) \) is an asynchronous, digital-like waveform with some instances of short-pulse rejection due to the continuous nature of Eq. (8). Different from the experiment, the PSD of \( s(t) \) shows small oscillations in its amplitude. We hypothesize that these oscillations are due to high-frequencies in \( v(t) \) that bleed through the model’s filter for the LGA. A higher order low-pass filter in the model can suppress these oscillations but requires an increase in the model’s complexity.

2. Projection of simulated attractor

Next, we examine the attractor and PWL nature of the physical model by constructing time-delayed embeddings using \( v_n \), the local maxima of \( v(t) \). Figure 7(a) shows that a 2D embedding forms a projection of the system’s attractor with two linear clusters that show qualitative agreement with the experiment, although the clusters produced by the model are less scattered. These quantitative differences occur because our model approximates the system as an ideal two-pole band-pass filter with symmetric rise and fall times in the switching state. Nevertheless, the dynamics of the state \( (v_n - T_n, v_n) \) in the model transitions between these two clusters as \( v_n \) switches states, as shown in Fig. 7(b).

We also examine the simulated probability density \( P(g_n) \), given in Fig. 7(c). It shows two primary peaks located at \( \sim g_{LH} \) in qualitative agreement with the experimental results. However, these two peaks have smaller widths due to the assumptions of the model.

 Taken together, the time series analyses and attractor projections demonstrate that our PWL model reproduces the fundamental features of the experimental system. Next, by constructing a discrete map that captures all of the complexity of the continuous system, we characterize the stability of the experiment and model to approximate their maximal Lyapunov exponents.

VI. DISCRETE MAP REPRESENTATION AND MAXIMAL LYAPUNOV EXPONENTS

In this section, we construct a time-delayed discrete map to describe the dynamics of the slow-varying amplitude of
the UHF waveform of our system. As observed in Secs. I–V, the system’s dynamical complexity is not present in the fast oscillations of \(v(t)\) but rather in the slow modulation of the local maxima \(v_\phi\). Thus, we consider the map for the dynamics of \(v_\phi\), which is a discrete representation of the system. Using this map, we derive a simple expression for the continuous system’s maximal Lyapunov exponent.

To construct the map, we note that the evolution of \(v_\phi\) is determined by \(v_{\phi-T_f}\), a local maxima in the past with \(T_f\) being the average number of local maxima in \([t, t + \tau_f]\), the gain of the system \(g_n\), and the bandwidth \(\Delta_{LH}\) of the VGA. Since the local maxima follow the envelope of oscillations in \(v(t)\), the dynamics in \(v_\phi\) is described by a discrete low-pass filter\(^{30}\)

\[
v_{\phi+1} = v_\phi + x_c(g_n v_{\phi-T_f+1} - v_\phi),
\]

where \(x_c = \gamma / (\gamma + \Delta^{-1}) = 0.75, \Delta = (\Delta_L + \Delta_H)/2, g_n = (s_n/A) (g_L - g_H) + g_H\), and \(\gamma = 0.5\)ns is the approximate spacing between local maxima (\(\sim 1/\Delta\)). The switching state of the discrete map is given by \(s_n = F_{LG}(v_\phi)\) where \(F_{LG}\) is the output of a discrete LGA. Equation (9) represents the feedback loop and is approximately PWL.

For the control loop, we construct a discrete version of Eq. (9), which is also a discrete low-pass filter

\[
\xi_{\phi+1} = \xi_\phi + x_\xi(F_{LG}(v_{\phi-T_f+1}) - \xi_\phi),
\]

where \(x_\xi = \gamma / (\gamma + \omega_{LP}^{-1}) = 0.06\) and \(T_c = T_f - 2\) (chosen based on experimental measurements). Thus, Eqs. (9) and (10) with functions \(F_s(x)\) and \(F_{LG}(x)\) represent our time-delayed discrete map for the local maxima \(v_\phi\). We plot in Fig. 8 the evolution of the map, its 2D and 3D phase space projections, and the probability density \(P(g_n)\) of the discrete feedback-loop gain. These results show similar characteristics to those of the experimental system and its continuous model.

We construct the map’s finite-dimensional Jacobian \((T_f + 1 \times T_f + 1)\) to analyze the stability of a small perturbation \(\delta v_\phi, \delta \xi_\phi\) along the map’s trajectory \((v_\phi, \xi_\phi)\). Its general expression is

\[
J = \begin{pmatrix}
\frac{\partial v_{\phi+1}}{\partial v_\phi} & \frac{\partial v_{\phi+1}}{\partial v_{\phi-T_f+1}} & \ldots & \frac{\partial v_{\phi+1}}{\partial v_{\phi-T_f+1}} & \frac{\partial v_{\phi+1}}{\partial \xi_\phi} \\
\frac{\partial \xi_{\phi+1}}{\partial v_\phi} & \frac{\partial \xi_{\phi+1}}{\partial v_{\phi-T_f+1}} & \ldots & \frac{\partial \xi_{\phi+1}}{\partial v_{\phi-T_f+1}} & \frac{\partial \xi_{\phi+1}}{\partial \xi_\phi}
\end{pmatrix}
\]

The stability of the map’s trajectories is determined by the magnitude of the largest eigenvalue \(|\lambda|_{\max}\) of \(J\) averaged over the entire attractor. Because of the piecewise-linear nature of Eq. (9), the majority of terms in \(J\) are constant or zero, and thus the Jacobian reduces to a simple form:

\[
J = \begin{pmatrix}
A_1 & 0 & \cdots & 0 & A_2 & A_3 \\
0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 1 & \cdots & 1 & 0 & 0 \\
0 & \cdots & 0 & 1 & 0 & 0 \\
0 & \cdots & 0 & 0 & A_4 & A_5
\end{pmatrix}
\]

where \(A_1 = 1 - x_\xi, A_2 = x_\xi g_n, A_3, A_4, A_5\) are complex functions of the map properties. \(J\) can be simplified further by approximating \(A_3 \sim 0\) because it is only nonzero when \(s_n\) switches (\(\sim 5\%\) of points). With \(A_3 = 0\), the magnitudes of the eigenvalues \(\lambda\) of \(J\) decrease along the trajectories of the map when compared to the true Jacobian’s eigenvalues. Thus, in the context of calculating a maximal Lyapunov exponent, which involves only the eigenvalue of largest magnitude, setting \(A_3 = 0\) gives a conservative estimate.

Under this approximation, the eigenvalues \(\lambda\) satisfy the roots of characteristic polynomial

\[
(\lambda - A_3)(\lambda^{T_f+1} - \lambda^{T_f} + A_2) = 0.
\]

Based on the factorization of Eq. (13), one of the eigenvalues is always \(\lambda = A_5\). Along the map’s attractor, \(A_5\) remains at

FIG. 7. Simulated attractor projections. (a) 2D and (b) 3D projections of the simulated attractor constructed using \(s_{n-T_f}, v_{n-T_f}\), and \(v_n\), where \(T_f = 87\). A dashed line in (a) indicates a reference line with unity slope. (c) The simulated probability density \(P(g_n)\) for \(g_n\).
~0.94 V for all $n$. We determine the other eigenvalues by solving numerically

$$(1 - a_v)\dot{x}_{n+1} - \dot{x}_n + \epsilon g_n = 0,$$  \hspace{1cm} (14)

and we find that $|\lambda_n|_{\text{max}}$ switches approximately between two values: $\lambda_L = 0.982$ (locally stable) and $\lambda_H = 1.005$ (locally unstable) for $g_n = g_H$ and $g_n = g_L$, respectively. Since $\lambda_{L,H} > \lambda_{S}$, these values govern the local rate of maximum expansion.

Interestingly, if we allow $a_v \rightarrow 1$, which removes the dependence of $x_{n+1}$ on $x_n$ (the bandwidth becomes large and Eq. (9) reduces to Eq. (2)), then Eq. (14) becomes $\dot{x}^2 = g_n$. With this second approximation, which also yields a conservative estimate for $|\lambda_n|_{\text{max}}$, we determine an analytical expression $|\lambda_n|_{\text{max}} = (g_n)^{1/2T} \sim \lambda_{L,H}$ for $g_n = g_{L,H}$. Thus, as the gain switches, the system switches between stable and unstable trajectories.

Therefore, perturbations in the map’s trajectories will grow (decay) if the net-gain is greater (less) than unity. This growth and decay in $x_n$ is a one-dimensional projection of the stretching and folding of the attractor in the full $(T_f + 1)$-dimensional phase space. Surprisingly, it appears that such a low-dimensional projection is always aligned along the direction of maximum expansion.

Furthermore, based on the similarities when comparing the map’s dynamics to the local maxima and switching state in the experiment and model, we conjecture that our circuit’s gain switching is also a one-dimensional projection of its attractor’s stretching and folding (this assumes that the Jacobian derived from the map is a good approximation for the local flow of the dynamics of the local maxima extracted from the continuous system). Under this assumption, we now estimate the maximal Lyapunov exponent of the experiment, model, and map.

Given $|\lambda_n|_{\text{max}}$, which are local measurements, the maximal Lyapunov exponent $h_{\text{max}}$ for the trajectories of the system is given by

$$h_{\text{max}} = \lim_{N \rightarrow 0} \frac{1}{N} \sum_{n=1}^{N} \ln(|\lambda_n|_{\text{max}}),$$  \hspace{1cm} (15)

where $h_{\text{max}} > 0$ is an indication of chaos. Substituting $|\lambda_n|_{\text{max}} = (g_n)^{1/2T}$ into Eq. (15) yields

$$h_{\text{max}} = \lim_{N \rightarrow 0} \frac{1}{T_f N} \sum_{n=1}^{N} \ln (|g_n|).$$  \hspace{1cm} (16)

Assuming ergodicity of the dynamics, Eq. (16) is equal to an integral over the distribution of the system’s gain

$$h_{\text{max}} = \frac{1}{T_f} \int \ln(|g_n|) P(g_n) \, dg_n,$$  \hspace{1cm} (17)

where $P(g_n)$ is the probability density of the gain states given by Figs. 5c, 7c, and 8e for the experiment, model, and map, respectively. The resulting $h_{\text{max}}$ are reported in Table I, showing quantitatively similar, positive values for the experiment, model, and map (using surrogate data of equal length, we have verified that each $h_{\text{max}}$ is significantly positive).

This analysis highlights that our feedback system is fundamentally different from the system in Ref. 13, which only requires a one-dimensional map to solve for its maximal Lyapunov exponent. We acknowledge that a rigorous calculation of the maximal Lyapunov exponent for our continuous system requires a more extensive study to further strengthen
the validity of our approximations. However, we believe these calculations give a deep physical insight into the underlying mechanisms that lead to the chaotic behavior observed in our system.

VII. CONCLUSIONS AND FUTURE DIRECTIONS

In conclusion, we demonstrate a UHF electronic device displaying PWL chaos using feedback and control loops that overcomes many of the hindrances of slower LRC oscillators. We exploit a simple VGA with time delayed feedback to generate growing and decaying multi-mode oscillations centered at frequency $\sim 2.1$ GHz. Using matched time delays for the feedback and control loops, our circuit performs fast control and avoids nonlinear saturation. As for the dynamics, this novel circuit generates simultaneously a chaotic signal with a corresponding switching state that partitions the system’s attractor in 2D and 3D projections, thus, demonstrating the foundation for UHF PWL-chaos with a real-time, readily available symbolic dynamics.

Alongside our experimental system, we presented a continuous model and a discrete map to demonstrate the key aspects from the observed dynamics. We measure the circuit’s time-delays and frequency dependent band-pass characteristics and simulate the dynamics using delay differential equations to describe the continuous system in detail and by iterating the map to characterize the system’s complexity. These simulations use a logarithmic function, low-pass filters, and continuous switching functions to approximate the piecewise-constant parameters. Our model and map demonstrate a fundamental understanding of the system dynamics and give insight into the non-ideal effects arising in the experiment.

For a balanced comparison, we acknowledge that, unlike the circuit design by Coron et al.,\textsuperscript{13,22} our UHF system does not contain an exact analytical solution for the dynamics. In addition, the asynchronous nature of our switching state presents additional challenges for digital communications when compared to standard techniques.\textsuperscript{32} We are also currently investigating a matched\textsuperscript{13} (or pseudo-matched\textsuperscript{33}) filter for chaos that can recover $s(t)$ from $v(t)$ in the presence of large noise. The simplicity of our design combined with these benefits represent a significant step towards inexpensive and robust chaos-based radars.

Finally, we present only the simplest case for our system where $\tau_f$ and $\tau_c$ are approximately matched. Preliminary experimental and numerical work shows that additional mismatch between these two delays can give rise to new chaotic dynamics with differently structured PSDs. Thus, the delay mismatch represents an unexplored degree-of-freedom for controlling the dynamics and studying the stability of this system.\textsuperscript{25,34} Potential applications for this additional control include realizing a chaos communications system\textsuperscript{35,36} and creating orthogonal communications channels.\textsuperscript{37}

ACKNOWLEDGMENTS

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APPENDIX A: CONTROL LOOP CHARACTERIZATION AND LATENCY TIME

In this section, we characterize the three main components of the control loop: the LGA, the TTL gate, and the voltage control port $f_{ctl}$ of the VGA, in order to estimate model parameters and approximate each component’s CLT.

In Fig. 9(a), we plot a typical output waveform from the LGA. Using a signal generator, we drive the LGA with a 2 GHz sinusoidal signal $x(t)$ that is pulsed for 10 ns. When zero power is present in $x(t)$, the LGA outputs a positive voltage $\zeta(t) \sim 1.5$ V. For oscillations in $x(t)$, $\zeta(t)$ decreases to a value that is proportional to the amplitude of $x(t)$. Hence, the output of the LGA inversely follows the envelope of input oscillations. These time series are measured using two high-impedance probes with identical propagation times after calibration. Thus, the time skew between $x(t)$ and $\zeta(t)$ accurately represents the causal behavior of the LGA and shows that the CLT$_{LGA}$ is approximately 8 ns.

Using Eqs. (6) and (7), we drive our model for the LGA using $x(t)$ and fit the output $\hat{\zeta}(t)$ with the model parameters $C_i$ (for $i = 1, 2, 3$) and $f_i$ using a reduced $\chi^2$ algorithm. We note that during the fitting, the CLT of our LGA model is not 8 ns due to the choice of a low-order filter. Thus, extra delay was added into the model LGA’s CLT. This extra delay will be discussed at the end of this section. The resulting fit is shown in Fig. 9(b).

We next characterize the TTL gate using the output from the LGA when it is driven by $x(t)$. Using the same high-impedance probes, we measure and plot the TTL output $s(t)$ alongside its input $\zeta(t)$ in Fig. 9(c). In the figure, $s(t)$ switches high (low) for $\zeta(t)$ below (above) the TTL threshold of $\sim$1 V. We mark CLT$_{TTL} = 1.1$ ns, the average delayed response of the gate switchings (rise and fall) in response to a threshold crossing.

We use Eq. (8) to fit the TTL gate parameters $A$, $m$, and $I$. The simple model does not produce the correct CLT for the TTL gate, as an additional 1.1 ns time delay is necessary to achieve a good fit (see discussion at the end of this section). The fit is shown in Fig. 9(d). Finally, we

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Model</th>
<th>Map</th>
<th>Surrogate data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_{max}$</td>
<td>$8.1 \times 10^{-5}$</td>
<td>$5.3 \times 10^{-5}$</td>
<td>$7.0 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

TABLE I. Maximal Lyapunov exponents.
where $CLT \approx 40$ ns.

analyze the processing time of the voltage control port of the VGA in response to a signal from the TTL gate. To do so, we drive the VGA with a 2 GHz low-amplitude sinusoidal signal and observe the output as $s(t)$ switches to measure $CLT_{\text{vga}} \sim 4$ ns. Since the model for the gain of the VGA does not include a processing time, additional delay is necessary in the model.

Therefore, in order to match and model the full control loop delay time $\tau_c$, we use the relation $\tau_c = CLT + \tau_{\text{c},\text{oa}}$, where $CLT = CLT_{\text{LGA}} + CLT_{\text{TTL}} + CLT_{\text{vga}}$. We model the coaxial cable delay as a time shift that compensates for any CLT that is not present in our models for the LGA, TTL gate, or voltage control port. As a result, we tune the value of $\tau_c$ in the model to match that of the experiment, which is approximately 40 ns.

**APPENDIX B: MULTI-MODE TRANSFER FUNCTION AND FITTING**

We determine the value of $f_{LH}^{(+,-)}$ (used in Eq. (4)) using a fit to the experimental spectral responses shown in Figs. 2(b) and 2(c). To do so, we derive the transfer function for the setup shown in Fig. 2(a) using the differential equation

$$\frac{\dot{v}(t)}{\Delta_{LH}} + v(t) + \frac{(\omega_{LH}^{(o)})^2}{\Delta_{LH}} \int v(t') dt' = a_{LH}(v(t - \tau_f) + v_{\text{in}}),$$

(B1)

where $a_{LH} = g_o g_{\text{in}}$ is the attenuated gain (to keep the system from saturating) and $\Delta_{LH}, \omega_{LH}^{(o)}$ are the band-pass filter parameters for low (L) and high (H) gain states. We note that this equation is similar to that Eq. (3) with an additional input driving term $v_{\text{in}}$ and an attenuation $g_o$ in the feedback. We then Fourier transform Eq. (B1) and solve for $H_{LH}(f) = \tilde{v}/\tilde{v}_{\text{in}}$, the system’s transfer function, where $\tilde{v}$ and $\tilde{v}_{\text{in}}$ are the Fourier transforms of $v(t)$ and $v_{\text{in}}(t)$, respectively. The functional form of $H_{LH}(f)$ is

$$H_{LH}(f) = \frac{a_{LH} \Delta_{LH}}{2\pi f + \Delta_{LH} + \left[\frac{2\pi a_{LH} f}{\pi f} - a_{LH} \Delta_{LH} e^{2\pi i f \tau_f}\right]},$$

(B2)

where $\Delta_{LH} = 2\pi (f_{LH}^{(+)} - f_{LH}^{(-)})$, and $f_{LH}^{(+)}/f_{LH}^{(-)} = \sqrt{f_{LH}^{(+)}/f_{LH}^{(-)}}$.

Using the magnitude of $H_{LH}(f)$, we fit the experimental data and obtain the fits shown in Figs. 10(a) and 10(b).

The fitting parameters are $f_{LH}^{(+)} = 2.41 \pm 0.01$ GHz, $f_{LH}^{(-)} = 1.85 \pm 0.01$ GHz, $f_{LH}^{(+)} = 3.29 \pm 0.1$ GHz, and $f_{LH}^{(-)} = 1.97 \pm 0.04$ GHz. We note that the uncertainties are higher for the frequency parameters $f_{LH}^{(+/-)}$ due to approximation of the resonances in Fig. 2(c) using Eq. (B2). The experimental system is not a true band-pass filter for $v_{\text{ctl}} = v_{\text{in}}$, and our approximation leads to discrepancies between Figs. 2(c) and 10(b), but this approximation is necessary for our simple model. The fitted gain parameters are $a_{LH} = 0.6 \pm 0.01$ and $a_{LH} = 0.2 \pm 0.01$. Also, in order to facilitate fitting, $\tau_f$ is a free fitting parameter, where its fitted values are $\tau_f = 41 \pm 1$ ns.

**APPENDIX C: TRANSIENT DYNAMICS AND UNDERLYING GRAMMAR**

In this section, we compare the experiment, continuous model, and discrete map by examining their transient dynamics as well as the grammar of their potential symbolic dynamics. These comparisons provide more details about the dynamics that were not considered previously.

We plot typical transient dynamics for $v(t)$ and $s(t)$ in Fig. 11. Electrical and additive noise seed the experiment (Fig. 11(a)) and simulations (Figs. 11(b) and 11(c)), respectively, which show multi-mode growth in $v(t)$ and that $s(t)$ begins switching at $0.5 \mu s < t < 0.75 \mu s$.

Next, to demonstrate the existence of a potential grammar in the system, we examine four-bit digital words that appear sequentially in the switching state. We sample $s(t)$ at a clock frequency of 1 GHz from the experiment and continuous model and $s_m$ from the map (assuming time steps of $\gamma n$) and convert the signal into a list of ones and zeros, labeled $s_m$: for $s(t_m) > 1$, $s_m = 1$ and for $s(t_m) < 1$, $s_m = 0$, where $t_m$ are the clock sampling times. Next, we construct
the pairings [0,0], [0,1], [1,0], and [1,1] by pairing $[\sigma_m, \sigma_{m+1}]$, $[\sigma_{m+2}, \sigma_{m+3}]$, … and then follow the evolution of the pairs. We depict the sequences of four-bit words in Fig. 11, where a point is drawn between two pairings on the (x, y) axes if the four-bit word $([\sigma_m, \sigma_{m+1}], [\sigma_{m+2}, \sigma_{m+3}])$ exists in the alphabet. Lines connect sequential four-bit words.

In the analysis of the experiment and map shown in Figs. 11(a) and 11(c), there are missing words and missing links, showing restricted alphabets and grammars. From the continuous model, all words are present but with a restricted grammar. We note that the simulated grammars in the model and map are sensitive to the system’s parameters. Thus, we conjecture that mismatches with the experiment lead to the observed differences. Also, we acknowledge that overlapping lines cover missing links and thus a higher-dimensional representation is needed to fully characterize the grammar.

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