

# Supplementary Material to “Predictability and suppression of extreme events in a chaotic system”

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## I. CHAOTIC ELECTRONIC CIRCUIT

A single chaotic electronic oscillator used in our study is identical to that used by Gauthier and Bienfang in their study of attractor bubbling [27]. Here, we summarize the properties of the circuit for completeness, which consists of a negative resistor wired in series with a capacitor, which is coupled to an inductor-resistor-capacitor tank circuit through a nonlinear conductance, as shown schematically in Fig. S1. Its behavior is governed by the set of dimensionless equations (3-5) in the Letter, where  $V_{1j}$  and  $V_{2j}$  represent the voltage drop across the capacitors (normalized to the diode voltage  $V_d = 0.58$  V),  $I_j$  represents the current flowing through the inductor (normalized to  $I_d = V_d/R = 0.25$  mA for  $R = 2,345$   $\Omega$ ),  $g[V] = V/R_2 + I_r(\exp(\alpha_f V) - \exp(-\alpha_r V))$  (Eq. (6)) represents the current (normalized to  $I_d$ ) flowing through the parallel combination of the resistor and diodes, and time is normalized to  $\tau = \sqrt{LC} = 23.5$   $\mu$ s. The circuit displays “double scroll” behavior for  $I_r = 22.5$   $\mu$ A,  $\alpha_f = \alpha_r = 11.6$ ,  $R_1 = 1.30$ ,  $R_2 = 3.44$ ,  $R_3 = 0.043$ ,  $R_{dc} = 0.15$  (the dc resistance of the inductor), and  $R_4 = R_3 + R_{dc} = 0.193$ , where all the resistances are normalized to  $R$ . The state vector is given in terms of these variables via

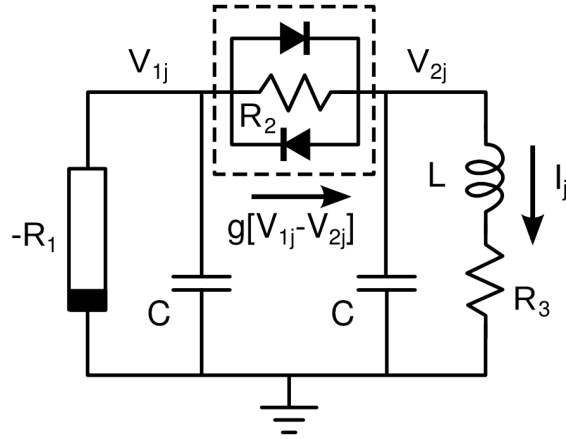


FIG. S1. **Chaotic electronic oscillator** with negative resistance  $-2,814 \Omega$ ,  $10 \text{ nF}$  capacitors,  $55 \text{ mH}$  inductor (dc resistance  $353 \Omega$ ) in series with resistance  $100 \Omega$ , and a passive nonlinear element (resistance  $8,067 \Omega$ , and back-to-back parallel diodes, type 1N914, dashed box).

the relation  $\mathbf{x}_j^T = (V_{1j}, V_{2j}, I_j)$ . The chaotic electronic circuits are coupled by measuring the differences of either  $V_{1j}$  or  $V_{2j}$  in each circuit with an instrumentation amplifier, converting to a current, and injecting the resulting signal to the slave node above one of the capacitors.

## II. LOCAL LYAPUNOV EXPONENTS

Attractor bubbling occurs when the largest transverse Lyapunov exponent  $\lambda_{\perp}$  is negative (the synchronized state is stable in the average sense), but there exist positive local transverse Lyapunov exponents  $\eta_{\perp}$  of invariant sets embedded in the chaotic attractor. The transverse Lyapunov exponents  $\lambda_{\perp}$  and  $\eta_{\perp}$  are determined from the variational equation

$$\frac{d\delta\mathbf{x}_{\perp}}{dt} = \{DF[\mathbf{s}(t)] - c\mathbf{K}\}\delta\mathbf{x}_{\perp}, \quad (\text{S1})$$

where  $DF[\mathbf{s}(t)]$  is the Jacobian of  $\mathbf{F}$  evaluated on  $\mathbf{s}(t)$ , and, for  $\lambda_{\perp}$ ,  $\mathbf{s}(t)$  is a typical orbit on the chaotic attractor, while, for  $\eta_{\perp}$ ,  $\mathbf{s}(t)$  is an ergodic orbit on the invariant set of interest embedded in the chaotic attractor. Determining the value of the largest Lyapunov exponent  $\lambda_{\perp}$  (or  $\eta_{\perp}$ ) involves integrating Eq. (S1) for a short time, renormalizing  $\delta\mathbf{x}_{\perp}$  to avoid computer overflow, and continuing the integration until the attractor (or the relevant invariant set) is largely covered.

For the purpose of understanding bubbling, it is the largest local (short-time) Lyapunov ex-

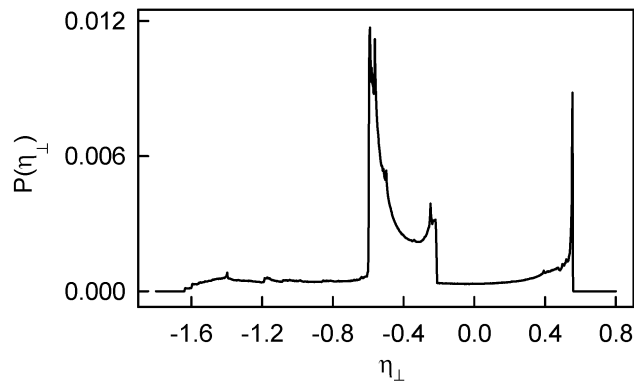


FIG. S2. **Probability distribution function of the short-time, or local Lyapunov exponents** for the coupled chaotic oscillators.

ponents that are of interest. It is the times when  $\eta_{\perp}$  is positive that the trajectory  $\mathbf{s}(t)$  is near an unstable set embedded in the attractor and a bubbling event is likely. Figure S2 shows the probability density for  $\eta_{\perp}$  using a short integration time of 0.005 dimensionless time units. We see that the distribution is very non-Gaussian at short times, and, most importantly, is sharply peaked around  $\eta_{\perp} \sim 0.55$ . Analysis shows that this large peak is associated with the unstable saddle embedded in the attractor at  $\mathbf{x}_{\perp} = 0$ . It is this unstable saddle that causes trajectories to transition between the double scrolls of the attractor.

The analysis so far assumes that the components of the master and slave electronic oscillators are identical and that there is no noise in the system. We took great care to match all components to better than 1%, and the noise level is small. In this case, it is useful to suppose the existence of an effective approximate invariant manifold, but these nonideal behaviors (i.e., noise and mismatch) are required to observe bubbling. When, at some time  $t_0$ , the trajectory gets close to an ideal unstable set (such as the unstable saddle), we can regard the nonideal effects as acting like a transverse initial perturbation, which subsequently leads the orbit to experience exponential growth according to

$$|\mathbf{x}_{\perp}| = |\mathbf{x}_{\perp}(t_0)| \exp[\eta_{\perp}(t - t_0)], \quad (\text{S2})$$

for  $t - t_0$  smaller than the characteristic time scale over which the trajectory remains within a neighborhood of the unstable set whose (local transverse) stability is given by  $\eta_{\perp}$ . This is the initiation of the bubble.

### III. THEORETICAL DERIVATION OF THE DISTRIBUTION

We now present a theoretical analysis for the observed power law exponent  $-2$ , and we also give an argument to explain why the distribution is truncated at a maximum event size and ends in a peak. In order to build the PDF of event sizes, let us follow the trajectories  $\mathbf{x}_M(t)$  of the master and  $\mathbf{x}_S(t)$  of the slave subsystems when they both start, at time  $t_0$ , in a volume of phase space close to the unstable, saddle-type fixed point at the origin. If the distance to the saddle-point is small enough, the dynamics inside this volume can be approximated by linearizing the evolution equations (Eqs. (3-5)) around the saddle-point. The structure of the linearized phase space depends on the values of the characteristic exponents of the saddle, which are obtained as the eigenvalues of the 3-D Jacobians for the master ( $DF[0]$ ) and slave ( $DF[0] - c\mathbf{K}$ ) subsystems: For the master subsystem,  $\gamma_{1M} = 0.517$  is the positive exponent;  $\gamma_{2M}$  and  $\gamma_{3M}$  are complex conjugates with negative real part,  $\gamma_{2M,3M} = -0.261 \pm 0.970i$ . For the slave subsystem,  $\gamma_{1S} = 0.491$ ,  $\gamma_{2S} = -0.422$ ,  $\gamma_{3S} = -4.47$  are all real. Each subsystem  $j = M, S$  has two eigenvalues with real part negative,  $\gamma_{2j}$  and  $\gamma_{3j}$ , which define a *stable plane*, a linearization of the stable manifold of the nonlinear dynamics close to the saddle-point. Transverse to the stable plane, there is an *unstable direction* that corresponds to the direction of the eigenvector associated to the positive eigenvalue  $\gamma_{1j}$ . The projection of each subsystem trajectory onto its respective stable plane decreases exponentially with an exponent  $\Re\{\gamma_{2M}\}$  for the master and at least  $\gamma_{2S}$  for the slave. The projection of each subsystem trajectory along its respective unstable direction grows exponentially with exponent  $\gamma_{1j}$ . This description is represented pictorially in Fig. S3.

Notice that the magnitudes of the unstable exponent  $\gamma_{1j}$  of the master and slave subsystems are nearly equal (5% difference). Thus, for simplicity, we take both positive exponents equal, and define  $\eta = \gamma_{1M} \cong \gamma_{1S}$ . The 1-D projections of the trajectories  $\mathbf{x}_M$  and  $\mathbf{x}_S$  along the unstable direction are denoted by  $u_M$  and  $u_S$ , respectively. The dynamics along these 1-D axes are depicted in Fig. S4. Written in terms of their initial center  $u_{\parallel}(t_0) = [u_M(t_0) + u_S(t_0)]/2$  and oriented half-separation  $u_{\perp}(t_0) = [u_M(t_0) - u_S(t_0)]/2$ , these projections obey the solutions of the unstable part of the linearized equations

$$u_M(t) = [u_{\parallel}(t_0) + u_{\perp}(t_0)] \exp[\eta(t - t_0)], \quad (\text{S3})$$

$$u_S(t) = [u_{\parallel}(t_0) - u_{\perp}(t_0)] \exp[\eta(t - t_0)], \quad (\text{S4})$$

and grow exponentially until the contribution of higher-order terms become significant in the

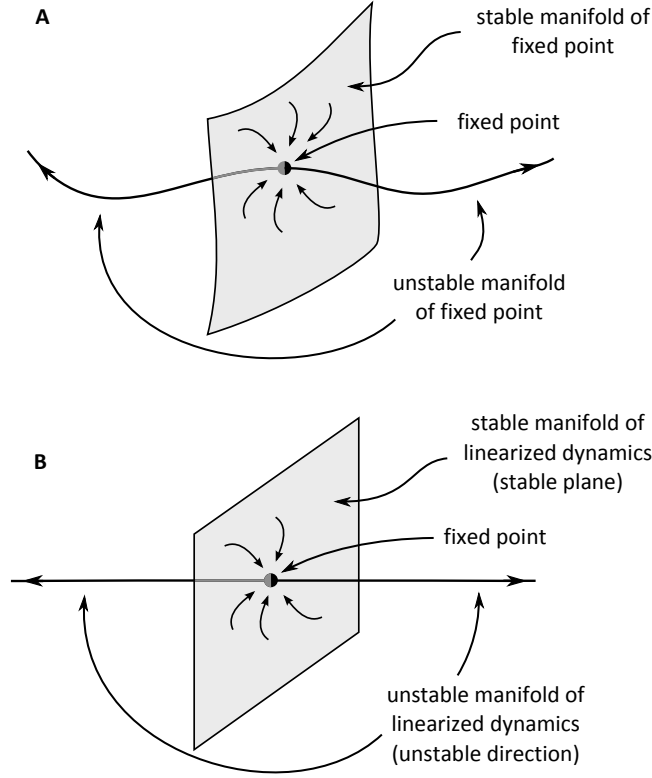


FIG. S3. **Manifolds of the fixed point** (A) for the nonlinear flow and (B) for the linearized dynamics.

dynamics and the linear approximation of the flow near the saddle-point is no longer valid. We call this latter phenomenon *saturation* of the exponential growth. After saturation, the trajectories can still grow at a different, nonexponential rate, and eventually are brought back by the nonlinear dynamics to a region near the saddle, in a process ending the burst away from the attractor that we call *rejection*. We want to calculate the distribution of the burst sizes  $B$ , where  $B$  is the maximum value of the separation between the trajectory and the invariant manifold during the burst.

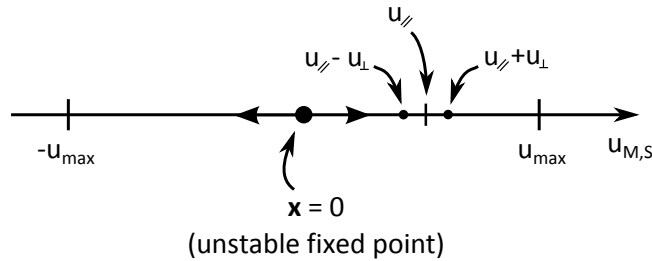


FIG. S4. **Linearized dynamics along the unstable direction.**

When the orbit is not in a bursting event, noise or parameter mismatch keeps  $|\mathbf{x}_{\perp}(t)|$  non-

zero. Thus, when the chaotic dynamics places the orbit near the fixed point at some time  $t_0$ , the linear approximation to the absolute separation between the real and ideal (no noise or mismatch) trajectories along the unstable direction is

$$\Delta(t) = 2|u_{\perp}(t)| = |u_M(t) - u_S(t)| \cong 2|u_{\perp}(t_0)| \exp[\eta(t - t_0)], \quad (\text{S5})$$

where  $|u_{\perp}(t_0)|$  is determined by the small noise and/or mismatch. In order to determine  $B$ , we neglect the components of the trajectories in the stable plane,  $|\mathbf{x}_{\perp}(t)| \cong |u_{\perp}(t)|$ , an approximation that is justified by the strong contraction of the linear flow along the stable plane — the stable component shrinks exponentially fast. We also assume that saturation happens when one of the trajectories  $u_M(t)$  or  $u_S(t)$  reaches a fixed distance  $u_{\max}$  from the stable plane, at time  $t_{\text{sat}}$ , either on the positive or on the negative side of the plane. That is, either  $|u_M(t_{\text{sat}})|$  or  $|u_S(t_{\text{sat}})|$  is equal to  $u_{\max}$ . With these approximations, the absolute values of  $u_M(t)$ ,  $u_S(t)$ ,  $u_{\parallel}(t)$ , and  $u_{\perp}(t)$ , all grow exponentially with the same exponent  $\eta$ , and the size of the burst that started at  $t_0$  is  $B = \Delta(t_{\text{sat}})$ .

To find an approximate value for  $\Delta(t_{\text{sat}})$  there are two cases to consider: i) when  $|u_{\parallel}(t_0)| > |u_{\perp}(t_0)|$ , the trajectories for the master and slave systems remain relatively close to each other, so they both saturate together and ii) when  $|u_{\parallel}(t_0)| < |u_{\perp}(t_0)|$ , the distance between the trajectories grows faster than their center and one of either the master or the slave trajectory saturates before the other one does. In case i), the trajectories are on the same side of the stable plane. Let us assume, without loss of generality, that the master trajectory saturates first, on the positive direction, hence,

$$B = \Delta(t_{\text{sat}}) = u_{\max} - [u_{\parallel}(t_0) - u_{\perp}(t_0)] \exp[\eta(t_{\text{sat}} - t_0)]. \quad (\text{S6})$$

Using the solution for the master unstable component, Eq. (S3), we can eliminate the time exponential in Eq. (S6)

$$\exp[\eta(t_{\text{sat}} - t_0)] = \frac{u_M(t_{\text{sat}})}{u_{\parallel}(t_0) + u_{\perp}(t_0)} = \frac{u_{\max}}{u_{\parallel}(t_0) + u_{\perp}(t_0)}, \quad (\text{S7})$$

to write

$$B = u_{\max} - u_{\max} \left( \frac{u_{\parallel}(t_0) - u_{\perp}(t_0)}{u_{\parallel}(t_0) + u_{\perp}(t_0)} \right), \quad (\text{S8})$$

$$B = \frac{2u_{\max}u_{\perp}(t_0)}{u_{\parallel}(t_0) + u_{\perp}(t_0)}. \quad (\text{S9})$$

Similar analysis for other possibilities in case i) gives

$$B = \frac{2u_{\max}|u_{\perp}(t_0)|}{|u_{\parallel}(t_0)| + |u_{\perp}(t_0)|}. \quad (\text{S10})$$

Notice that the maximum value of  $B$  in case i) corresponds to  $u_{\max}$ , when  $|u_{\perp}(t_0)| \lesssim |u_{\parallel}(t_0)|$ .

In case ii) the trajectories are always on opposite sides of the stable plane and the distance between them has a larger absolute value than their center, hence it grows faster (with the same exponent, the growth rate is proportional to the initial size) and both trajectories saturate nearly at the same time, one of them at  $u_{\max}$  and the other at  $-u_{\max}$ , so that their final separation is  $B = 2u_{\max}$ . If saturation happens close to the edge of the 3-D chaotic attractor,  $2u_{\max}$  is the maximum size possible for a burst. Hence, the PDF for the burst sizes in case ii) is roughly  $2u_{\max}$ , which can be interpreted as the narrow peak at the end of the distribution observed experimentally in Fig. 2. In principle, it is also possible that once one of the trajectories saturates it will quickly be reinjected, before the other one has had the time to reach saturation. In this situation the burst size is smaller than  $2u_{\max}$ , but larger than  $u_{\max}$  for trajectories that remain on opposite sides of the stable plane.

Using Eq. (S9) for the burst size, we find an approximate form for the distribution of bursts that are of the type described by case i) above ( $|u_{\parallel}(t_0)| > |u_{\perp}(t_0)|$ ). We write the probability density for a burst of size  $B$  as

$$p_B(B) = \iint \delta\left(B - \frac{2u_{\max}u_{\perp}(t_0)}{u_{\parallel}(t_0) + u_{\perp}(t_0)}\right) p_{u_{\parallel}(t_0)}[u_{\parallel}(t_0)] p_{u_{\perp}(t_0)}[u_{\perp}(t_0)] du_{\parallel}(t_0) du_{\perp}(t_0), \quad (\text{S11})$$

where  $\delta(\cdot)$  is a Dirac delta function,  $p_{u_{\parallel}(t_0)}[u_{\parallel}(t_0)]$  and  $p_{u_{\perp}(t_0)}[u_{\perp}(t_0)]$  are the probability densities, respectively, for the initial center and initial separation between the trajectories of the master and slave subsystems after reinjection. Equation (S11) is equivalent to

$$p_B(B) = \iint \frac{2u_{\max}u_{\perp}(t_0)}{B^2} \delta\left(u_{\parallel}(t_0) + u_{\perp}(t_0) - \frac{2u_{\max}u_{\perp}(t_0)}{B}\right) p_{u_{\parallel}(t_0)}[u_{\parallel}(t_0)] p_{u_{\perp}(t_0)}[u_{\perp}(t_0)] du_{\parallel}(t_0) du_{\perp}(t_0). \quad (\text{S12})$$

Assuming that the distribution of the center of reinjection is a smooth and slowly varying function of the distance to the plane, and recalling that the initial separation between the trajectories  $u_{\perp}(t_0)$  is small,

$$p_{u_{\parallel}(t_0)}\left(\frac{2u_{\max}u_{\perp}(t_0)}{B} - u_{\perp}(t_0)\right) \cong p_{u_{\parallel}(t_0)}(0), \quad (\text{S13})$$

in Eq. (S12), such that

$$p_B(B) \cong \frac{2u_{\max}}{B^2} p_{u_{\parallel}(t_0)}(0) \int u_{\perp}(t_0) p_{u_{\perp}(t_0)} [u_{\perp}(t_0)] du_{\perp}(t_0), \quad (\text{S14})$$

$$= \frac{2u_{\max}}{B^2} p_{u_{\parallel}(t_0)}(0) \bar{u}_{\perp}(t_0), \quad (\text{S15})$$

$$= CB^{-2}, \quad (\text{S16})$$

where  $\bar{u}_{\perp}(t_0)$  is the average value of the distance between reinjected trajectories, which is a constant, and  $C = 2u_{\max} p_{u_{\parallel}(t_0)}(0) \bar{u}_{\perp}(t_0)$  is also constant. This last equation (Eq. (S16)) gives the power law with exponent  $-2$  observed experimentally.

#### IV. NUMERICAL RESULTS

All the results shown in this article are experimental observations, for which the model discussed in the previous section is in excellent agreement. Here we show a comparison of the experiment and predictions of the model, obtained by numerical integration of Eqs. (3-5). In order to induce bubbling, we modified the parameters of the slave circuit by 1%. Figure S5 shows both numerical prediction and experimental observations of the temporal evolution of  $|\mathbf{x}_{\perp}|$ . Over a short time scale, we observe that the shapes of the individual pulses of desynchronization events are very similar in both theory and experiment, as shown in Fig. S5(a), while Fig. S5(b) shows that, over a longer time scale, the statistics of these pulses in the numerical simulation also appears to be similar to the experimental observation.

In Fig. S5 colored circles show the positions and values of the largest peaks  $|\mathbf{x}_{\perp}|_n$  detected within each desynchronization burst, which we define as the event-size. To simplify the algorithm to locate those points we run a search for local maxima over a smoothed time series, where a maximum is defined as a point which is higher than the 3 previous and the 3 consecutive points. Once a maximum higher than a certain limit ( $|\mathbf{x}_{\perp}| > 1$ ) is located, any maximum appearing shortly after (7 time units) is rejected, to prevent single burst from contributing with multiple event-sizes. This procedure is applied to both experimental and numerical data to generate the sequence  $|\mathbf{x}_{\perp}|_n$ .

With regards to the event-size distributions, there is great similarity between experimental observations and theoretical predictions, as shown in Fig. S6. In particular, the slope of the power law for intermediate-size events is  $-2.0 \pm 0.1$  for both the experiment and for the theoretical model. Furthermore, both distributions show pronounced dragon-kings. When controlling



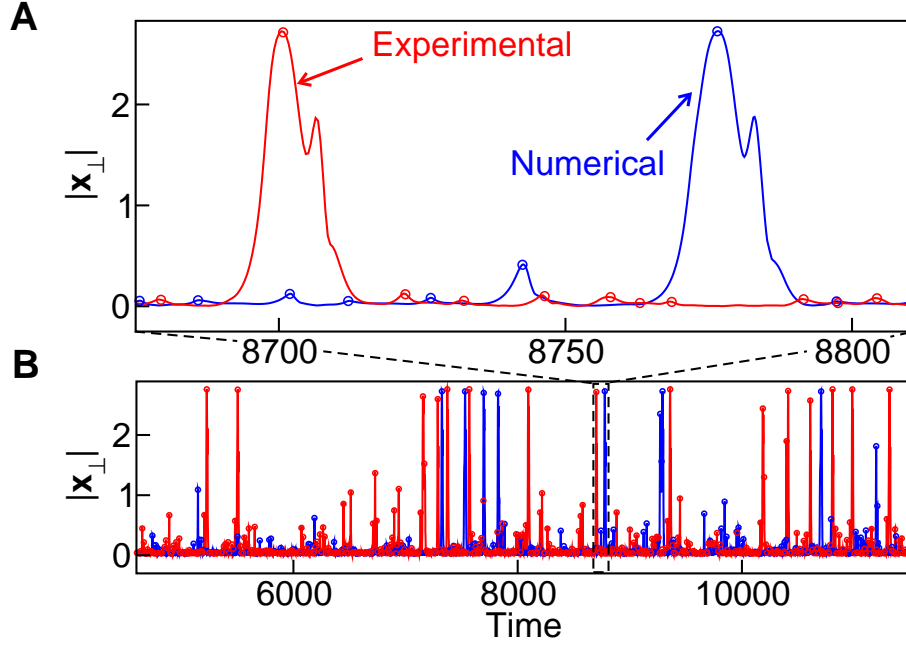


FIG. S5. **Comparison between experiment and the predictions of Eqs. (1–3)** **A**, Temporal evolution of  $|\mathbf{x}_\perp|$  showing the detail of a large bubble observed in the experiment (red) and resulting from the model (blue). Circles indicate the largest peaks  $|\mathbf{x}_\perp|_n$ , detected within each burst. **B**, The same time series viewed over a longer time scale. The values of the model parameters for the master oscillator are:  $V_d = 0.58 \text{ V}$ ,  $R = 2,345 \ \Omega$ ,  $I_d = V_d/R = 0.25 \text{ mA}$ ,  $I_r = 22.5 \ \mu\text{A}$ ,  $\alpha_f = 11.60$ ,  $\alpha_r = 11.57$ ,  $R_1 = 1.298$ ,  $R_2 = 3.44$ ,  $R_3 = 0.043$ ,  $R_{dc} = 0.150$ ,  $R_4 = R_3 + R_{dc} = 0.193$ , and for the slave oscillator are:  $V_d = 0.58 \text{ V}$ ,  $R = 2,345 \ \Omega$ ,  $I_d = V_d/R = 0.25 \text{ mA}$ ,  $I_r = 22.4 \ \mu\text{A}$ ,  $\alpha_f = 11.50$ ,  $\alpha_r = 11.71$ ,  $R_1 = 1.308$ ,  $R_2 = 3.47$ ,  $R_3 = 0.043$ ,  $R_{dc} = 0.152$ ,  $R_4 = R_3 + R_{dc} = 0.195$ ,  $c = 4.6$ ,  $K_{ij} = 1$  for  $i = j = 2$  and 0 otherwise, and  $c_{DK} = 0$ .

dragon-kings, we find that precise value of the cut-off in the distribution is quite sensitive to the value of  $|\mathbf{x}_M|_{th}$ . We therefore take this as an adjustable parameter in the model and obtain close agreement between the observations and the model when we take the threshold at 0.32 for the model.

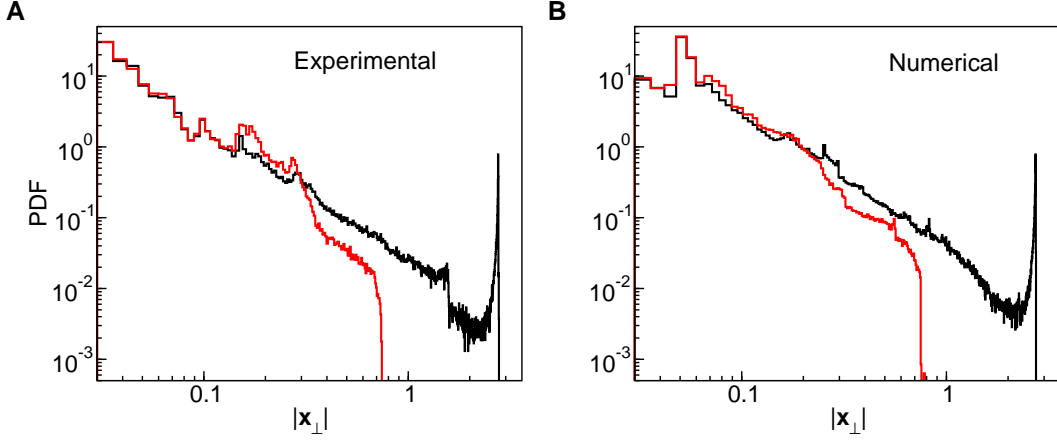


FIG. S6. **Comparison of the PDFs for the maxima of  $|x_{\perp}|$**  obtained (A) in the experiment and (bf B) numerical integration of the model (Eqs. (1–3)). The values of the model parameters are the same given in the caption of Fig. S5 except for  $c_{DK}$ , which is zero for the black curves and  $c_{DK} = 0.55$  for the red curves,  $|x_{M}|_{th} = 0.50$  in the experiment and  $|x_{M}|_{th} = 0.32$  in the model, and  $(K_{DK})_{ij} = 1$  for  $i = j = 1$  and 0 otherwise.

## V. TEST OF STATISTICAL CONVERGENCE

The probability density function (PDF) of the experimental event sizes was constructed by normalization of a histogram with a total of nearly one million events. These events will be referred here as the “full sample” and the number of events is the full sample size. We observed that histograms constructed with smaller subsamples, with one quarter of the total number, deviate only slightly from the histogram constructed from the full sample. An approximate functional form for the PDF of event sizes was derived in Sec. III. The hypothesis that we want to test here is that the values of the normalized histograms converge to some exact PDF which is not known from first principles with the increase of the number of points in the sample. This happens if each value of the histogram at a given bin is a random variable with finite average. To assess this convergence quantitatively we calculate the square of the residuals, defined as the difference between the PDF obtained with one of the subsamples and the PDF of the full sample, average over the samples and take the square root of this value, integrated over the whole domain of the PDFs, as an estimate of the total error in the probability calculated with one of the subsamples. The numerical value for this deviation in our case is 3%. This confirms that the reported PDF is well-defined and characterizes the stationary statistical properties of

the random variable  $|\mathbf{x}_\perp|_n$ . The normalized histograms constructed with 4 subsamples and the full sample that originated Fig. 2, and the square of the residuals and the cumulative square of the residuals are shown here in Fig. S7.

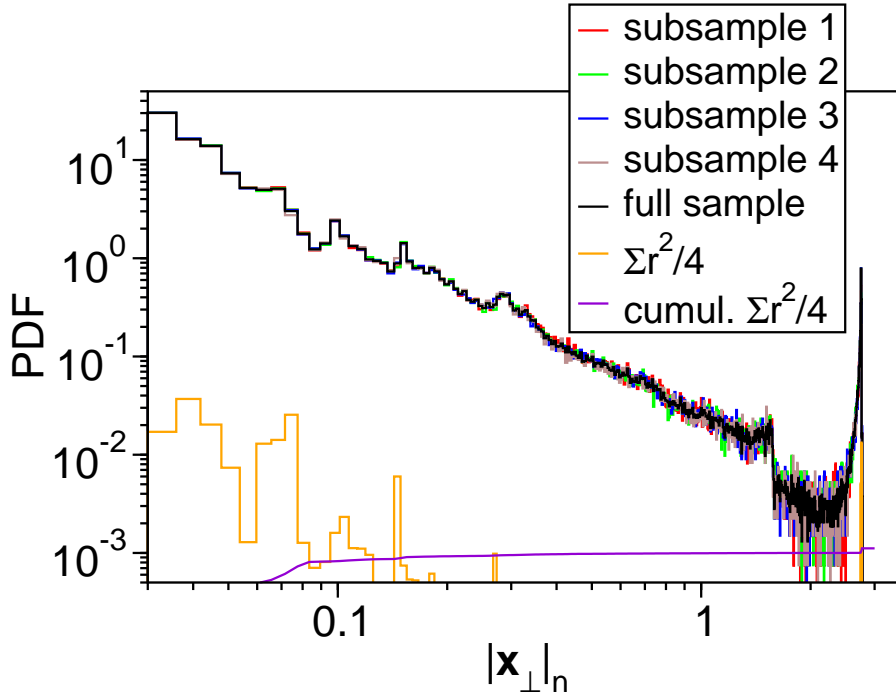


FIG. S7. **Probability Density Functions obtained with increasing sample size** converge to a stationary value. Here the full sample has  $N = 10^6$  points and produces the normalized histogram in black. The histograms constructed with 4 subsamples comprised of  $N/4$  points each deviate from the histogram of the full sample with a residual whose average square value is plotted in the orange curve. The cumulative mean-square deviation is shown the violet curve, ending in  $\sim 10^{-3}$ , which corresponds to a cumulative error in probability of 3%.

## VI. STATISTICAL TEST OF DISTRIBUTION

The theoretical analysis presented in Sec. III suggests that the distribution of burst sizes in our system follows a power law with exponent  $-2$ , within a certain range of burst sizes, and ends in a statistically significant deviation upward from a power law, that we identify as due to the dragon-kings. There are a number of statistical tests in the literature that aim to verify whether data observed experimentally is consistent with a given statistical model (see supple-

mental Refs. [31, 32]). The idea behind all those tests is to check the hypothesis (called the *null hypothesis*, in the language of statistics) that the empirical data was drawn from a reference distribution, prescribed by the model. We use the method known as Kolmogorov-Smirnov (K-S) test [31, 32], which is based on the maximum value of the absolute difference between the cumulative reference distribution and the cumulative empirical distribution (a normalized cumulative histogram constructed from the observed data). Figure 2 suggests the our data is nearly a power law for event sizes in the interval  $0.04 < |\mathbf{x}_\perp|_n < 1.80$ , Thus, in order to perform the K-S test, we first generate the normalized histogram of event sizes restricted to this interval. Our hypothesis is that, for the data restricted to this interval, the reference distribution is a truncated power law. Figure S8 shows the empirical distribution of the original, unrestricted data (black, the same curve shown in Fig. 2); the normalized histogram constructed using only the maxima that happen to fall within the range  $0.04 < |\mathbf{x}_\perp|_n < 1.80$  (blue); and the reference distribution (red) which is a truncated power law. The integral of each of these PDFs is normalized to unit probability, hence all these curves show equal area in linear scale. We construct our empirical histograms using 500 bins with uniform sizes in a linear scale, and we have no empty bins in the resulting histograms. It is usually suggested to use logarithmic bin sizes to avoid empty bins in heavy-tailed distributions, but this does not affect our case, because of the large amount of bins and data points. Using a least-squares fit to a straight line in log-log scale, we find the exponent of the truncated power law is always within the range  $-1.89$  to  $-2.14$ , for different choices of the endpoints of the interval of  $|\mathbf{x}_\perp|_n$ , consistent with the theoretical value  $-2$ . We fix the endpoints at the values given and use the exponent  $-2.0$  to define the reference distribution. To check that data drawn from the reference distribution passes the test, we produced synthetic data (false maxima) following the reference distribution and construct an “empirical” distribution from a sample of these false maxima with the same size  $N$  as the true data. The PDF of false maxima is shown in green in Fig. S8, where we see that it follows the reference distribution, but with large fluctuations that appear larger at the tail of the empirical distribution.

The one-sample K-S test uses the maximum absolute value of the vertical distance  $D$  between the empirical cumulative distributions functions (ECDF) and the reference cumulative distribution function (RCDF) as a statistic to decide whether the data was drawn from the reference distribution, thus, we proceed by integrating the empirical and reference PDFs up to size  $|\mathbf{x}_\perp|_n$  to generate the corresponding empirical CDF (ECDF) and reference CDF (RCDF), shown

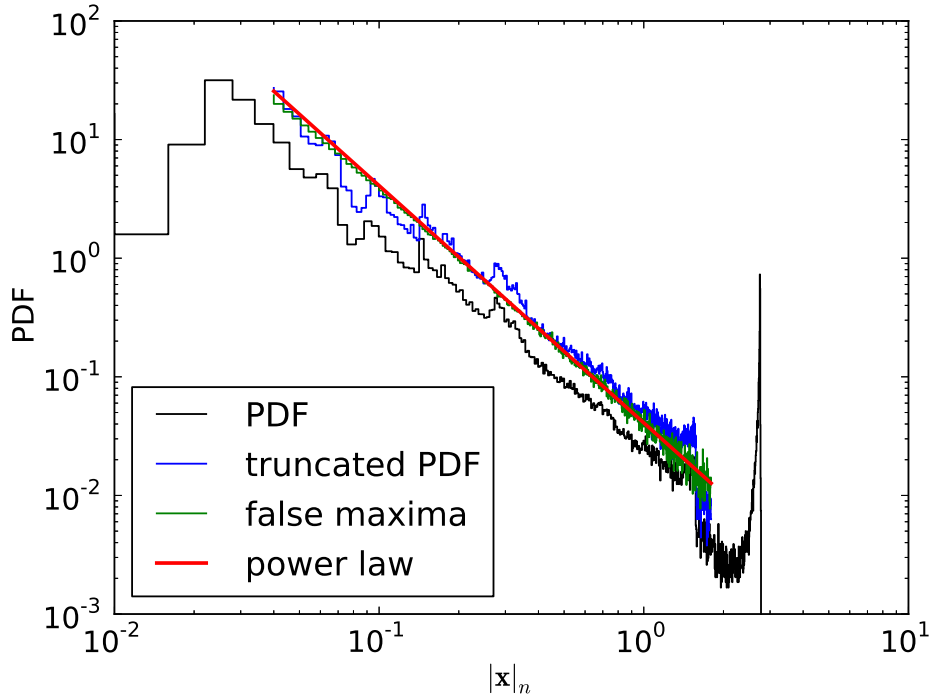


FIG. S8. **Probability Density Functions** for all the experimental event-sizes  $|\mathbf{x}_\perp|_n$  (black); for events only in the range  $0.04 < |\mathbf{x}_\perp|_n < 1.80$  (blue); the reference distribution (red), which is a truncated power law with exponent  $-2$ ; and false maxima (green), randomly drawn according to the reference distribution. The number of data points in the truncated distributions is  $N \cong 3 \times 10^5$ .

in Fig. S9. We see in Fig. S9 that the ECDF (blue) deviates significantly from the RCDF (red) in the range  $|\mathbf{x}_\perp|_n \in [0.1, 0.3]$ . The deviation is large enough that it enables one to reject the null hypothesis. The p-value for the distance  $D$  observed in our ECDF ( $\sqrt{ND} \sim 34$ ) is negligible (0.0). In Fig. S9 we also see that the distribution for false maxima follows the reference distribution very accurately ( $\sqrt{ND} \sim 0.72$ ) passing the test with p-value of 68%. Synthetic data generated from power laws with different exponents  $-1.5$  and  $-2.5$  are also rejected, with even larger values of the statistics:  $\sqrt{ND} \sim 124$  and  $\sqrt{ND} \sim 92$ , respectively. These results enable us to say that the bumps and valleys seen in the empirical distributions of the experimental data are not statistical fluctuations of a pure power law, but rather statistically significant structures in a more complicated distribution. The large peak at the end of the distribution of unrestricted data causes an even larger deviation.

Notice that these results do not invalidate the analytical model presented in Sec. III. The

purpose of this model is to give a qualitative picture of the phenomenon of bubbling with a solvable distribution. The approximations and hypothesis made, although good for most of the events, do not take into account details of the nonlinear dynamics happening far from the unstable saddle point, which are determinant for the details of the distribution.

In order to verify the consistency of our data with a power law we resample the raw data by decimating the sequence. The decimated sequence discards 700 maxima for each maximum kept. Thus we have only  $N \sim 10^3$  in the decimated sample. The K-S statistic for this decimated sequence is  $\sqrt{ND} \sim 1.3$ , which gives a p-value of 8%, thus passing the test. Synthetic data drawn from power laws with exponents  $-1.5$ ,  $-2.0$ , and  $-2.5$ , and the same sample sizes as the decimated data give statistics  $\sqrt{ND} \sim 5.3$ ,  $\sqrt{ND} \sim 0.60$ , and  $\sqrt{ND} \sim 3.7$ , respectively, which correspond to p-values  $1.9 \times 10^{-24}$ , 86%, and  $2.3 \times 10^{-12}$ , respectively. It is clear that only the data drawn from the power law with exponent very close to  $-2$  and our empirical data from the decimated sequence pass the test. The resulting cumulative distributions are shown in Fig. S10, where we can see that the typical fluctuations of the empirical CDF obtained with the decimated data have sizes similar to the deviations observed in CDF of the synthetic data, but the distributions with wrong exponent lie much above or below the reference distribution.

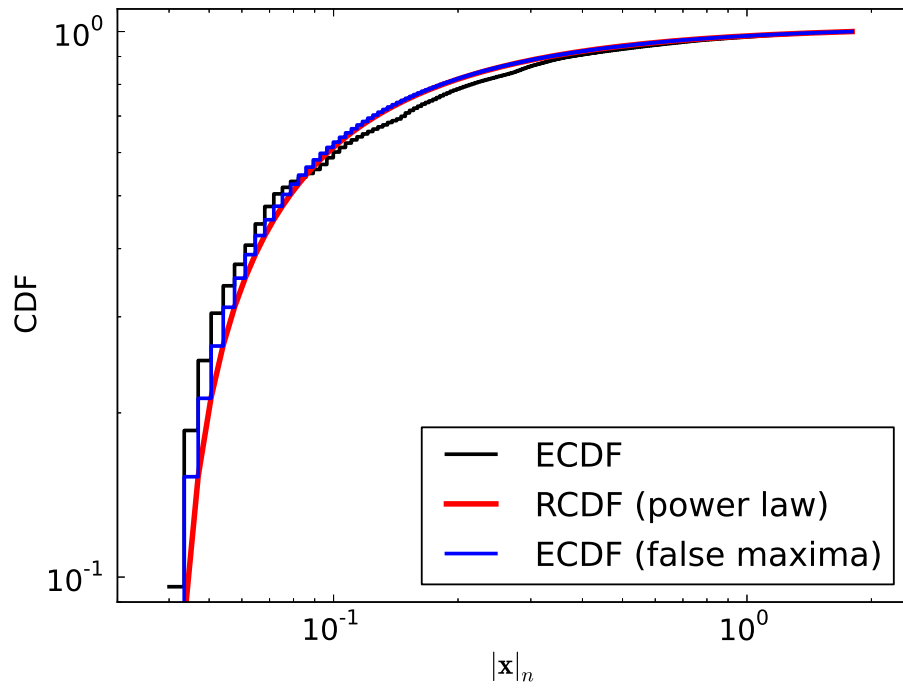


FIG. S9. **Cumulative Density Functions (CDFs)** for maxima in the truncated distribution limited to the interval  $0.04 < |x_{\perp}|_n < 1.80$  (black); for the reference distribution (red), which is a power law with exponent -2 truncated in this interval; for the artificially generated samples following the reference distribution (blue). The number of data points in the truncated distributions is  $N \cong 10^5$ .

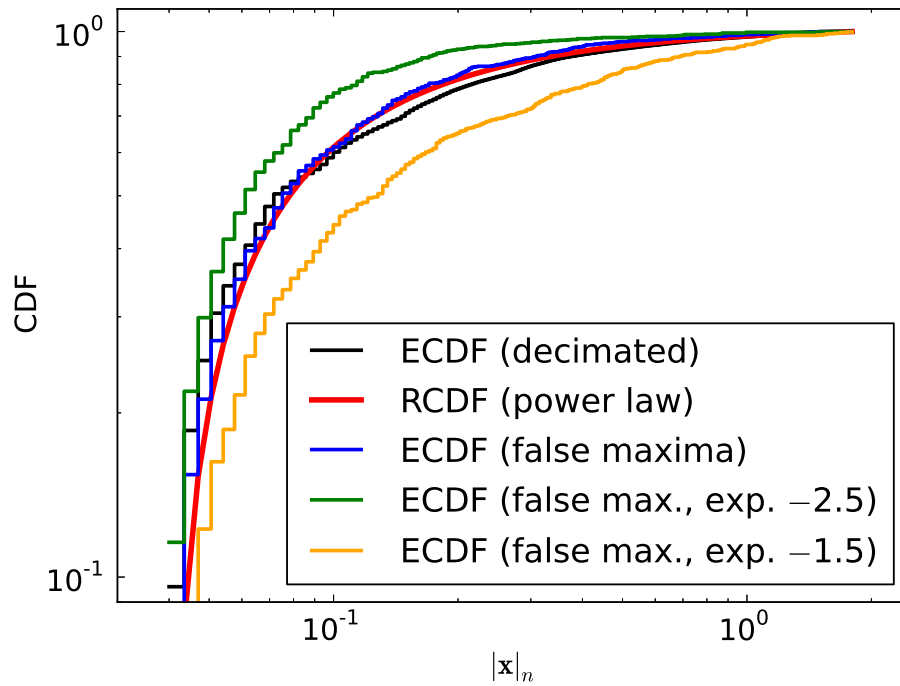


FIG. S10. **Cumulative Density Functions (CDFs)** for decimated maxima in the truncated distribution limited to the interval  $0.04 < |\mathbf{x}_\perp|_n < 1.80$  (black); for the reference distribution (red), which is a power law with exponent  $-2$  truncated in this interval; for the artificially generated samples following the reference distribution (blue); and for artificially generated samples following truncated power laws with exponent  $-1.5$  (orange) and  $-2.5$  (green). The number of data points in the truncated distributions is  $N \cong 10^3$ .



**ADDITIONAL REFERENCES**

[31] A. Clauset, C. R. Shalizi, and M. E. J. Newman, *SIAM Review* **51**, 661 (2009).

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