Polarization dynamics of two-photon and cascade lasers in the presence of an arbitrarily directed magnetic field

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Abstract

We study theoretically the polarization dynamics of a new type of quantum oscillator that is based on the two-photon stimulated emission process in the presence of a magnetic field of arbitrary orientation. Both cases of cascade (small intermediate-state atomic detuning) and two-photon (large atomic detuning) lasers are considered. The primary goal of this work is to investigate the origin of recently observed polarization instabilities in a two-photon laser (Pfister et al 2001 Phys. Rev. Lett. 86 4512) using a highly simplified model. It is found that the two-photon laser can emit linearly polarized radiation with its plane of polarization orthogonal to the direction of the magnetic field at small magnetic field strengths. It can also emit elliptically polarized radiation over a large range of magnetic field strengths and orientations. When the magnetic field deviates from a direction perpendicular to the laser cavity axis periodic instabilities can appear through a Hopf bifurcation. This dynamic regime could have contributed to the polarization instabilities observed in the experiment.

Keywords: Polarization dynamics, vector lasers, instabilities

1. Introduction

The two-photon laser [1–3] is a new type of quantum oscillator based on the two-photon stimulated emission process whereby two photons incident on an excited atom stimulate it to a lower energy state and two new photons identical to the incident ones are added to the light beam. The laser is expected to display unusual behaviour at both the microscopic [4] and macroscopic [5] regimes because the two-photon stimulated emission rate depends quadratically on the incident photon flux, resulting in an inherently nonlinear light–matter interaction. In particular, the unsaturated gain is proportional to the photon flux and thus the intracavity photon number undergoes a run-away process once the laser is brought to threshold, growing rapidly until the two-photon transition is saturated [6]. Therefore, the two-photon laser operates in a high-intensity saturated regime (a source of optical nonlinearity) even at the laser threshold, giving rise to the possibility that the laser will display dynamical instabilities [2, 5].

A recent breakthrough in two-photon laser research was reported by Pfister et al [7] where they used a multi-photon Raman scattering process in spin-polarized $^{39}$K atoms to achieve a large two-photon stimulated emission rate in the absence of competing nonlinear optical processes. Lasing was achieved by combining the Raman-driven potassium atoms with a high-finesse optical resonator. They observed that the laser displays complex dynamical instabilities in its state of polarization, which they attributed to the Zeeman degeneracy of the potassium atoms and competition between degenerate quantum pathways. The instabilities depended on the strength of an applied dc magnetic field and the laser possibly displayed chaotic behaviour for moderate field strengths of the order of 2 G. They suggested that the laser might be a bright polarization-entangled twin-beam source, but the potential
The primary goal of this paper is to investigate theoretically the origin of polarization instabilities observed in the two-photon laser by Pfister et al [7] using a highly simplified model. The analysis of the model will allow us to determine the structure of the phase space, the role of different bifurcations in the laser dynamics, the sensitivity of the dynamics on the presence and orientation of an externally applied magnetic field, and make predictions for future experiments. In general, we find that polarization instabilities exist in such a system and the laser behaviour is very sensitive to the orientation of the magnetic field. For the specific case of a dc magnetic field applied in a direction transverse to the cavity axis (the approximate orientation used in the experiment), we find that the generated laser beam is, in general, elliptically polarized (EP). For a sufficiently strong transverse magnetic field, all instabilities are suppressed and the beam is linearly polarized (LP) with its polarization vector parallel to the magnetic field. In addition, we find that the nonlinear anisotropy induced by a small transverse magnetic field can favour the generation of a LP beam where the polarization vector is perpendicular to the magnetic field. This latter finding is analogous to the behaviour observed in early experiments with He–Ne lasers [8], although the microscopic origin of the stimulated emission process for He–Ne lasers is very different than that of the two-photon and cascade lasers considered here.

To set the stage for the development of our simplified model of the two-photon and cascade Raman lasers, we briefly summarize the mechanisms giving rise to two-photon stimulated emission. In the experiment by Pfister et al [7], potassium atoms are illuminated by an intense ‘driving’ field that induces a multi-photon process involving simultaneous absorption of two drive-field photons and stimulated emission of two cavity photons. By tuning the drive-field frequency close to the potassium $D_1(2S_{1/2} \rightarrow 2P_{1/2})$ transition, the two-photon stimulated emission process can be selectively enhanced while suppressing competing nonlinear optical processes. Such a composite atom–field system is a breakthrough in two-photon laser research, but complicates accurate theoretical analysis of the laser. In addition, the laser-driven potassium system is exceedingly rich because of the degenerate magnetic sublevels participating in the interaction. For example, figure 1 shows the possible multi-photon interactions connecting the initial atomic state (a $|F = 2, M = 2\rangle$ state of the ground-level manifold) with the three possible final states (states $|F = 1, M = 0\rangle$ and $|F = 1, M = +1\rangle$ and $|F = 1, M = -1\rangle$ of the ground-level manifold). It is seen that there are multiple final states and multiple quantum pathways that exist because of the geometry of the experimental set-up. Since the magnetic field was directed nearly orthogonal to the cavity axis and the optical resonator was quasi-isotropic, all pathways contribute to the laser behaviour and must be considered in a theoretical analysis of the laser dynamics.

While our eventual goal is to develop a complete model of the scattering processes shown in figure 1, we find that their complexity defies a simple interpretation of the results. As a step toward a better understanding of the physical situation, we consider a simplified model of the two-photon stimulated emission process that allows for multiple final laser states and multiple states of polarization of the photons scattered into the cavity modes as shown in figure 2(a), but does not take into account the drive-field photons. Such a simplified model is plausible because it is known that the multi-photon scattering processes shown in figure 1 can be viewed in the dressed-state basis whereby the effects of the pump-field photons are accounted for and the generated laser beam is, in general, elliptically polarized (EP). Such a composite atom–field system is a first step toward the development of a detailed understanding the dynamic behaviour of the laser system. To our knowledge, the only reported vector laser model where also two generated fields are involved is the cascade $J'' = 0 \rightarrow J = 1 \rightarrow J' = 0$ laser, which was studied in [9, 10]. In [9] the polarizations of the two fields were kept fixed and thus they were not described by dynamical variables. In [10] the polarization degrees of freedom were taken into account. We note that benefit can also be taken of comparisons with the case of optically pumped $J'' = 0 \rightarrow J = 1 \rightarrow J' = 0$ [11] and $J'' = 1 \rightarrow J = 0 \rightarrow J' = 1$ [12] vector laser models, where the first photon is a drive photon instead of a generated photon.

2. A laser in the presence of a magnetic field

One of our goals is to determine the importance of an applied dc magnetic field and its orientation relative to the cavity axis on the laser polarization states and the laser dynamics, so we must take into account all transitions shown in figure 2(a) and the Zeeman shift of the atomic energy sublevels. In general, researchers have been interested in the effects of magnetic fields on laser dynamics since the early 1960s (see [13] and references therein) because of the fundamental aspects of the problem and because it provides an effective way to control the laser dynamics.
Figure 2. Level scheme and field components for the cascade laser in (a) atomic coordinate system and circular (Zeeman) basis \((x' - iy')/\sqrt{2}, (x' + iy')/\sqrt{2}, z'\), and (b) atomic coordinate system and rectangular (Cartesian) basis \(x', y', z'\). The sublevels \([-\rangle, [2]\) and \([+\rangle\) are degenerate if no magnetic field \((\Lambda = 0)\) is applied.

There are two ways the magnetic field can act. In one case the magnetic field creates anisotropy in the gain medium, which generally depends on the laser field intensity, polarization azimuth and ellipticity, and is usually referred to as a problem of the Zeeman laser. In the other case the magnetic field is applied to an absorbing (passive) intracavity element (which is not the case of the present work), giving rise to what is called the Faraday laser. The difference between these devices is that in the latter case the anisotropy induced by the magnetic field is independent of the laser field parameters.

Most previous research has focused on Zeeman lasers, primarily for the case of a longitudinal magnetic field. Since only \(\sigma_+\) and \(\sigma_-\)-polarized photons contribute to the cavity field in this situation, the model is essentially identical to the case of a laser in the absence of a magnetic field, considerably simplifying the analysis. Much less studied are lasers in which the magnetic field is not parallel to the cavity axis so that all the dipole transitions \((\Delta M = \pm 1, 0)\) are allowed. An overview of the experimental and theoretical research on Zeeman lasers, along with extensive theoretical analysis of the case of a transversely oriented magnetic field, can be found in [13]. In these studies, it is found that a non-parallel magnetic field can result in selection of a single LP mode \([8, 13–15]\).

This is in contrast to what has been found for the case of lasers subject to the action of a longitudinal magnetic field, where only circularly polarized (CP) \(\sigma_+\) or \(\sigma_-\) modes are emitted. Moreover, while a large non-parallel magnetic field can favour generation of a LP beam where the polarization vector is parallel to the magnetic field, a small magnetic field can favour generation of a LP beam where the polarization vector is orthogonal to the magnetic field \([8, 13, 15]\).

These phenomena have been found in He–Ne lasers with different types of lasing transitions. In particular, Culshaw and Kannelaud predicted theoretically the preference of a laser to generate a LP field propagating along the cavity axis and aligned parallel or orthogonal to the magnetic field for the case of a \(J = 0 \rightarrow J' = 1\) transition and neglecting anisotropy of the cavity. They also demonstrated this preference experimentally in a He–Ne laser operating on a \(J = 1 \rightarrow J' = 2\) at a wavelength of \(\lambda = 1.153\ \mu m\) [8]. The effect of the cavity anisotropy and detuning was addressed rigorously for a \(J = 0 \rightarrow J' = 1\) transition as well as for transitions involving larger angular momentum quantum numbers in \([13, 15]\).

Heer et al. [14] studied the case of arbitrary orientation of the magnetic field for ring lasers and amplifiers. In [16], a theory for a weakly anisotropic, multi-polarization-mode ring laser subject to the action of an arbitrarily directed magnetic field was developed.

One feature of most of these studies is that they concentrated on lasers where the population and coherence decay rates of the atoms comprising the gain medium were much larger than that of the laser field. The scalar counterpart of such lasers are the so-called class A lasers [17], which are the simplest from the viewpoint of nonlinear dynamics because all material variables can be eliminated adiabatically. In contrast, more complex behaviour is expected from class C lasers for which all material dynamics must be retained. In this work, we consider the most general case of class C lasers because the population, coherence and laser field decay rates are of the same order of magnitude in the experiments of Pfister et al. [7] and hence adiabatic elimination of the material dynamics can be hardly justified.

In this paper, we generalize the approach of [14, 15] to class C devices, and in addition we enlarge the model to take into account the fact that two atomic transitions in a cascade scheme, instead of a single transition, can generate photons as shown in figure 2(a). We will study the cascade and two-photon \(J'' = 0 \rightarrow J = 0 \rightarrow J' = 1\) lasers with arbitrary orientation of the applied magnetic field. Possibly, the presence of the upper field in figure 2(a) will influence the intensity and polarization dynamics of the lower field.

We will use two different coordinate systems to analyse the laser dynamics: one aligned with the cavity axis (‘cavity coordinates’) for describing the propagation of light within the cavity via the wave equation, and the other aligned with the magnetic field \(B\) (‘atomic coordinates’) for describing the light–matter interaction via the density matrix equations. Our strategy is to express the field in the cavity coordinates, transform the electric field vectors to the atomic coordinates to analyse the effects of the field on the atoms and determine the induced polarization of the medium, and transform the induced polarization vector back to the cavity coordinates to analyse the effects of the atoms on the cavity field.

The remainder of this paper is as follows. In section 3, the theoretical approach taking into account the effects of a magnetic field of arbitrary orientation on the dynamics of a class C cascade and two-photon \(J'' = 0 \rightarrow J = 0 \rightarrow J' = 1\) laser (see footnote 4) is presented. Section 4 is devoted to the study of the laser in the off-state and its stability, and the behaviour of the lasing steady states is investigated and discussed in section 5. Main results are summarized in section 6.

3. Theory for arbitrary magnetic field in a class C cascade laser

As discussed above, two different coordinate systems are used to explore the dynamics of the two-photon laser; one relates to

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\[ J'' = 0 \rightarrow J = 0 \rightarrow J' = 1 \]

For the case of a...
The circular basis as |−⟩ ≡ |−⟩ and denoted as |−⟩ and |⟩ instead of the circular basis. If the magnetic field is parallel to the z-axis, the Cartesian basis in the atomic coordinate system coincides with the Cartesian basis in the cavity coordinate system, which we define as x, y, z (figure 3).

As indicated, when the laser system is subject to the action of a magnetic field B, the quantization axis will be chosen to be parallel to B (atomic coordinate system). When the magnetic field is not directed along the cavity axis, transitions with ΔM = ±1 and 0 are allowed in the electric-dipole approximation. Hence, the laser can emit P- and R-polarized photons in such a case (see figure 2(a)). In this work we will consider that the strength of the magnetic field is small (Zeeman shift is much less than the gain-profile width). This means that the frequencies of all the lower fields E_2, E_2', E_4, associated with transitions |J = 0, M = 0⟩ ↔ |J′ = 1, M = 0⟩, |J = 0, M = 0⟩ ↔ |J′ = 1, M = +1⟩, respectively, can be taken to be equal, see figure 2(a). In such a case E_2' and E_2' will represent CP components of a new field. We can then define new atomic states x' ≡ (|−⟩ + |+)}/√2 and y' ≡ i(|−⟩ − |+⟩)/√2 and the fields polarized along x' (E_x' = (E_2' + E_4')/√2) and y' (E_y' = i(E_2' - E_4')/√2) axes in the atomic coordinate system and Cartesian basis x', y', z' (figure 3). The energy level diagram of the atomic system in this basis is depicted in figure 2(b).

When the magnetic field is orthogonal to the cavity axis, the propagation vector of the electric field is perpendicular to B. Choosing the y' axis along the cavity, the field component E_y' is equal to zero because the laser field is transverse. Hence, it is possible to simplify the problem in this case by using the Cartesian basis (x', y', z') for the field and the gain medium instead of the circular basis. If the magnetic field B is strictly orthogonal to the cavity axis, the Cartesian basis in the atomic coordinate system coincides with the Cartesian basis in the cavity coordinate system, which we define as x, y, z (figure 3). Unfortunately, misalignment of the magnetic field forces us to use two different coordinate systems. Moreover, the gain medium response is given most conveniently in the atomic coordinate system and the electric field in the cavity coordinate system. Hence, to obtain a self-consistent system of Maxwell–Bloch equations for the general case, one of the two strategies outlined in section 2 must be employed. We find that the most straightforward approach is to use Cartesian basis states in the cavity coordinate system so that only x and z LP components of the cavity field are needed.

Next, let us construct the Hamiltonian describing the atoms. The unperturbed (atomic) and interaction (atom + field) parts of the Hamiltonian in the atomic coordinate system and circular basis (figure 2(a)) take the following form:

$$\hat{H}_{\text{cir}}^{\mu} = \hbar \begin{pmatrix} \Omega_2 & 0 & 0 & 0 \\ 0 & \Omega_0 & 0 & 0 \\ 0 & 0 & \Omega_2 + \Lambda & 0 \\ 0 & 0 & 0 & \Omega_2 - \Lambda \end{pmatrix},$$

$$\hat{H}_{\text{int}}^{\mu} = \begin{pmatrix} \bar{F}_{02} & 0 & 0 & 0 \\ -\bar{F}_{3} & 0 & 0 & 0 \\ -\bar{F}_{20} & 0 & 0 & 0 \\ -\bar{F}_{40} & 0 & 0 & 0 \end{pmatrix}.$$  \(1\)

Here, \(\Lambda = \mu_B G L |B|/\hbar\) is the frequency splitting between the sublevels |+⟩ and |−⟩ induced by the magnetic field B, \(\mu_B\) is the Bohr magneton, \(G L\) is the Landé factor, \(\Omega_2\) is the frequency of level |i⟩ and \(\Omega_{ij} = \Omega_i - \Omega_j\) (see figure 2(a)). Parameters \(F^\prime\) are proportional to the Rabi frequencies, \(F^\prime = E^\prime \mu^\prime/\hbar\). The electric field vector E' associated with the lower transition in figure 2(a) and the electric-dipole moment operator \(\mu^\prime\) can be conveniently represented in the atomic coordinate system and circular basis as \(\mu^\prime = \mu^\prime (E_2' + E_4'\sqrt{2}) + \mu^\prime (E_2' - E_4'\sqrt{2})\) and \(E' = E_x' (x' - iy')/\sqrt{2} + E_y' (x' + iy')/\sqrt{2} + \mu^\prime z' + c.c.,\) where c.c. denotes the complex conjugate.

Transforming the Hamiltonian to the Cartesian basis in the atomic coordinate system\(^5\), the interaction part of the

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\(^5\) In general, this procedure presents a problem because it is not always possible to define |x'⟩ = (|+⟩ - |−⟩)/√2 and |y'⟩ = i(|−⟩ - |+⟩)/√2 even when frequencies of the CP fields are equal \(\omega_{ij}\). Fortunately, this is not the case for our system, in which |−⟩ and |+⟩ states can be combined giving |x'⟩ and |y'⟩ states.
Hamiltonian takes the form

\[ \hat{H}^{\text{cart}} = \frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \times \begin{pmatrix} \phi_f \\ \phi_f \\ \phi_f \\ \phi_f \end{pmatrix} \]

where \( F_{\pm} = \frac{(F_{\pm}^{(1)} + F_{\pm}^{(2)})/\sqrt{2}}{\sqrt{2}} \) and \( F_{\pm}^{(1)} = \frac{i(F_{\mp} + F_{\pm})/\sqrt{2}}{\sqrt{2}} \). Note that the form of this matrix is identical to that in the Zeeman basis.

On the other hand, the unperturbed Hamiltonian \( \hat{H}^{\text{cart}} \), in which only the diagonal elements are non-zero in the circular basis and atomic coordinate system becomes non-diagonal in the Cartesian basis when the magnetic field is non-zero and is given by

\[ \hat{H}^{\text{cart}} = \begin{pmatrix} h\Omega_4 & 0 & 0 & 0 \\ 0 & h\Omega_0 & 0 & 0 \\ 0 & 0 & h\Omega_2 & 0 \\ 0 & 0 & 0 & -i\hbar\Lambda \end{pmatrix}. \]  

It follows from (3) that there is no simple method for drawing an energy level diagram for the case when the fields interact with the atom described in the Cartesian basis in the presence of the magnetic field. If a magnetic field \( B \) is applied to the atoms along the \( z' \) axis, the sublevels \( \{+\} \) and \(-\) are shifted in energy with respect to sublevel \( 2 \) by an amount \( \pm\Lambda \), respectively (figure 2(a)), while the Cartesian sublevels \( \{x'\} \) and \( \{y'\} \) are a linear combination of \( \{+\} \) and \(-\) sublevels.

In the cavity coordinate system, the electric fields can be naturally expressed as

\[ E_{m}(y, t) = \hat{E}_m \rho_{m} M_{m} \eta_{m}(t) \exp[i(k_{m} y - \omega_{m} t - \phi_{m}(t))] + \text{c.c.}, \]

\[ E_{k}(y, t) = \hat{E}_k \rho_{k} M_{k} \eta_{k}(t) \exp[i(k_{k} y - \omega_{k} t - \phi_{k}(t))] + \text{c.c.}, \]

where \( m = x, y, z \); \( \mu \) are the electric-dipole matrix elements which can be taken real; \( \eta_{m}(t) \) are the involute of the (real) Rabi frequencies, \( k \) and \( \omega \) are the wavenumbers and reference frequencies for the laser fields, respectively, and \( \phi_{m}(t) \) are the phases of the laser fields so that \( \omega_{m} + \phi_{m}(t) \) represent the instantaneous laser field frequencies. To proceed further, we transform the electric fields into the atomic coordinate system and Cartesian basis, which is accomplished using a three-dimensional (3D) rotation. The most useful description of 3D rotation is in terms of Euler angles. Because of invariance of the problem under a 2D rotation in the plane orthogonal to the magnetic field, it remains general if we consider only two of the three Euler rotations. Let the first rotation be made about the \( z \) axis through an angle \( \alpha \) and the second rotation is made about the new \( x \) axis through an angle \( \beta \) (figure 3). Then, the transformation matrix is given by

\[ \hat{T}_{\text{cart}} = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 & 0 \\ -\sin \beta \sin \alpha & \cos \beta \cos \alpha & \sin \beta \cos \alpha & \sin \beta \end{pmatrix}, \]

Transforming the fields from the cavity coordinate system and Cartesian basis into the atomic coordinate system and Cartesian basis gives

\[ E' = (\hat{T}_{\text{cart}})^{-1} E, \]

where

\[ E'_{\pm} = E_{\pm} \cos \beta, \]

\[ E'_{x} = E_{x} \sin \beta \sin \alpha, \]

\[ E'_{y} = E_{y} \sin \beta \cos \alpha, \]

It is worth pointing out that now we are faced with only three field components, which we are really interested in. Such a transformation also establishes a reference frame: i.e. when \( \alpha = \beta = 0 \), both bases coincide, magnetic field is transverse and \( E'_{\mp} = E_{\mp} = 0 \).

The equation for the fields in the atomic Cartesian basis in vector form reads

\[ \hat{\Gamma}' + \hat{\Gamma}' E' = iP', \]

where \( \hat{\Gamma}' \) is the loss tensor, \( P' \) is the polarization vector of the gain medium and \( \dot{m} \) denotes the time derivative.

The polarization of the medium is determined from the density matrix in the atomic coordinate system and Cartesian basis for which it has the following relevant elements: five sublevel populations, which will be denoted as \( \rho_{kl}^{(k, l)} = \{0, 2, 4, x', y'\} \), four induced dipole moments (or one-photon coherences) \( \rho_{02}^{(0, 2)}, \rho_{04}^{(0, 4)}, \rho_{02}^{(0, 2)}, \rho_{04}^{(0, 4)} \), three intersublevel coherences \( \rho_{x'x'}, \rho_{y'y'}, \rho_{x'y'} \), and three Raman coherences \( \rho_{x'x}, \rho_{x'y}, \rho_{y'y} \). We assume that the relaxation processes are isotropic and that the population decays to a reservoir of external states. The decay rates for the level populations will be denoted as \( \gamma_{k} \). The decay rates for the one- and two-photon coherences between states \( |a\rangle \) and \( |b\rangle \) are denoted as \( \gamma_{ab} \), and \( \gamma_{abc} \) is taken into account through corresponding rates \( \lambda_{k} \).

We obtain the system of the field equations in the cavity coordinate system and Cartesian basis by multiplying equations (8) from the left by \( \hat{T}_{\text{cart}} \) and incorporating additively terms describing the cavity linear anisotropy [12].

\[ E_{00} = (1 + i\omega_{02} + \kappa_{2} E_{00} + i\rho_{02}^{(0, 2)} \cos \beta + (p_x e^{-i\lambda} - 1) E_{00}), \]

\[ E_{02} = (1 + i\omega_{02} + \kappa_{2} E_{02} + i\rho_{02}^{(0, 2)} \sin \beta + (p_x e^{-i\lambda} - 1) E_{02}), \]

where \( \omega_{0k} \) and \( \omega_{2k} \) are the isotropic-cavity resonance frequencies; \( g_{x}, g_{y}, g_{z} \), and \( g_{a} \) are the gain parameters (see appendix); \( \kappa_{2} \) and \( \kappa_{4} \) are the cavity decay rates; \( p_{x} \) and \( p_{z} \) are the amplitude transmission factors for the fields polarized in
and \( z \) directions, respectively, and \( \Delta \) is the phase anisotropy (expressed in radians).

It is convenient to extract explicitly the frequencies \( \omega_{\pm4} \) and phases \( \phi_{\pm4} \) of the laser field components from all complex elements of the density matrix, introducing slowly varying density matrix elements \( R \) as defined in the appendix. We thereby obtain the Maxwell–Bloch equations describing the laser dynamics, in the usual rotating-wave and slowly-varying envelope approximations, and in their most general form which allows for possible presence of cavity anisotropies and a magnetic field of arbitrary orientation (see appendix).

Because we have assumed that the frequencies of the \( x \)- and \( z \)-polarized fields interacting with the lower-energy transitions are equal, these fields have to be considered as components of a single mode whose state of polarization can vary. This is in contrast to the field interacting with the \([4] \leftrightarrow [0]\) transition, which has only \( z \)-polarized component. For instance, when \( \xi = \pi \nu \) (\( \nu \) is an integer) the lower field is LP with azimuth \( \Phi = \arctan(n_{\perp}/n_{\parallel}) \); it is CP when \( n_{\perp} = n_{\parallel} \) and \( \xi = (2n + 1)\pi/2 \). In all other cases, the lower field is EP with azimuth and ellipticity given, respectively, as \([19]\)

\[
\Phi = \frac{1}{2} \arctan[\tan(2\varphi) \cos \xi], \\
\xi' = \tan[\frac{1}{2} \arcsin[\sin(2\varphi) \sin \xi]],
\]

where \( \varphi = \arctan[n_{\perp}/n_{\parallel}] \). Hence, in general, the laser can emit \( z \)-polarized light on the upper transition and an arbitrarily polarized field on the lower transition whose specific state depends on factors such as the linear (i.e. assumed to be a linear function of the laser field parameters) anisotropies of the cavity and of the gain medium (induced by the magnetic field) and nonlinear anisotropy of the gain medium.

Inspection of system \((A.1)\) shows that the equations for the phase of the upper mode and overall phase \((\phi_{\parallel} + \phi_{\perp})\) of the lower mode are not coupled with the atoms and thus can be determined separately. Moreover, for the case of a transverse magnetic field \((\alpha = \beta = 0)\), the equation for the phase difference \( \xi \) is also not coupled with the remaining equations. However, even with these simplifications, system \((A.1)\) is very complex and only a few further analytical steps are possible.

4. The off-solution and its stability

The off-\((\text{zero field})\) solution of the system \((A.1)\) is

\[
R_{ll} = \lambda_{l}/\gamma_{l}, \\
R_{l'y'} = (\lambda_{l}g_{l} + 2\Delta R_{l'2y'}/\gamma_{l}g_{l}), \\
R_{l'y'}' = \frac{\Lambda(\gamma_{l}g_{l}\lambda_{l} - \lambda_{l}g_{l})}{\gamma_{l}g_{l}g_{l}' + 2\Lambda^{2}(\gamma_{l}g_{l} + \gamma_{l}'g_{l}')},
\]

where \( l = 0, 2, 4 \). Other variables not listed in \((11)\) are zero. It should be pointed out that the distribution of the population between \([x']\) and \([y']\) sublevels is unequal only in the presence of an anisotropy due to either the relaxation mechanism or the incoherent pumping mechanism.

The Jacobian matrix governing the stability of the off-solution can be factored into two submatrices. One of them is a \(2 \times 2\) matrix, which governs the stability of the off-solution in the \([E_{02}, R_{02}]\) subspace and the other one is a \(5 \times 5\) matrix, which corresponds to the \([E_{02}, E_{04}, R_{02}, R_{04}, R_{02}']\) subspace. Hence, we can conclude that the local bifurcations of the trivial solution lead only to single-photon laser emission (in the upper or lower transitions), but they cannot generate two-photon emission; i.e. the lasing solution in the two-photon laser is detached from the zero solution \([9]\).

A \(2 \times 2\) matrix provides the characteristic exponents describing the growth or decay of perturbations of the upper mode and induced one-photon coherence \(R_{02}\). We find that this matrix admits a solution corresponding to a pitchfork bifurcation (PB) given by the zero of the determinant, leading to the appearance of the upper field solution with a well-defined frequency \(\omega_{4}\) at the first laser threshold \(\lambda_{4}^{0}\)

\[
\delta_{4}^{0}y_{40} = \kappa_{4}(\epsilon_{24} - \delta) = \kappa_{4}(\omega_{4} - \Omega_{40}),
\]

\[
\lambda_{4}^{0} = \frac{\lambda_{0} + \gamma_{4}k_{4}y_{40}^{2} + (\omega_{4} - \Omega_{40})^{2}}{g_{4}y_{40}^{2} \gamma_{40}^{2} \cos^{2} \beta},
\]

where the parameters are defined in the appendix. Reasonable analytical solutions for the other \(5 \times 5\) matrix can be obtained only in the case of a transverse magnetic field \((\alpha = \beta = 0)\), where it can be factored into \(2 \times 2\) and \(3 \times 3\) submatrices. The \(2 \times 2\) submatrix is similar to that for the upper field and is associated with the growth of the \(z\)-polarized lower field, determining its frequency \(\omega_{2}\) and lasing threshold \(\lambda_{0}^{0}\)

\[
\delta_{2}^{0}y_{20} = \kappa_{2}\delta, \\
\lambda_{0}^{0} = \frac{\lambda_{2}^{0}}{\gamma_{2}^{0} + \gamma_{0}^{0}k_{2}^{0}} - \frac{\delta^{2} + \gamma_{2}^{0}}{g_{2}y_{20}^{2}},
\]

The \(3 \times 3\) submatrix provides PB for the \(x\)-polarized lower state (exact expressions are given in the appendix). Assuming that the \([x']\) and \([y']\) sublevels are equally populated \((\lambda_{x'} = \lambda_{y'})\), the gain factors are identical \(\gamma_{x'} = \gamma_{y'}\), and the relaxation mechanism is isotropic \((\gamma_{x'} = \gamma_{y'} = \gamma_{0}^{0})\), we obtain

\[
\delta_{2}^{0}y_{20} = \frac{\delta^{2} + \gamma_{2}^{0}}{\gamma_{x'}^{0} + \gamma_{y'}^{0}},
\]

\[
\lambda_{0}^{0} = \frac{(\delta + \Lambda_{2}^{0})^{2} + \gamma_{0}^{0}(\delta - \Lambda_{2}^{0})^{2} + \gamma_{2}^{0}}{g_{x'}^{0}(\delta^{2} + \gamma_{x'}^{2} + \Lambda_{2}^{0})},
\]

In the absence of the phase anisotropy expressions \((12)\) and \((14)\) coincide with the well-known dispersion relations for a scalar Lorenz–Haken laser \([20]\). Phase anisotropy generalizes them to the case of an anisotropic-cavity (vector) laser leading to a frequency separation between the anisotropic- and isotropic-cavity eigenfrequencies, which is equal to the cavity birefringence \(p_{0} \sin \Delta\). Equations \((16)\) and \((17)\) show that the frequency of the \(x\) LP mode is affected by the magnetic field. For \(B = 0\) \((\Lambda = 0)\), expression \((16)\) takes the form of a generalized dispersion relation such as those of \((12)\) and \((14)\) \(\text{‘frequency pulling’ effect}\. Simple inspection of \((16)\) shows that the offset of the actual laser frequency from the anisotropic-cavity resonance becomes maximal either when \(\Lambda = 0\) or \(\Lambda^{2} \gg \delta^{2} + \gamma_{0}^{0}\). These maximal offsets are equal in magnitude but of opposite signs. A change of sign of the cavity detuning occurs at \(\Lambda^{2} = \delta^{2} + \gamma_{0}^{0}\), when the laser frequency takes the value of the anisotropic-cavity eigenfrequency.

The lasing threshold for the \(x\) LP mode coincides with \(\lambda_{0}^{0}\) when \(\Lambda = 0\) and there is no anisotropy either in
the pumping/gain medium or in the relaxation mechanism associated with the lower sublevels. For small atomic detuning,
\[
|\delta| \leq \delta_S \equiv |\gamma_0|/\sqrt{3},
\]
the magnetic field increases the lasing threshold $\lambda_0^x$, causing the $x$ LP mode to appear first. However, the behaviour of $\lambda_0^x$ changes qualitatively when $|\delta| \geq \delta_S$. In particular, the threshold $\lambda_0^x$ decreases when $\Lambda$ is increased for small magnetic field. It takes its minimum value at
\[
\Lambda^2 = \delta^2 - \left(\sqrt{\delta^2 + \gamma_0^2} - \delta\right)^2
\]
and then begins to increase. Hence, in contrast to intuitive expectations, there is a domain of magnetic fields
\[
0 < |\Lambda| < \sqrt{3\delta^2 - \gamma_0^2},
\]
for $|\delta| \geq \delta_S$, in which the $x$ LP mode will be excited at the first laser threshold instead of the $z$ LP lower mode. This phenomenon is somewhat similar to the preference of a class A laser with incoherent pumping to emit $x$ LP field at small magnetic field strengths [15]. It can be understood by taking into account the fact that the applied magnetic field creates frequency-dependent linear (i.e. independent of the laser field intensity) and nonlinear anisotropies. The nonlinear anisotropy is more pronounced at small $\Lambda$ and it is this anisotropy which favours the field polarized orthogonally to the magnetic field [8]. For large magnetic field strengths, the linear anisotropy, which is related with shifts of the $|\rangle$ and $|-\rangle$ sublevels (figure 2(a)) out of the atomic resonance, becomes a dominant factor. It determines the conditions for the appearance at the first laser threshold the $z$ LP lower mode. Because the anisotropy induced by the magnetic field is frequency-dependent, this phenomenon is sensitive to the atomic detuning, as clearly seen in (19).

Increasing $\delta$ so that $\delta^2 > \gamma_0^2$, the thresholds of all the single-mode solutions increase and tend to infinity as $\delta \to \infty$. This reflects the underlying feature of the two-photon laser, in which the off-solution is always stable. Physically, as follows from (12), (14) and (16), this occurs due to the off-resonance conditions for the monomode solutions.

Expressions (18)–(20) suggest a simple method for measuring the one-photon coherence decay rates. In particular, the critical value of the detuning $\delta_S$ at which the polarization state of the lasing mode at threshold changes directly gives the value of $\gamma_0^x = \sqrt{3}\delta_S$. Another strategy can be employed for large $\delta$ ($\delta > \delta_S$). For instance, varying the magnetic field at fixed detuning $\delta$, we can find either the minimum value of the lasing threshold for the $x$-polarized mode (which will give the magnitude of $\gamma_0^x$, in accordance with (19)), or the point in which the $x$ LP mode is no longer excited (see (20)). Note that the system becomes isomorphic to a conventional $J = 0 \to J' = 1$ incoherently pumped class C laser with selective pumping of the intermediate level $|0\rangle$, which simplifies experimental testing of the method.

5. Discussion of dynamics

Let us assume the situation of selective pumping of the upper level [4] alone, which is the case of the experiment in [7]. Since our primary goal is to investigate the two-photon laser, we start from the condition when both the upper and lower field solutions are already present, i.e. we do not consider dynamics when only one of these two fields is excited. To follow as closely as possible the experiment [7], we adopt the following values for the laser parameters: $\gamma_1 \equiv \gamma_0 = \gamma_0 = \gamma_0 = 6\text{ MHz}$, $\gamma_{42} = \gamma_{42} = \gamma_4 = \gamma_4 = \gamma_2 = 6\text{ MHz}$, $\gamma_1 \equiv \gamma_4 = \gamma_0 = \gamma_2 = \gamma_1 = 5.9\text{ MHz}$, $\lambda_0 = \lambda_0 = \lambda_0 = \lambda_0 = \epsilon_2 = \delta_2 = \delta_4 = \Delta = 0$, $p_1 = p_2 = 1$, $g_4 = g_0 = g_2 = g_0 = g_4 = g_2 = 720\text{ MHz}$, $\kappa \equiv \kappa_4 = \kappa_2 = 1.2\text{ MHz}$, $\lambda_4 = 200\text{ MHz}$. Note in particular that we consider two-photon resonance. The magnetic field $\Lambda$, one-photon detuning $\delta$, and the Euler angles $\alpha$ and $\beta$ will be the main control parameters.

At the atomic resonance, the system (A.1) admits two- and three-field solutions. We find that in the laser the $|\eta_4, 0, \eta_1\rangle$ and $|\eta_4, \eta_1, 0\rangle$ two-field solutions are equally likely in the absence of a magnetic field. This is a consequence of our choice of the parameter setting under which $\eta_4 = \eta_2$. The three-field solution consists of the upper field $\eta_4$ and two lower fields with the same amplitudes $\eta_1 = \eta_2$. As we pointed out earlier, the phases of all the fields in the case of $\Lambda = 0$ are arbitrary. Hence, superposition of the lower fields constitutes a LP mode oriented by $45^\circ$ in the $xy$ plane (note, the phases define the field ellipticity but not the polarization azimuth). The fields $\eta_4$ and $\eta_2$ remain equal to each other, independently of the detuning $\delta$. Applying a magnetic field, however, breaks the perfect symmetry between the $\eta_4$ and $\eta_2$ fields due to linear and nonlinear frequency-dependent anisotropies induced in the gain medium.

The bifurcation diagram of figure 4(a) displays two- and three-field solutions. We find that in the laser the $|\eta_4, 0, \eta_1\rangle$ and $|\eta_4, \eta_1, 0\rangle$ two-field solutions are equally likely in the absence of a magnetic field. This is a consequence of our choice of the parameter setting under which $\eta_4 = \eta_2$. The three-field solution consists of the upper field $\eta_4$ and two lower fields with the same amplitudes $\eta_1 = \eta_2$. As we pointed out earlier, the phases of all the fields in the case of $\Lambda = 0$ are arbitrary. Hence, superposition of the lower fields constitutes a LP mode oriented by $45^\circ$ in the $xy$ plane (note, the phases define the field ellipticity but not the polarization azimuth). The fields $\eta_4$ and $\eta_2$ remain equal to each other, independently of the detuning $\delta$. Applying a magnetic field, however, breaks the perfect symmetry between the $\eta_4$ and $\eta_2$ fields due to linear and nonlinear frequency-dependent anisotropies induced in the gain medium.
transitions to be different. Because we consider now the limit makes the amplitudes of the laser fields connecting the same see, the system admits two bimode and one three-field stable

\[ \eta(\text{dashed–dotted–dotted curve—because} \]

Thus superposition of the lower fields

\[ \text{and, consequently, on the hyperfine structure of the atomic levels. This solution, as well as the three-field solution} \]

\[ [\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}], \text{exists only in certain domains of } \Lambda. \text{ As in} \]

the case of the absence of a magnetic field, the three-field solution is stable for small \( \Lambda \). However, the amplitudes \( \eta_2^{(1)} \) and \( \eta_3^{(1)} \) are very sensitive to \( \Lambda \): \( \eta_2^{(1)} (\eta_3^{(1)}) \text{ rapidly goes toward} \]

zero (\( \eta_4^{(1)} \)) as \( \Lambda \) increases. Approaching these limit values, the three-field solution disappears at a saddle-node bifurcation point (figure 5(b), SN1). At such value of \( \Lambda \), there is also a PB of the bimode solution \( [\eta_1^{(2)}, \eta_2^{(2)}, 0] \) (depicted as PB1 in figure 5(a)), which makes such bimode solution stable until the next PB (figure 5(a), PB2) where the stable three-field solution reappears. Finally, the three-field solution disappears due to the off-resonance condition at a saddle-node bifurcation for large \( \Lambda \) (figure 5(b), SN3). This bifurcation coincides with the PB of the bimode scalar solution (figure 5(a), PB3). Thus these results show again that there is a strong influence of the nonlinear gain medium anisotropy, which can even stabilize the \( [\eta_1^{(2)}, \eta_2^{(2)}, 0] \) solution. Because of the nonlinear nature of the anisotropy, the stability domain of this solution is very sensitive to the atomic decay rates. In particular, decreasing \( \gamma_4 \), this domain rapidly shrinks and vanishes at \( \gamma_4 \approx 5.8 \text{ MHz} \).

The predominance of the nonlinear frequency-dependent gain medium anisotropy at small \( \Lambda \) causes the azimuth of the lower laser mode to turn toward the \( x \) direction. Increasing \( \Lambda \), the nonlinear anisotropy decreases, while the linear one increases. This changes the sense of turning of the polarization azimuth at \( \Lambda \approx 4 \text{ MHz} \): for larger \( \Lambda \) it turns toward the \( z \) axis and, finally, aligns with it. In general, this auxiliary anisotropy makes the lower mode to be EP (figure 5(b)). However, interplay between the linear and nonlinear anisotropies can result in linear polarization and change of the sign of the ellipticity (as in the case of 5(b) at \( \Lambda \approx 16.5 \text{ MHz} \)).

Misalignment of the magnetic field changes qualitatively both the laser behaviour and the structure of the phase space. We found that the laser behaviour is only weakly sensitive to variation of the angle \( \alpha \). In contrast, it is very sensitive to misalignment in the \( yz \) plane, which reveals the importance of the ‘degree’ of non-orthogonality (i.e. misalignment of the magnetic field from a direction orthogonal to the cavity axis direction). A typical bifurcation diagram of the laser field amplitudes as a function of \( \Lambda \) for such a case is shown in figure 6 for \( \alpha = 1^\circ \) and \( \beta = 10^\circ \). A first remarkable fact seen in the figure is that the former three-field solution becomes periodic, affecting the field intensity as well as the field polarization state. It appears as a result of a Hopf bifurcation which replaces the PB (see figure 5). Note, at small \( \Lambda \) the periodic solution is unstable, while the steady states are stable (see caption of figure 6). Second, there is a new stable solution at moderate \( \Lambda (12 < \Lambda \lesssim 16 \text{ MHz}), \) which is not present when \( \alpha = \beta = 0 \). The stable branches of this solution for the fields \( \eta_4 \) and \( \eta_5 \) are less intense than the unstable ones, which violates the ‘maximum emission principle’ (a principle which is often—but not always—verified in laser physics). Third, there are no longer bimode solutions. Indeed, both for large and small \( \Lambda \), neither one of the stable fields \( (\eta_4, \eta_5, \text{ or } \eta_6) \) is zero. And fourth, the misalignment mixes the former \( [\eta_1^{(2)}, \eta_2^{(2)}, 0] \) and \( [\eta_1^{(1)}, 0, \eta_3^{(1)}] \) solutions giving a more complicated behaviour with bistability at small \( \Lambda \).

Hence, small misalignment can destabilize the laser output even when it is well below the so-called ‘bad cavity’ limit (given by \( \kappa = \gamma_\perp + \gamma_\parallel \approx 11.9 \text{ MHz} \)), which is known

Figure 4. (a) Bifurcation diagram showing stable (dots) and unstable branches of the off-solution \( \eta_4 = \eta_3 = \eta_5 = 0 \) and \( \eta_4 \) (continuous curve), \( \eta_4 \) (dashed curve), and \( \eta_1 \) (dashed–dotted curve) as a function of the one-photon detuning for a laser with transverse \( (\alpha = \beta = 0) \) magnetic field \( \Lambda = 0.5 \text{ MHz} \). ‘1’ and ‘2’ distinguish two different solutions \( \eta_1^{(1)} \) and \( \eta_2^{(1)} \), respectively. SN and PB denote saddle-node and PB points, respectively. Stable (continuous curve) and unstable (dashed curve) branches of the phase difference \( \xi = \phi_2 - \phi_1 \) of the lower fields are depicted in (b). Other parameters are listed at the beginning of section 5.
stays for bimode solution with nearly equal amplitudes \( \eta_2^{(0)} \approx \eta_3^{(0)} \). \( \delta \) denotes bimode solution of the scalar two-photon laser with \( \alpha = 1 \) and \( \Lambda = 1 \) MHz. Stable steady states are depicted by dots. Dashed–dotted–dotted vertical line distinguishes Hopf bifurcation. The meaning of the other curves is as in figure 4 (a).}

large misalignment (on the right of the dashed–dotted–dotted curve in figure 7) instabilities disappear at the Hopf bifurcation and the laser dynamics becomes stable.

6. Conclusions

We have studied theoretically the polarization dynamics of a new type of quantum oscillator which is based on a two-photon stimulated emission process and subject to the action of a magnetic field of arbitrary orientation. This investigation has been performed for both cases of a cascade laser, which is characterized by a small intermediate-state atomic detuning, and a two-photon laser (large atomic detuning). In this work we intended to study the origin of recently observed polarization instabilities in the two-photon laser [7] using a highly simplified model. Specifically, we investigated a class C laser system implemented on a \( J' = 0 \rightarrow J = 0 \rightarrow J' = 1 \) atomic transition (see footnote 4).

It has been shown that the trivial solution is connected by local bifurcations with the lasing solution only in the case of the cascade laser. Hence, the trivial solution cannot be destabilized by a local bifurcation in two-photon lasers [9]. When a dc magnetic field is applied in a direction transverse to the cavity axis (the approximate orientation used in the experiment), we find that the laser beam generated on the lower (\( J = 0 \rightarrow J' = 1 \)) transition is EP over a large domain of the magnetic field strengths. It is also found that at small magnetic field strengths the two-photon laser can emit LP radiation with its plane of polarization orthogonal to the direction of the magnetic field. We find that the laser stationary states are very sensitive to misalignment of the magnetic field in the \( yz \) plane, which reveals the importance of the ‘degree’ of non-orthogonality. In particular, for a slightly misaligned laser, the model predicts the existence of periodic instabilities which appear due to a Hopf bifurcation and affect both the intensity and polarization of the laser field. The misalignment can destabilize the laser output even when it is well below the ‘bad cavity’ limit. For a sufficiently strong transverse magnetic field, all instabilities are suppressed and the beam is LP with its polarization vector parallel to the magnetic field.
The fact that the magnetic field may not have been aligned exactly orthogonal to the laser cavity in the experiments of [7] suggests that dynamic instabilities as those described by means of our simplified model could be responsible for the observed polarization fluctuations. The present theoretical results can help in defining operating conditions where the dynamic instabilities are absent and in determining to what extent observed instabilities can be due to deterministic dynamic effects.

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Appendix. Lasing equations of the off-state

The explicit relations between \( \rho' \) and \( R \) are

\[
\rho'_{00} = R_{00} e^{-i k_{00} t - i \phi_0}, \\
\rho'_{01} = R_{01} e^{-i k_{01} t - i \phi_1}, \\
\rho'_{12} = R_{12} e^{-i k_{12} t - i \phi_2}, \\
\rho'_{21} = R_{21} e^{-i k_{21} t - i \phi_3}. \\
\rho'_{22} = R_{22} e^{-i k_{22} t - i \phi_4}. \\
\rho'_{0'1} = R_{0'1} e^{-i k_{0'1} t - i \phi_5}, \\
\rho'_{1'2} = R_{1'2} e^{-i k_{1'2} t - i \phi_6}, \\
\rho'_{2'1} = R_{2'1} e^{-i k_{2'1} t + i \phi_7}, \\
\rho'_{2'2} = R_{2'2} e^{-i k_{2'2} t + i \phi_8}. \\
\]

For the remaining parameters \( \rho' = R \).

The gain parameter is defined as follows:

\[
g = N \frac{\mu^2 \mu_0}{2 \hbar \bar{\hbar} V},
\]

where \( \mu \) denotes projection of the dipole moment on the direction of the corresponding field, \( \omega_0 \) denotes frequency of the corresponding cavity mode, \( \bar{\hbar} \) is permittivity of vacuum, and \( N \) represents the total number of atoms in the laser medium of volume \( V \).

The Maxwell–Bloch equations for the laser are as follows:

\[
\begin{align*}
\dot{R}_{44} &= \lambda_4 - \gamma_4 R_{44} + 2 \eta_4 R_{10}^\dagger \cos \beta, \\
\dot{R}_{22} &= \lambda_2 - \gamma_2 R_{22} - 2 \eta_2 R_{10}^\dagger \cos \beta, \\
\dot{R}_{00} &= \lambda_0 - \gamma_0 R_{00} - 2 \eta_1 \eta_4 R_{10}^\dagger R_{0}^\dagger \cos \beta, \\
&+ 2 \eta_1 \eta_2 R_{00}^\dagger \cos \alpha + R_{00}^\dagger \sin \alpha, \\
&+ 2 \eta_1 \eta_3 A_{3}^\dagger \sin \alpha - A_{1}^\dagger \cos \alpha \sin \alpha, \\
\dot{R}_{44}' &= \lambda_4' - \gamma_4 R_{44}' - 2 \Lambda R_{44}' \cos \beta, \\
&- 2 \eta_1 \eta_4 R_{44}^\dagger \cos \alpha - 2 \eta_2 A_{1}^\dagger \cos \alpha \sin \alpha, \\
\dot{R}_{22}' &= \lambda_2' - \gamma_2 R_{22}' - 2 \Lambda R_{22}' \sin \beta, \\
&- 2 \eta_1 \eta_4 R_{22}^\dagger \sin \alpha + 2 \eta_2 A_{3}^\dagger \sin \alpha \cos \alpha, \\
\dot{R}_{00}' &= \lambda_0' - \gamma_0 R_{00}' - 2 \Lambda R_{00}' \sin \beta, \\
&- 2 \eta_1 \eta_4 R_{00} \sin \alpha + 2 \eta_2 A_{3}^\dagger \sin \alpha \cos \alpha,
\end{align*}
\]

\[
\begin{align*}
\dot{R}_{40} &= -\gamma_4 R_{40} + i(\phi_0 + \varepsilon_{42} - \delta) R_{40} - i(D_{40} \eta_4 \cos \beta \\
&+ R_{42} \eta_2 \cos \beta + A_{p} R_{44} + A_{M} \Delta R_{44}'), \\
\dot{R}_{02} &= -\gamma_0 R_{02} + i(\phi_0 + \delta) R_{02} + i(D_{02} \eta_2 \cos \beta \\
&+ R_{22} \eta_4 \cos \beta + A_{p} R_{22} + A_{M} \Delta R_{22}'), \\
\dot{R}_{01} &= -\gamma_0 R_{01} + i(\phi_0 + \delta) R_{01} - i(D_{01} \eta_1 \cos \beta \\
&+ i((\eta_2 R_{44}' + R_{22} \eta_4 \cos \beta + A_{M} \Delta R_{44}'). D_{01} A_{p} - \Lambda), \\
\dot{R}_{01}' &= -\gamma_0 R_{01}' + i(\phi_0 + \delta) R_{01}' + i(D_{01} \eta_1 \cos \beta \\
&+ i((\eta_2 R_{44}' + R_{22} \eta_4 \cos \beta + A_{p} R_{22}' + D_{01} A_{M} - \Lambda), \\
\dot{R}_{42} &= -\gamma_4 R_{42} + i(\phi_0 + \varepsilon_{42} + \varepsilon_{42}') R_{42} + i(\eta_2 R_{44}' + R_{22} \eta_4 \cos \beta + A_{M} \Delta R_{44}'), \\
\dot{R}_{12} &= -\gamma_2 R_{12} + i(\phi_0 - \phi_0') R_{12}' - \Lambda R_{12}', \\
&+ i(A_{p} \Delta R_{44}'. A_{M} \Delta R_{44}'), \\
\dot{R}_{21} &= -\gamma_2 R_{21} + i(\phi_0 - \phi_0') R_{21}' - \Lambda R_{21}', \\
&+ i(\eta_2 R_{44}'. A_{M} \Delta R_{44}'), \\
\dot{R}_{41} &= -\gamma_4 R_{41} + i(\phi_0 + \phi_0') R_{41}' + \Lambda R_{41}', \\
&+ i(\eta_2 R_{44}' + R_{22} \eta_4 \cos \beta - A_{M} \Delta R_{44}'), \\
\dot{R}_{22}' &= -\gamma_2 R_{22}' - i(\phi_0 - \phi_0') R_{22}' + \Lambda R_{22}', \\
&+ i(\eta_2 R_{44}' + \Lambda R_{44}'), \\
\dot{\eta}_4 &= -\kappa_4 \eta_4 - g_4 R_{40}^\dagger \cos \beta, \\
\dot{\eta}_4 &= -\kappa_4 \eta_4 - g_4 R_{40}^\dagger \cos \beta - g_5 R_{00}^\dagger \sin \alpha, \\
\dot{\eta}_4 &= -\kappa_4 \eta_4 - g_4 R_{20}^\dagger \sin \beta \sin \alpha, \\
&+ g_4 A_{3} \sin \alpha \cos \sin \beta - g_5 R_{00}^\dagger \cos \beta, \\
\dot{\eta}_4 &= \delta_4^x - g_4 R_{00}^\dagger \cos \beta / \eta_4, \\
\dot{\eta}_4 &= \delta_4^x - (g_4 R_{00}^\dagger \cos \beta \sin \alpha + g_4 R_{00}^\dagger \cos \alpha \eta_4^{-1}), \\
\dot{\eta}_4 &= \delta_4^x + (g_4 R_{00}^\dagger \cos \beta \sin \alpha - g_4 R_{00}^\dagger \cos \alpha \sin \alpha). \\
\end{align*}
\]
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