DIRECT OBSERVATION OF OPTICAL PRECURSORS IN A COLD POTASSIUM GAS

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Physics in the Graduate School of Duke University

2006
ABSTRACT

(Physics)

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Abstract

This thesis considers how an electromagnetic field propagates through a dispersive linear dielectric in the case when the field is turned on suddenly. It has been predicted nearly 100 years ago that the point in the waveform where the field first turns on (the front) propagates precisely at the speed of light in vacuum. Furthermore, it is predicted that distinct wave-packets develop after the front, but before the arrival of the main part of the field (the main signal). These wave-packets are known as optical precursors. It was believed that precursors are an ultra-fast phenomena, persisting only for a few optical cycles, and that they have an exceedingly small amplitude.

I describe a method to increase the duration of optical precursors into the nanosecond range using a dielectric with a narrow resonance. I also show how to increase the precursor amplitude by tuning the carrier frequency of the field near the resonance frequency of the oscillators making up the dielectric medium. The field emerging from the dielectric consists of a several-nanosecond-long spike occurring immediately after the front with near 100% transmission, which subsequently decays to a constant value expected from Beer’s Law of absorption. I demonstrate, using a modern asymptotic theory, that the spike consists of both the Sommerfeld and Brillouin precursors. Thus, my measurement is the first direct observation of optical precursors. The precursor research might be useful for imaging applications requiring penetrating optical radiation, such as in biological systems, or in optical communication systems.

While the asymptotic theory explains qualitatively my observations, I find that
there are large quantitative disagreements. I hypothesize that these errors are due to the fact that I use a weakly-dispersive narrow-resonance medium for which this theory has never been tested. I suggest empirical fixes to the theory by comparison to my data. I also compare the asymptotic theory and data to a second theory that is known to describe well my experimental conditions, but was believed by some researchers not to predict optical precursors. I demonstrate that this belief is incorrect.
Acknowledgments

Past six years here at Duke has not only been a time for pursuing a Ph.D. degree in physics but also been a time for learning valuable lessons in my life. I realize that solving problems in one’s life is not much different from the procedure of researching to get an answer for a question. As a result, I am about to graduate, which is only a beginning of a real journey in my life. At this transitional point, I would like to thank number of people who have influenced and shaped me along this path. What I can do here is to merely thank a few.

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Chapter 1

Introduction

Many modern optical technologies, such as optical communication and medical imaging, require an understanding of how optical pulses propagate through a dispersive medium. In this regard, techniques for tailoring the dispersion of materials enable us to control pulse propagation characteristics better than before. In 1989, Harris showed that absorption of a probe beam propagating through a medium can be cancelled by illuminating the atoms with a strong laser beam, the so-called process of “electromagnetic induced transparency” (EIT) [1]. For the last decade, his work has initiated many new avenues of research related to this novel technique and its applications. EIT is now one of the fastest growing topics in quantum optics, atomic physics, and even condensed matter physics.

Using EIT, for example, it is now possible to achieve an exceedingly slow or fast group velocity $v_g$ compared to the speed of light in vacuum $c$ for pulses of light propagating through a gas of atoms [2]. For example, Hau et al. in 1999 were able to slow down extremely the group velocity of a pulse propagating through an ultracold atomic gas, achieving $v_g \sim 17$ meters per second [3]. At the opposite extreme, superluminal pulse propagation ($v_g > c$ or $v_g < c$) was observed by Wang et al. [4] in 2000, and light storage in atomic gas was accomplished by two groups [5, 6] almost at the same time in January 2001. By 2005, there were over 6,000 papers published on slow- and fast-light research.
The achievement of slow and fast light has triggered fundamental questions about the information velocity as it relates to causality in Einstein’s special theory of relativity. Chiao and collaborators postulated that information encoded on optical wave form is carried by points of non-analyticity, which travel precisely at $c$ [7,8]. In a related work, Parker and Walker showed that encoding information on a pulse results in non-analytic points [9] by considering the thermodynamics of the information processing. Stenner et al. first confirmed these ideas experimentally in both slow- and fast-light experiments [10,11]. They encoded one bit of information on a pulse by changing rapidly the pulse intensity. By tracking when it is first possible to identify the bit, they showed that the information velocity is equal to $c$ regardless of the dispersive properties of the medium.

It is generally believed that the rapid change of intensity on a pulse creates optical precursors, the transient behavior of the propagated field [12–14]. In the experiment by Stenner et al., however, it was difficult to distinguish the precursors from the signal pulse because they used a pulse with a relatively smooth turn-on; that is, the light intensity is non-zero before the rapid change. To address this shortcoming, an experiment needs to be conducted that clearly tests for the existence of optical precursors. The observation of optical precursor is the primary goal of my thesis.

By generating a sharp edge on an input pulse (sudden turn-on from a zero light intensity), I measure optical precursors with a significant amplitude. This measurement is facilitated by extending the time scale of the precursors up to the order of tens of nanoseconds using a narrow-resonance absorber, such as cold atoms contained in a magneto-optical trap (MOT). My measurement is the first direct observation of optical precursors.

In the first chapter of my thesis, I discuss more about the information velocity
research performed by Stenner et al. [11], and how it motivates my thesis project. I then introduce the fundamentals of optical precursors, explain why the measurement of precursors has been regarded as difficult based on the history of precursor theory, and I review the few indirect observations. Finally, I present the first direct observation of optical precursors that were performed here at Duke.

1.1 Light interaction with dielectric material

![Figure 1.1: An electromagnetic field propagates through a dispersive dielectric medium.](image)

Before discussing the concept of the group velocity for a propagating optical pulse, I discuss how light interacts with a dielectric material from a conceptual point of view. I present the medium response to the incident light, which is characterized through the complex index of refraction $n(\omega)$ and its connection to $v_g$. Then, I will discuss so-called “fast light,” which occurs when $v_g > c$ or $v_g < 0$.

What happens when an electromagnetic field propagates through a linear dispersive dielectric material, as shown schematically in Figure 1.1? Before answering this question, let us think about the most trivial case: light propagation through vacuum. An electromagnetic wave, or pulse, does not change its shape or speed when
it passes through vacuum. The situation is more complicated for the case when a pulse propagates through a medium because different frequency components interact with medium differently, thereby experiencing different phase velocities or absorption when they pass through the medium. This dispersive phenomena results from the material’s dielectric properties due to their dipole moments. Materials consist of dipole moments, as shown in Figure 1.2, which can be treated approximately as a classical harmonic oscillator. This is known as Lorentz model of a dielectric (Lorentz oscillator) [13].

In Figure 1.2, an electron is bound to a fixed nucleus by a distance $r$ by a force obeying Hooke’s Law. The distance $r$ oscillates in time due to an external electric field $E(r, t)$. From now on, I consider only the electric field because the magnetic field only interacts weakly with the atoms; its coupling to be down by at least the fine-structure constant. The force acts to restore the electron with a characteristic frequency $\omega_0$. The oscillation experiences damping characterized by a damping rate $\delta$. These properties are quantified by the equation of motion describing the Lorentz
model, which is given by

\[ m \frac{d^2 r}{dt^2} = -eE(r, t) - m\omega_0^2 r - 2m\delta \frac{dr}{dt}, \]  \hspace{1cm} (1.1)

\[ \frac{d^2 r}{dt^2} + 2\delta \frac{dr}{dt} + \omega_0^2 r = -\frac{e}{m} E(r, t). \]  \hspace{1cm} (1.2)

As we will see later, it is convenient to obtain the solution of Eq. (1.2) in the frequency domain \((\omega)\) instead of the time domain \((t)\). This transformation is achieved by Fourier transforming the equation. The Fourier transform of an arbitrary function \(f(z, t)\) is defined as

\[ F(z, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z, t)e^{i\omega t} d\omega, \]  \hspace{1cm} (1.3)

and its inverse Fourier transform is given by

\[ f(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(z, \omega)e^{-i\omega t} d\omega. \]  \hspace{1cm} (1.4)

Using these definitions, Eq. (1.2) is Fourier transformed to obtain

\[ r(\omega) = \frac{eE(r, \omega)/m}{\omega^2 - \omega_0^2 + i2\delta \omega}. \]  \hspace{1cm} (1.5)

The macroscopic polarization of the medium \(P(r, \omega)\) is obtained using Eq. (1.5)

\[ P(r, \omega) = Np(r, \omega) = -Ne\rho(\omega) = \chi(\omega)E(r, \omega), \]  \hspace{1cm} (1.6)

where \(N\) is the number of molecules (atoms) per unit volume, and \(p(r, \omega) = -e\rho(\omega)\) is the microscopic dipole moment. \(P(r, \omega)\) is related to the electric displacement (in
Figure 1.3: The complex refractive index $n(\omega)$. (a) Real part of $n(\omega)$, and (b) imaginary part of $n(\omega)$. The box denotes anomalously dispersive region.

cgs units) as

$$D = \epsilon(\omega) E(r, \omega) = E(r, \omega) + 4\pi P(r, \omega) = [1 + 4\pi \chi(\omega)]E(r, \omega). \quad (1.7)$$

From Eqs. (1.5)-(1.7), the dielectric constant (electric permittivity) $\epsilon(\omega)$ is given by

$$\epsilon(\omega) = 1 + 4\pi \chi(\omega) = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i2\delta\omega}, \quad (1.8)$$

where $\omega_p = \sqrt{4\pi Ne^2/m}$ is the plasma frequency. The complex refractive index $n(\omega)$ is related to the dielectric constant as

$$n(\omega) = \sqrt{\epsilon(\omega)} = \sqrt{1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + 2i\omega\delta}}. \quad (1.9)$$

The frequency dependance of $n(\omega)$ in the vicinity of $\omega_0$ is shown in Figure 1.3.

This classical model of a Lorentz oscillator has its quantum counterpart, a dipole transition in a two-level atom in the presence of an electromagnetic field. The density
matrix equation of motion for a two level atom [15] provides exactly the same relation as given by Eq. (1.8); therefore, its dispersion and absorption relations are identical.

The complex refractive index in Eq. (1.9) is expressed as the real and the imaginary part as \( n(\omega) = n_r(\omega) + i n_i(\omega) \), as shown in Figure 1.3. The imaginary part \( n_i(\omega) \) is related to the absorption of incident light as it propagates through the medium at a distance \( z \). If the incident plane wave is given as

\[
E(z, t) = A_0 e^{i(k(\omega)z - \omega t)},
\]

where \( k(\omega) = \omega n(\omega)/c \) is the wave vector, the normalized intensity transmission of the incident light is given as

\[
T(z, t) = |E(z, t)|^2/A_0^2 = e^{-2\omega n_i(\omega) z/c} = e^{-\alpha(\omega) z},
\]

where \( \alpha(\omega) \equiv 2\omega n_i(\omega) z/c \) is the absorption coefficient. Equation (1.11) is known as Beer’s Law of absorption, which indicates the absorption of light intensity through a medium at distance \( z \). On the other hand, the real part of \( n(\omega) \) is related to medium dispersion, the variation of the (real) index of refraction in terms of frequency. Figure 1.3 shows that the medium dispersion is an increasing function of the frequency except for the region indicated by the dashed rectangle, where the slope of dispersion is negative.

The frequency range of negative slope is the so-called region of anomalous dispersion, and the region of positive slope is known as normal dispersion. The slope is related to the group velocity of pulse in a way that

\[
v_g \equiv \left[ \frac{dk_r(\omega)}{d\omega} \right]_{\omega_c}^{-1} = \frac{c}{n_r(\omega) + \omega \frac{dn_r}{d\omega}} \bigg|_{\omega_c}.
\]
As we see in the denominator of Eq. (1.12), $\nu_g < c$ for $dn/d\omega > 0$ where the medium is normally dispersive, and $\nu_g > c$ or $\nu_g < 0$ for $dn/d\omega < 0$ in the anomalously dispersive region. Figure 1.4 shows one of the Stenner’s fast-light data, which shows the comparison of Gaussian pulses propagating through vacuum and fast-light medium when the carrier frequency is set to the anomalous dispersion $\omega_c = \omega_0$. In the anomalously dispersive medium, the pulse advancement is about 27.4 ns compared to the pulse propagating through a vacuum. Note that, in his experiment, he uses a so-called “gain doublet” to create a region of anomalous dispersion (negative slope) using amplifying resonances rather than the absorption resonance I discuss here.

Figure 1.4: Fast-light pulse propagation through a vacuum (solid-line), and through a medium (dashed line). (Figure 1.4 from Ref. [16].)
1.2 Information velocity research in fast-light medium

At this point, one may ask the question “Is causality, predicted by Einstein’s special theory of relativity, preserved in Stenner et al.’s experiment?” More specifically, “Does information travel faster than the speed of light in vacuum?” To answer this question, Stenner first developed an experimental technique to measure the information velocity. He encoded ideal symbols “1” and “0” as information (Figure 1.5) on his input pulse, then sent the information through both a fast-light medium and vacuum. Figure 1.6 shows the result of his experiment. The pulse advancement (dashed line) of the front side of the pulse is as large as that shown in Figure 1.4(a). The arrival time of information is, however, almost identical for the fast-light medium or vacuum, as shown in Figure 1.6(b). He determined the arrival time of information
statistically in terms of “the fraction of incoming bits that are mis-identified,” which is also known as the “bit error rate (BER).” His results helped address controversial debates about causality and confirmed the theoretical prediction that the information velocity is equal to $c$ [7]. This first measurement of the information velocity in a fast-light medium was a real breakthrough.

This novel experiment, however, has shortcomings. One cannot clearly see optical precursors due to the point of non-analyticity created when information was encoded in the pulses because of the smooth turn-on of the Gaussian pulse. Precursors might exist, but may be smeared out by the smooth turn-on. Hence, this shortcoming motivates me to undertake an experiment to directly observe optical precursors.

1.3 Fundamentals of optical precursors

In the previous section, by introducing Stenner’s information velocity research, I present a motivation for my research. In this section, I describe the origin and characteristics of optical precursors.

Without great detail, it is useful to imagine a simple cartoon of the generation of optical precursors. Figure 1.7 shows an example, as described in electromagnetism text books or old literatures [12,13,17]. When a sharp edge on a pulse propagates through a dispersive media, the transmitted field has a transient signal followed by a main signal. The transient behavior (indicated by the dashed oval in Figure 1.7) is known as the precursors.

To understand the origin of this transient behavior, let us imagine the transient interaction between light and matter as shown in Figure 1.8. An incident pulse has a front where the field is suddenly turned on. When it enters a dispersive medium, the
Figure 1.6: Fast symbol pulses. Each symbol shown here is an average of 50 individual symbols. (a) The two symbol pulses after traveling through the fast-light medium (dashed lines) and vacuum (solid lines). (b) A close-up of the data in (a). (Figure 1.6 from Ref. [11].)
Figure 1.7: The generation of optical precursors, which are the transient part of the transmitted field.

Figure 1.8: Self-consistent description of the interaction of light with matter.
incident field polarizes a collection of microscopic dipoles, thereby creates a macroscopic polarization. The macroscopic polarization generates a radiative field, which interferes with the incident light. The modified field interacts with the collection of dipoles again, and the above events are repeated. As a result, it takes a finite time for the steady-state polarization to build up. Related to this, Sommerfeld predicted, in the early 1900’s, that the first part of a steep edge-pulse passes straight through the medium [12]. Consequently, the front of a pulse travels at $c$. He called the first part of the field right after the front a “forerunners,” which is also known as a “precursors.”

To place these heuristic arguments on a solid mathematical foundation, I recall the derivation of the exact expression of the transmitted field through a causally dispersive media. The electromagnetic wave equation is formed from Maxwell’s equations in dielectric materials [13], and is given by

$$\nabla^2 \vec{E}(z, t) - \frac{1}{c^2} \frac{\partial^2 \vec{E}(z, t)}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial^2 P(z, t)}{\partial t^2}. \quad (1.13)$$

I assume the field is a plane-wave whose propagation vector is along the $z$-direction. Furthermore, I assume that the beam of light is polarized and that the medium is isotropic. Thereby the propagation of light is described by the scalar 1-dimensional wave equation, which is given as

$$\frac{\partial^2 E(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E(z, t)}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial^2 P(z, t)}{\partial t^2}. \quad (1.14)$$

To solve this equation, we need to know the polarization of the medium $P(z, t)$. As we have seen earlier [Eq. (1.6)], the polarization can be obtained in the frequency $\omega$ domain. Therefore, we need to transform the wave equation into the frequency do-
main. The Fourier transform of the one-dimensional electromagnetic wave equation is given as

\[
\frac{\partial^2 E(z, \omega)}{\partial z^2} + \frac{\omega^2}{c^2} E(z, \omega) + \frac{4\pi\omega^2}{c^2} P(z, \omega) = 0,
\]

(1.15)

where \(k^2(\omega) = \varepsilon(\omega)\omega^2/c^2\). The solution of Eq. (1.15) is given as

\[
E(z, \omega) = E(0, \omega) e^{ik(\omega)z}.
\]

(1.16)

A physical interpretation of this equation is as follows. As the incident pulse \(E(z = 0, t)\) propagates through a medium distance \(z\), each frequency component of the incident pulse \(E(0, \omega)\) interacts with the medium in a different manner, causing changes in the spectrum \(E(z, \omega)\). The transfer function \(e^{ik(\omega)z}\) characterizes the medium response: the real (imaginary) part of the complex wave number \(k(\omega)\) contains the information about the medium dispersion (absorption). The complex wave number \(k(\omega)\) is related to the complex index of refraction \(k(\omega) = \omega n(\omega)/c\).

The temporal evolution of transmitted field at a medium penetration depth \(z\) is obtained by the inverse Fourier transform of the transmitted spectrum \(E(z, \omega)\) through the relation

\[
E(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(0, \omega) e^{i(k(\omega)z-\omega t)} d\omega.
\]

(1.17)

When the incident beam is taken as a step-modulated sinusoidal electric field of the form \(E_{in}(z = 0, t) = A_0 \Theta(t) \sin(\omega_c t)\), where \(\Theta(t)\) is the unit step function, the
transmitted field is given as,

\[ E(z, t) = \frac{A_0}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\omega - \omega_c} e^{i\phi(\omega, \theta)/c} d\omega, \] (1.18)

where the spectrum of the incident pulse is \( E(0, \omega) = iA_0/\sqrt{2\pi}(\omega - \omega_c) \), the phase of the integrand is \( \phi(\omega, \theta) = i\omega(n(\omega) - \theta) \), and \( \theta = ct/z \) is the retarded time. Equation (1.18) is the starting point of the theories discussed in Chapters 3 and 4. As will be seen later, this transmitted field consists of transient (the precursors) and steady-state responses (the main signal).

### 1.4 History of precursor research

In the previous section, I introduced the fundamentals of optical precursors, and derive the mathematical expressions of the wave equation to start the study of optical precursors. In this section, I will review the theoretical study of optical precursor related to solving the integral form of the wave equation Eq. (1.18), as well as a few experimental demonstrations.

Optical precursors were first introduced by Sommerfeld and Brillouin when they studied a step-modulated pulse propagating through a dispersive medium. The purpose of their study was to demonstrate that signals cannot travel faster than \( c \) even in regions of anomalous dispersion for which \( v_g > c \) or \( v_g < 0 \) [12]. To characterize the signal propagation, they considered a step-modulated pulse, which is suddenly turned-on at a time before which there is no light. They denote the moment when the light is turned on as a “front.”

Sommerfeld and Brillouin considered the dispersive medium as a collection of Lorentz oscillators possessing a single resonance. The refractive index for a single-
resonance Lorentz dielectric (or two-level atoms) is given by Eq. (1.9). To obtain
the transmitted pulse through this causally-dispersive medium, they used asymp-
totic methods because there is no known analytic solution to Eq. (1.18). The carrier
frequency of the pulse was different from the resonance frequency (off-resonance con-
dition) through most of their analysis. They also assume a broad medium resonance
(large $\delta$) and a large plasma frequency $\omega_p$. With these assumptions, Sommerfeld
showed that the front edge of the pulse travels at $c$ [12], thereby proving that signals
propagate in a relativistically causal manner. After the front, Brillouin found that
the pulse breaks up into two small wave packets, followed by a large wave packet
(the ‘main signal’) as shown in Figure 1.7. The first (second) small wave packet is
now known as the Sommerfeld (Brillouin) precursor, which arises from the spectral
components of the incident pulse above (below) the resonance.

Brillouin predicted that it is hard to detect precursors because of their small
intensities compared to the main-signal. The Sommerfeld (Brillouin) precursor is
predicted to have a maximum intensity of $10^{-7}$ ($10^{-4}$) of the eventual main signal
intensity. Although Brillouin’s analysis is quoted in many electricity and magnetism
textbooks, several researchers have identified mistakes in his early work, as reviewed
by Oughstun and Sherman [14]. Most notable is the prediction that the precursors
are much larger than previous thought; their amplitude can be similar to or larger
than the amplitude of the main signal. Even though the precursors are predicted to
be large, the conventional wisdom is that they are a strictly an ultra-fast phenomena,
lasting for only a few optical cycles and hence indirect detection methods are
required. This conclusion, however, is based on the fact that the theories have only
been evaluated for a medium with a broad resonance.

Precursor-like behavior was first observed for microwave pulse propagation in
a ferrimagnetic coaxial waveguide near cut-off [18]. This experiment used artificially controlled dispersion characteristics, and hence did not test the theories of pulse propagation through a Lorentz dielectric. By virtue of modern experimental technologies, optical precursors have been indirectly observed for pulses propagating through a semiconductors when the pulse carrier frequency is set near an exitonic resonance [19, 20], and in pure water [21]. However, none have made a quantitative comparison with the modern asymptotic theory by Oughtsun and Sherman [14] for the case of pulse propagation through a resonant dielectric in a region of anomalous dispersion.

1.5 The first direct measurement of optical precursors

From the investigation of previous precursor research, I came up with experimental conditions for observing large-amplitude optical precursors that persist for a long time. The condition for large-amplitude precursor is shown in Figure 1.9, one of the results from the modern asymptotic analysis by Oughtsun and Sherman [14]. They predict large-amplitude precursors, especially when they set the carrier frequency of input pulse to the medium resonance frequency ($\omega_c = \omega_0$). They did not seem to pay attention to the fact that the on-resonance carrier frequency results in large-amplitude precursor, which is an important fact.

Despite the large-amplitude of the predicted precursors, one might not be able to observe them directly using the parameters used to obtain Figure 1.9 because the precursor time scale is still about 1 fs. Figure 1.9 shows that the main signal arrives at about 1 fs ($t = \theta z/c = 31.00 \times 10^{-6} \div 3 \times 10^{10}$) after the front edge emerges.
Figure 1.9: Precursors predicted for the pulse propagation through a resonant dielectric obtained using Oughstun and Sherman’s modern asymptotic methods. They used the parameters: $\omega_c = \omega_0 = 4 \times 10^{16} \text{ s}^{-1}$, $\delta = 0.28 \times 10^{16} \text{ s}^{-1}$, and $z = 1 \times 10^{-6} \text{ cm}$. (Figure 1.9 from Ref. [14].)
from the medium. I noticed that this time scale is roughly the inverse of the broad resonance linewidth ($\delta \sim 0.1\omega_0$) they used in their analysis. The time scale of the precursors appears to be inversely related to the linewidth $\delta$.

To understand the inverse relation, we should remember the fact that the precursor time scale is the time scale of the transient response of the medium to the incident light. Figure 1.10, for example, shows a two-level atom characterized by a resonance linewidth $\delta$ (half width at half maximum), which is the quantum counterpart of the damping rate in the classical Lorentz model. As in many quantum optics text books, the resonance linewidth $\delta$ is related to the decay time of the dipole transition. As I will discuss later, time scale of the precursor field is proportional to $1/\delta$. Therefore, if I have narrow medium resonance, the precursor time scale is extended.

Obtaining a narrow medium resonance $\delta$ is the first stage of my experiment. I prepare a dielectric medium consisting of a cloud of cooled and trapped potassium ($^{39}$K) atoms generated in a vapor-cell magneto-optic trap (MOT). The cloud has

**Figure 1.10:** Monochromatic light interaction with two level atoms (a) energy level diagram, and (b) frequency-dependent transmission of a very weak probe laser beam as it passes through the gas.
a diameter $L \sim 1$-2 mm, and total atomic number density of $\sim 1 - 2 \times 10^{10}$ cm$^{-3}$. The temperature of the trap is $\sim 400$ µK, which is cold enough to freeze out the atomic motion and avoid Doppler broadening. Therefore, a magneto-optical trap is a useful tool for extending the precursor time scale up to tens of ns. Once a cold potassium gas is prepared, the next step is to prepare a single- or double-Lorentz resonance by optical pumping. Once the medium is prepared, a weak continuous wave laser beam is sent through an electro-optical switch that is driven by an edge-generator to create a step-modulated pulse. The step-modulated pulse is sent to the cold potassium atoms, and its transmitted intensity reaches a detector as shown in Figure 1.11. The experimental procedure will be discussed in Chap. 2.

![Experimental setup for direct observation of optical precursors.](image)

**Figure 1.11:** Experimental setup for direct observation of optical precursors.

### 1.6 Results

In the previous section, I introduced two conditions for the first direct observation of optical precursors and the experimental techniques. In this section, I will briefly highlight my experimental results.
Figure 1.12 shows the temporal evolution of the step-modulated pulse propagating through the cloud of atoms for the case of different absorption path lengths $\alpha_0 L$ and the case when $\omega_c = \omega_0$, where the dispersion is anomalous and $\alpha_0 = \alpha(\omega_0) = 2\omega_0 n_i(\omega_0)/c$. For moderate absorption ($\alpha_0 L = 0.41$), the pulse intensity rises immediately to $\sim 95\%$ of the incident pulse height and then decays to the steady-state value expected from Beer’s Law. As discussed below, the initial high-transmission transient spike is made up of both the Sommerfeld and Brillouin precursors. The time scale of the transient spike, defined as the time from the initial turn-on to the $1/e$ decay of the precursors, is found to be $\sim 32$ ns. For the maximum absorption path length ($\alpha_0 L = 1.03$), the initial transmission also reaches $\sim 95\%$, decays more rapidly ($\sim 16$ ns), and reaches the steady-state value expected from Beer’s Law.

It can also be seen from Figure 1.12 that the leading edge of all the pulses occur at the same time. This observation is consistent with Sommerfeld’s prediction that the pulse front propagates at $c$. If the group velocity predicted the speed of the pulse front, I would have expected that the leading edge would be advanced by 6.8 ns for the case when $\alpha_0 L = 0.41$ and 17.0 ns for $\alpha_0 L = 1.03$ as discussed in greater detail in Ch.2. Thus, my experiment is consistent with relativistically causal information propagation in a region of anomalous dispersion [11].

The data is taken not only for $\omega = \omega_0$, but also for $\omega \neq \omega_0$. For the case of $\omega_c \neq \omega_0$, Figure 1.13 shows the temporal evolution of the step-modulated pulse propagating through the medium when the detuning is near $\Delta \equiv \omega_c - \omega_0 \sim 5\delta$ or $\Delta \sim \delta$. In both cases, the absorption depth is fixed at $\alpha_0 L = 1.03$. For the case of $\Delta \sim 5\delta$, the transient transmission rises to $\sim 90\%$ right after the front and shows oscillation until it reaches to the steady state expected by Beer’s law. For $\Delta \sim \delta$ case, it still reaches $\sim 90\%$ peak and shows oscillation until it reaches its steady state,
Figure 1.12: Direct observation of optical precursors for the case of $\omega_c = \omega_0$. Experimentally observed optical transmission through the cloud of atoms (solid line). Dots are theoretical predictions with no adjustable parameters. For the theory plot (dots), the finite rise-time of the electronic-system [Eq. (3.59)] is included.
Figure 1.13: Direct observation of optical precursors for different carrier frequencies. Experimentally observed optical transmission through the cloud of atoms (solid line). Dots are theoretical predictions.

but the duration of the oscillation is longer than $\Delta \sim 5\delta$ case. As I will discuss later, the oscillations in the data have a frequency that are directly related to the detuning $\Delta$. The precursor amplitude is decreased and the main signal is increased as the carrier frequency is tuned far from the medium resonance.
1.7 Two theories of optical pulse propagation:

The asymptotic theory and the SVA theory

I briefly introduced the experimental data in the previous section. The data shows the transmitted pulse intensity, yet it is not clear which part of the waveform is the precursors or the main signal. To identify each part, I will discuss two theoretical approaches in Chs. 3-4, and compare both of the results to my data. In this section, a general description of two theories will be introduced for the simple case when $\omega_c = \omega_0$ (the “on-resonance” condition). Extension of theories for the off-resonance situation ($\omega_c \neq \omega_0$) will be discussed in Chs. 3-4.

To determine which part of the observed signals correspond to the precursors, I evaluate the asymptotic analysis presented in Ref. [14] for my experimental conditions. In this analysis, the incident beam is taken as a step-modulated sinusoidal electric field of the form $E_{in}(z = 0, t) = A_0 \Theta(t) \sin(\omega_c t)$. The transmitted field can be written in three distinct parts: the Sommerfeld precursor $E_S(z, t)$, the Brillouin precursor $E_B(z, t)$, and the main signal $E_C(z, t)$ so that the total transmitted field is given as

$$E(z, t) = E_S(z, t) + E_B(z, t) + E_C(z, t).$$  \hspace{1cm} (1.19)

Figure 1.14 (a) shows the envelope of the Sommerfeld (Brillouin) precursors as a dots (dashed line), and the total precursor is indicated as a solid line. The precursor amplitudes are of the same order and are out of phase, resulting in a partial cancellation of their combined amplitudes. The frequency of the Sommerfeld (Brillouin) precursor experiences a rapid chirp from infinite (zero) frequency to $\omega_0 \pm \delta$ within
∼10 fs, which is comparable to a few optical periods (2.6-fs-long optical period). On the several hundred nano-second scale, the femto-second order chirp is almost instantaneous, so that the frequencies of both precursors are the same for the all times beyond ∼10 fs. It is seen that the precursors and the main signal arrive immediately after the front and that the Sommerfeld and Brillouin precursors decay exponentially. This result confirms my earlier statement that the persistence of the precursors is governed by the resonance linewidth δ. Furthermore, in Figure 1.14 (b), It can be seen that the main signal is just the incident step-modulated field reduced in amplitude by an amount expected from Beer’s Law. Based on these findings, I conclude that the transient spike observed in my experiments (Figure 1.12) is composed of both the Sommerfeld and Brillouin precursors, which sit on top of the main signal. Figure 1.12, therefore, constitutes the first direct measurement of precursors in the optical domain for a step-modulated field propagating through a
medium characterized approximately as a single-resonance Lorentz dielectric.

The asymptotic analysis is valid in the so-called mature dispersion limit, where \( \alpha(\omega_c) z \gg 1 \). In my experiment, the largest absorption depth borders on this limit. Thus, I expect that it will only provide a qualitative understanding of my results. A detailed analysis will be discussed in Ch. 3. Some problems arise when the modern asymptotic analysis is applied to my experimental conditions. For example, the analysis predicts huge-amplitude precursors (of the order of \( \sim 10^{10} \)), as seen in Figure 1.14 (a). This problem persists even for the mature-dispersion regime. These unrealistic problems might indicate limits or errors in the asymptotic analysis. These problems are addressed using an empirically-derived scale factor. I do not know the origin of these problems. I have worked with K. Oughstun and we have been unable to find the error in the asymptotic analysis.

Another theoretical approach for understanding my experiments is to assume that the plasma frequency is small \((\omega_p \ll \sqrt{\omega_0 \delta})\), the material resonance is narrow \((\delta \ll \omega_0)\), the field is nearly resonant with the material oscillators \((\omega_c \sim \omega_0)\), and that the field varies slowly \([15, 22]\). Under these assumptions, known as the slowly-varying amplitude (SVA) approximation, it is possible to obtain an analytic solution describing the propagation of the step-modulated field through the single-Lorentz dielectric \([23, 24]\). The field is given by

\[
    E(z, t) = E_{SB}(z, t) + E_C(z, t),
\]

where \( E_{SB}(z, t) \) is the transient response of the propagated field and hence should be equal to the sum of the two precursors \( E_S(z, t) + E_B(z, t) \). It is not easily separated into the individual Sommerfeld and Brillouin precursor fields, yet I can easily
Figure 1.15: Envelope of each transmitted field by SVA theory. (a) The envelope of the total precursor $E_{SB}(z, t)$. (b) the main signal $E_{C}(z, t)$.

compare the total precursor field predicted by the two theories. The second term represents the main signal and is identical to $E_{C}(z, t)$ in the asymptotic theory. The SVA analysis reveals that the intensity transmission jumps to essentially 100% immediately after the front.1

As seen in Figure 1.16 (a), both theories2 agree well for $L = 0.2$ cm corresponding to the experimental data shown in Figure 1.12. As the path length increases [Figure 1.16 (b)-(d)], the SVA theory predicts oscillations in the precursor field envelope that are not present in the asymptotic theory. The oscillations are due to the absorption of the central part of the pulse spectrum by the material resonance and subsequent beating between the remaining sidebands [22–24]. Over the three-orders-of-magnitude change in path length shown in Figure 1.16, I see that the

---

1The data shows ~95% transmission rather than 100%, because the finite bandwidth of the electronics in my experiment smooths out the peak.

2The asymptotic theory is modified empirically.
Figure 1.16: Temporal evolution of the total precursor field envelope for the SVA theory (solid line) and the scaled asymptotic theory (dotted line) for (a) $z=0.2$ cm, (b) 2 cm, (c) 20 cm, and (d) 200 cm.
maximum amplitude of the oscillations is in reasonable agreement with the modified asymptotic theory, indicating that my empirical scale factor captures most of the discrepancy between the theories.

The two theories also explain the observed oscillation appearing in the transmitted intensity for the case of \( \omega_c \neq \omega_0 \) and \( L = 0.2 \) cm (Figure 1.13). According to both theories, the modulation results from beating between the total precursors and the main signal. The central frequency of the main signal is the same as the \( \omega_c \), while the central frequency of the precursors is \( \omega_0 \). The modulation frequency appearing in the data agree well with \( (\omega_c - \omega_0)/2\pi \).

Both theories for the single-resonance case can be extended to the case of a double-resonance. The solution for double resonance case is assumed to be the sum of the solution for two independent single-resonances media. This assumption is valid when both resonance peaks are well separated, the medium is weakly dispersive, and the resonances are narrow. These assumption describes my data reasonably well as will be discussed in Ch. 6.

### 1.8 Summary

I have introduced a new experimental technique for the first detection of optical precursors, its results, and theories explaining the data. In later chapters, the details of these works will be described. In Chapter 2, I will discuss the experimental methods and the results in detail. The modern asymptotic analysis will be presented in Chapter 3 to identify each part of the transmitted field. In Chapter 4, the slowly varying amplitude approximation (SVA) theory will also provide analytic expression for the transmitted field. The two theories will then be compared to my data and
to each other in Chapter 5. In Chapter 6, I extend both theories to study the case of a double-resonance medium. Finally, I will conclude my work in Chapter 7.
Chapter 2

The Precursor Experiment

In this chapter, I will present my experimental methods for the first direct observation of optical precursors and its results. To directly measure optical precursors, three steps are used. First, I prepare a cloud of cold potassium ($^{39}$K) atoms to obtain a dielectric medium with a narrow resonance, which is the key to extending the precursor duration. The cold atoms are optically pumped into one of the lower-energy ground states\(^1\) ($F = 1$ or $F = 2$) of the potassium $4S_{1/2}$ state, thereby preparing essentially a single- or double-resonance Lorentz dielectric. Once atoms are optically pumped, I send a weak-intensity, step-modulated pulse (carrier frequency $\omega_c$) through the medium and measure the intensity of the transmitted pulse. In the next three sections, I will discuss further the details of each experimental step as shown in Figure 2.1. The results will then be presented.

\(^1\)\(F\) or $F'$ denotes the hyperfine energy levels, which result from the interaction between the total angular momentum of the electron $J = L + S$ and the nuclear spin $I$, where $L$ is the orbital angular momentum of the electron and $S$ is the electron spin. See Appendix A for more details about the addition of angular momentum.
Figure 2.1: Three time sequences of the experiment. Stage 1 is a MOT preparation. Stage 2 is for optical pumping. Stage 3 is the optical precursor experiment performed at D\textsubscript{1}. The time sequence shows 20 $\mu$s turn-off time for Stage 2.
2.1 Stage 1: Preparation of a narrow-resonance Lorentz dielectric

A narrow atomic resonance is a crucial ingredient to extend the precursor time scale up to tens of nano-seconds. To achieve a narrow-resonance, we need to remove broadening mechanism such as Doppler broadening or collision broadening. Resonance in condensed matter systems, for example, are heavily effected by its neighborhood via phenomena of phonon interactions, resulting in a short coherence decay time, which causes a broad resonance line width. Such broad resonances were used in the previous experiments on optical precursors [19–21]. On the other hand, an atomic system has a longer coherence time scale with sharp energy levels because each atom is almost isolated from the neighbor. In a warm vapor cell, there is still the Doppler effect arising from atomic motion even when collisional broadening can be removed. Hence, atom cooling and trapping techniques are one way to achieve a narrow resonance in an atomic system. In my experiment, I prepare cold potassium atoms generated in a vapor-cell magneto-optic trap [25–27], as shown in Figure 2.2. More details of the trap are given in Appendix A.

The magneto optic-trap for neutral potassium (\(^{39}\text{K}\)) atoms is realized using laser beams tuned to the \(4S_{1/2} \leftrightarrow 4P_{3/2}\) transition (\(D_2\) transition, 767-nm transition wavelength) and anti-Helmholtz coils to create a gradient magnetic field. In the presence of this spatially inhomogeneous magnetic field, the atoms are captured using trapping and repumping lasers tuned to the \(4S_{1/2}(F = 2) \leftrightarrow 4P_{3/2}\) and \(4S_{1/2}(F = 1) \leftrightarrow 4P_{3/2}\) transitions, respectively, as shown in Figure 2.3 (e). (See Appendix A for more details.)

The trapped atoms have a temperature of \(\sim 400 \mu\text{K}\) measured by the release-and-
Figure 2.2: Experimental setup. EG: Edge generator, OSC: oscilloscope, APD: avalanche photo diode, PMT: photo multiplier tube, MZM: Mach-Zehnder modulator.
recapture method [28]. At such a cold temperature, the motion of the potassium atoms is essentially frozen, and the resonance linewidth becomes of-the-order of the natural linewidth (full width at half maximum) \(2\Gamma/2\pi = 1/2\pi t_{sp} \sim 6 \text{ MHz}\), where \(t_{sp}\) is spontaneous decay time of \(^{39}\text{K}\). This width is much narrower than the resonance width of a Doppler-broadened potassium vapor at room temperature, which is typically 800 MHz. Therefore, the estimated precursor time scale is \(1/2\delta \sim 1/2\Gamma \sim 26 \text{ ns}\). The diameter of the MOT, which I take to be the length \(L\) of the medium is determined to be in the range of \(\sim 1-2 \text{ mm}\) by measuring the \(1/e\) of the fluorescence of the MOT. The atomic number density is \(\sim 1-2 \times 10^{10} \text{ cm}^{-3}\) by measuring the absorption of a weak probe beam passing through the MOT.

The optical precursor experiment is conducted on the \(4\text{S}_{1/2} \leftrightarrow 4\text{P}_{1/2}\) transition (\(D_1\) transition, 770-nm transition wavelength). It is useful to define a notation associated with the \(D_1\) transition for the precursor experiment, as shown in Figure 2.4. At the \(D_1\) transition, there are two ground states \((F=1,2)\) and two excited states \((F'=1,2)\). The ground state splitting is \(\Delta_g = 462 \text{ MHz}\) and the excited state splitting is \(\Delta_e = 58 \text{ MHz}\). The four possible transitions between the ground states and the excited states are given as \(F'F=12,22,11,\) and \(21\), where \(F'F\) denotes the \(4\text{S}_{1/2}(F) \leftrightarrow 4\text{P}_{1/2}(F')\) transition. There are two pairs of double-resonance peaks separated by the ground state splitting \(\Delta_g\). Each pair of double-resonance peaks are separated from one another by the excited state splitting \(\Delta_e\). The four resonance peaks are denoted as \(\omega_{F'F}\), as seen in Figure 2.3. Note that the two absorption peaks at \(\omega_{11}\) and \(\omega_{21}\) are associated with transitions between the lower ground state \(F = 1\) and the two excited states \(F' = 1,2\). Similarly, the other two peaks at \(\omega_{12}\) and \(\omega_{22}\) are related to the transition between \(F = 2\) and \(F' = 1,2\). Therefore, the four peaks can be grouped as two kinds \((\omega_{12}, \omega_{22})\) or \((\omega_{11}, \omega_{21})\). To define a detuning \(\Delta_F \equiv \omega_{c} - \omega_{2F}\)
Figure 2.3: Transmission scan through four-resonance peaks of \( D_1 \) transition for the case of (a) both MOT beams on for 80 \( \mu s \), and (b) optical pumping into \( F=1 \) state with trapping beam off for 20 \( \mu s \). Energy level diagrams of \( D_1 \) transition and population distribution in the ground states explaining the transmission scan by weak probe beams (c) for the case of (a), and (d) for the case of (b). Energy level diagrams of \( D_2 \) transition showing MOT beams and optical pumping (e) in the case of (a), (f) in the case of (b).

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Figure 2.4: The notations associated with the $D_1$ transition.
within each pair, there are two reference resonance frequencies $\omega_{2F}$ for each pair; one is $\omega_{21}$, and the other is $\omega_{22}$. These definitions are summarized in Figure 2.4.

Figure 2.3(a) shows the measured frequency-dependant steady-state transmission $T$ for a weak probe beam propagating through the medium. The four dips correspond to the allowed D$_1$ transitions, as shown in Figure 2.3 (c). The absorption $A$ is directly related to the transmission through the relation $A = 1 - T = 1 - \exp(-\alpha_{F'F}L)$ (Beer’s law), where $\alpha_{F'F}$ is the line-center absorption coefficient at $\omega_{F'F}$. The macroscopic absorption coefficient is related to the microscopic absorption cross section $\sigma_{F'F}$ as $\alpha_{F'F} = N \sigma_{F'F}$, where $N$ is atomic number density. The absorption cross section $\sigma_{F'F}$ can be evaluated using the density matrix, which contains information about the dipole transition strengths [15]. All of the dipole matrix elements can be obtained using the addition rules of angular momentum theory (See Appendix A).

Finally, from the measured peak absorption $A$, the path length $L$, and theoretically determined $\sigma_{F'F}$, I can estimate the number density $N$.

At Stage 1, the two MOT beams are on, and the ground states are equally populated so that the absorption from the ground states to the excited states are well balanced, as shown in Figure 2.3 (c). In Stage 2, we are left with a single- or double-resonance peaks [Figure 2.3 (b)] from the four-resonance dips [Figure 2.3 (a)] by optical pumping, which I will describe in the next section.

2.2 Stage 2: Preparation of a single- or double-Lorentz dielectric

To obtain a single- or double-resonance Lorentz medium from the balanced four absorption peaks at $\omega_{F'F}$ [Figure 2.3 (a)], only one of the two ground states need
to be filled with atoms. The \((\omega_{11}, \omega_{21})\) pair is essentially a single resonance at \(\omega_{21}\) because the weak absorption peak at \(\omega_{11}\) does not affect the substantial propagation of a pulse whose carrier frequency is tuned to the center of the other strong peak, as will be discussed in later chapters. Hence, the gas of atoms behaves essentially as a single-resonance Lorentz dielectric. The other pair \((\omega_{12}, \omega_{22})\) has equal oscillator strength, which can be treated as double-resonance Lorentz medium. In addition, the two pairs are separated from one another by the ground state splitting \(\Delta_g = 462 \text{ MHz} \sim 100\delta\), so they never effect each other. That is, if one pair is used as a single- or double-resonance and the laser is tuned to this pair, the other does not effect the nature of the precursors. The atoms associated with the other pair need to be optically pumped into the pair used as a Lorentz medium to increase number density. The maximum number density of the Lorentz medium limits absorption depth \(\alpha_0 z\).

Optical pumping is based on absorption followed by spontaneous emission. Absorption and spontaneous emission take place when an incident light beam has a frequency resonant with a two energy-level system. An atom in the ground state is excited to the excited state by absorbing a photon from the light beam. The atom in the excited state then decays to the ground state via spontaneous emission of a photon. When we consider the situation in Figure 2.3 (e), for example, there are three energy levels (considering the excited states as one energy level). The two MOT beams are tuned near \(D_2\) transition \(4S_{1/2} \leftrightarrow 4P_{3/2}\). One beam is the so-called “trapping beam,” whose frequency \(\omega_{\text{trap}}\) is red-detuned near the \(4S_{1/2}(F = 2) \leftrightarrow 4P_{3/2}\) transition. The other is the “repumping beam,” whose frequency \(\omega_{\text{repump}}\) is red-detuned near the \(4S_{1/2}(F = 1) \leftrightarrow 4P_{3/2}\) transition. (See Appendix A for more details of the MOT.) If the repumping laser beam is turned off temporally, as shown
in Figure 2.3 (f), the trapping beam only excites atoms from the $F = 2$ state to the $F'$ state, atoms then decay spontaneously to either of two ground states $F = 1$ or $F = 2$. Finally, the number of atoms in the state $F = 2$ decreases while the number in the $F = 1$ state increases via repeated excitation followed by spontaneous decay.

To achieve optical pumping that places atoms in the $4S_{1/2}(F = 1)$ state, the repumping beam is repeatedly switched off for 20 $\mu$s (Figure 2.3 (f)) and turned back on for 80 $\mu$s, returning to Stage 1 (Figure 2.3 (e)). The optical pumping time interval 20 $\mu$s is determined to balance two conditions. One is to have enough time for the atomic states to reach equilibrium after optical pumping and to perform the optical precursor experiment. At the same time, I need to keep the total number of atoms in the trap constant during the optical pumping time. While the repumping beam ($4S_{1/2}(F = 1) \leftrightarrow 4P_{3/2}$) is off, the potassium atoms are optically pumped into the $4S_{1/2}$ ($F=1$) hyperfine level by the trapping beam as in Figure 2.3(f). Once in this state, the optical precursor experiment is conducted and the process is repeated.

Once a single- or double-Lorentz medium is prepared by optical pumping, the medium resonance properties are characterized by measuring the resonance linewidth $\delta_{F'F}$, and the line-center absorption coefficient $\alpha_{F'F}$. To determine the width of the resonance $\delta_{F'F}$ (half-width at half maximum) and the strength of the resonance (quantified by the plasma frequency $\omega_{p_{F'F}}$, which is related to the line-center absorption coefficient through the relation $\alpha_{F'F} = \omega_{p_{F'F}}^2 / 2c\delta_{F'F}$), I measure the frequency-dependent steady-state transmission of a very weak probe laser beam as it passes through the medium.

Note that I only focus on optical pumping to the $F = 1$ state accomplished by the trapping beam as in Figure 2.3 (f). In the same analogy, the $F = 2$ state is optically pumped by turning off the trapping beam ($4S_{1/2}(F = 2) \leftrightarrow 4P_{3/2}$) instead of the repumping beams.

The subscript $F'F$ denotes the $F \leftrightarrow F'$ transition ($D_1$). The resonance linewidth $\delta_{F'F}$ depends slightly on each transition as shown in Table 2.1 because of broadening due to the gradient magnetic field.
though the gas as shown in Figure 2.5. I observe a Lorentzian-like resonance with a width $2\delta_{F'F}/2\pi \sim 8.0$-9.6 MHz (full width half maximum), which is broader than the 6-MHz natural linewidth.

The broadened linewidth consists of residual Doppler broadening (<1 MHz), Zeeman splitting arising from the gradient magnetic field (~2-3 MHz), and the laser linewidth (~200 kHz). The residual Doppler linewidth\(^5\) is given as $\delta_D = 2\sqrt{m/2}/T_\star^2 \sim 896$ kHz, where $T_\star^2 = \lambda\sqrt{m_K}/2\pi\sqrt{2k_B}T \sim 296$ ns, mass of potassium is $m_K = 6.47 \times 10^{-23}$ [g], the wavelength of light is $\lambda = 766.7 \times 10^{-7}$ [cm], and Boltzmann’s constant is $k_B = 1.38 \times 10^{-16}$ [erg/K]. Simply speaking, $\delta_D$ is obtained when the Maxwell distribution of velocities is convolved with the natural Lorenzian linewidth.

Zeeman splitting results from the magnetic field gradient along the probe-beam propagation direction, which is measured as $\partial_z B = 4$ Gauss/cm, where $\partial_z B \equiv \partial B/\partial z$. The frequency shift per centimeter is given by the Zeeman energy shift $\partial_z \omega = g_J\mu_B\partial_z B\Delta M_F$, where the Landé g-factor $g_J \simeq 1$, $\mu_B = 1.4$ MHz/Gauss, and $\Delta M_F = 2$. In my case, $\partial_z \omega = 11.2$ MHz/cm, resulting in frequency shift by 2-3 MHz for 2 mm.

From measurements of the probe-laser-beam transmission $T$ at line center, I determine the line-center absorption coefficient through Beer’s Law: $T = \exp(-\alpha_{F'F}L)$. Table 2.1 contains characteristic parameters for each of the four transitions.

Up to now, the characterization of the medium has been performed using a steady-state transmission scan in the frequency domain. For temporal transient transmission, a fixed frequency is used and is the carrier frequency of the incident pulse. First, I set the carrier frequency at $\omega_c = \omega_{21}$ (the 4S\(_{1/2}\) (F=1) $\leftrightarrow$ 4P\(_{1/2}\) (F=2)

\(^{4}\)The same method to measure the atomic number density $N$ in the previous section.

\(^{5}\)Although the atoms are essentially frozen, they have residual motions at the low temperature of 400 $\mu$K.
Table 2.1: Parameters for four-resonance lines

<table>
<thead>
<tr>
<th>peak ID ($F'F$)</th>
<th>$2\Delta F/2\pi$ [MHz]</th>
<th>$\omega_{pp'}$ [s$^{-1}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>9.8</td>
<td>$2.09 \times 10^9$</td>
</tr>
<tr>
<td>22</td>
<td>9.8</td>
<td>$2.09 \times 10^9$</td>
</tr>
<tr>
<td>11</td>
<td>8.0</td>
<td>$1.43 \times 10^9$</td>
</tr>
<tr>
<td>21</td>
<td>9.8</td>
<td>$3.05 \times 10^9$</td>
</tr>
</tbody>
</table>

Figure 2.5: Optically pumping atoms into the two different ground states. Nine different frequencies chosen for (a) ($\omega_{11}$, $\omega_{21}$) and (b) ($\omega_{12}$, $\omega_{22}$). Energy level diagram for (c) ($\omega_{11}$, $\omega_{21}$), and (d) ($\omega_{12}$, $\omega_{22}$) for the case of optical pumping.
transition) for the case of an on-resonance carrier frequency, where the laser beams are interacting essentially with a single resonance whose frequency is $\omega_{21}$.

At the fixed carrier frequency $\omega_c = \omega_{21}$, I have two difference cases of the number of atoms $n$, thereby having two different absorption depths $\alpha_{21}L$. Related to this, I obtain two different sets of values, $T$ and $L$, by changing the total intensity of cooling and trapping laser beams. At the maximum intensity ($\sim 33$ mW/cm$^3$), I have the maximum atomic number density $1.16 \times 10^{10}$ cm$^{-3}$ and propagation length ($L=0.20$ cm) resulting in the minimum transmission of $T=0.36$ corresponding to $\alpha_{21} = 5.14$ cm$^{-1}$, $\alpha_{21}L = 1.03$, and $\omega_{p21} = 3.05 \times 10^9$ s$^{-1}$. By reducing the total intensity by $\sim 20\%$ of the maximum, I obtain a smaller atomic number density $7.72 \times 10^9$ cm$^{-3}$ and a propagation length ($L=0.12$ cm) resulting in $T=0.66$, corresponding to $\alpha_{21} = 3.41$ cm$^{-1}$, $\alpha_{21}L = 0.41$ and $\omega_p = 2.48 \times 10^9$ s$^{-1}$. With the trapping beams off, there are no trapped atoms and the transmitted pulse is identical to the incident pulse, which serves as a reference.

I also investigate the case of a detuned laser, I chose nine different frequencies near the $4S_{1/2}$ (F=1) $\leftrightarrow$ $4P_{1/2}$ (F′=1,2) transition for the second data set, as shown in the Figure 2.5 (a). These data sets are used to test two different models. One model being tested is a single-resonance case near $\omega_{21}$. In this case, frequencies “1” and “2” are used because there should be a symmetry between “1,5” and “2,4” from Figure 2.5 (a). The other is a double-resonance case, as seen in Ch. 6. Comparison of a single- and double- resonance model confirms my idea that the strong absorption line acts as a single-resonance while the weak one does not.

The detuning parameters are measured using a frequency-dependent steady-state transmission scan, as shown in Figure 2.7 (b). The value at each point is equal to the steady-state value in Figure 2.7 (a) obeying Beer’s Law. The transmission at each
Table 2.2: Experimentally determined detuning parameters for the double resonance pairs, $\omega_{F'F} = (\omega_{12}, \omega_{22})$ and $\omega_{F'F} = (\omega_{11}, \omega_{21})$.

<table>
<thead>
<tr>
<th>point ID</th>
<th>$\Delta_1(F = 1 \leftrightarrow F')$ [MHz]</th>
<th>$\Delta_2(F = 2 \leftrightarrow F')$ [MHz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$27.1^{+3.2}_{-2.1}$</td>
<td>$33.6^{+12.4}_{-4.8}$</td>
</tr>
<tr>
<td>2</td>
<td>$5.3^{+0.5}_{-0.4}$</td>
<td>$6.5^{+1.6}_{-1.3}$</td>
</tr>
<tr>
<td>3</td>
<td>$0^{+2.3}_{-2.3}$</td>
<td>$0^{+3.1}_{-3.1}$</td>
</tr>
<tr>
<td>4</td>
<td>$-5.3^{+0.4}_{-0.5}$</td>
<td>$-3.9^{+1.3}_{-1.6}$</td>
</tr>
<tr>
<td>5</td>
<td>$-29.2^{+7.8}_{-6.5}$</td>
<td>$-39.9^{+9.3}_{-3.0}$</td>
</tr>
<tr>
<td>6</td>
<td>$-49.5^{+0.1}_{-0.1}$</td>
<td>$-53.3^{+0.3}_{-0.3}$</td>
</tr>
<tr>
<td>7</td>
<td>$-59.9^{+1.5}_{-1.4}$</td>
<td>$-58.6^{+2.6}_{-2.6}$</td>
</tr>
<tr>
<td>8</td>
<td>$-64.4^{+1.5}_{-2.0}$</td>
<td>$-63.4^{+0.3}_{-0.6}$</td>
</tr>
<tr>
<td>9</td>
<td>$-82.8^{+3.4}_{-10.0}$</td>
<td>$-74.1^{+5.9}_{-25.0}$</td>
</tr>
</tbody>
</table>

Point is mapped onto the detuning value along the horizontal axis of Figure 2.7(b).

The horizontal axis is calibrated using the excited state splitting $\Delta_e = 58$ MHz. Table 2.2 shows all the experimental detuning parameters. The errors associated with each measurement in Table 2.2 are determined by mapping the experimental errors in the absorption values onto the frequency-calibrated horizontal axis. The asymmetry in the error bars originated from the different slope at each point in the absorption scan [Figure 2.7(b)]. This method is used in measuring the detuning parameters for other data sets taken near the $(\omega_{11}, \omega_{21})$ pair.

Finally, by shifting the frequencies to a lower value equal to the ground state splitting $\Delta_g$, the third data set is taken for $(\omega_{11}, \omega_{21})$ pair. Nine different frequencies are also shifted by $\Delta_g$ and are chosen as shown in Figure 2.5(b). This third data set is taken to test a balanced double-resonance medium.
2.3 Stage 3: Step modulated pulse propagation

In the previous section, I discussed how I prepared the Lorentz dielectric medium, and determined the carrier frequency $\omega_c$ of the incident pulse with respect to the medium resonance. In this section, I will present the preparation of a step-modulated pulse.

A step-modulated field is created by passing a weak continuous-wave laser through a 20-GHz-bandwidth Mach-Zehnder modulator (MZM, EOSpace, Inc.) driven by an electronic edge generator (Data Dynamics, Model 5113) whose edge is steepened using a back recovery diode (Stanford Research Systems, Model DG535), resulting in an edge rise time of $\sim$100-200 ps. The peak intensity of the pulse is $\sim$64 $\mu$W/cm$^2$, which is much less than the saturation intensity of the transition ($\sim$3 mW/cm$^2$). A weak-intensity incident pulse is important to avoid nonlinear optical phenomenon, and to detect optical precursors in the linear-optics regime. The pulse is detected with a fast-rise-time (0.78 ns) photomultiplier tube (PMT, Hamamatsu, Model H6780-20) and the resulting electrical signal is measured with a 1-GHz-analog-bandwidth digital oscilloscope (Tektronix, Model TDS 680B). In the absence of the atoms, I measure an edge rise time (10%-90%) of $\sim$1.7 ns for the complete system, corresponding to a bandwidth of $\sim$206 MHz. This time is short in comparison to the expected duration of the precursors.

2.4 Results

In this section, I present experimental results for the temporal evolution of the transmitted field in three different data sets, as mentioned in the previous section. One is the data for the case of an on-resonance carrier frequency $\omega_c = \omega_{21}$ for two
different optical depths $\alpha_{21}L$. Next, data is taken at nine different carrier frequencies near essentially a single-resonance (with a weak nearby resonance), obtained by optically pumping to the $F = 1$ state. Finally, data is taken at nine different frequencies near the equally-balanced double resonance (via optical pumping into the $F = 2$ state).

2.4.1 Transient transmission for $\omega_c = \omega_{21}$ with different absorption depths

Figure 2.6 shows the temporal evolution of the step-modulated pulse propagating through the cloud of atoms for the case of different absorption path lengths $\alpha_{21}L$ and the case when $\omega_c = \omega_0$, where the dispersion is anomalous. For moderate absorption ($\alpha_{21}L= 0.41$), the pulse intensity rises immediately to $\sim$95% of the incident pulse height and then decays to the steady-state value expected from Beer’s Law. As discussed later, the initial high-transmission transient spike is made up of both the Sommerfeld and Brillouin precursors. The time scale of the transient spike, defined as the time from the initial turn-on to the $1/e$ decay of the precursors, is found to be $\sim$32 ns. For the maximum absorption path length ($\alpha_{21}L$=1.03), the initial transmission also reaches $\sim$95%, decays more rapidly ($\sim$16 ns), and reaches the steady-state value expected from Beer’s Law.

I also see from Figure 2.6 that the leading edge of all the pulses occur at the same time. This observation is consistent with Sommerfeld’s prediction that the pulse front propagates at $c$. If the group velocity $v_g$ predicted the speed of the pulse front as in Eq. (1.8), the pulse advancement is given as

$$t_{adv} = L\left(\frac{1}{c} - \frac{1}{v_g}\right), \quad (2.1)$$
Figure 2.6: Direct observation of optical precursors transmission through the cloud of atoms. $\omega_c$ is fixed at point 3.
where

\[
v_g = \left[ \frac{dk_r(\omega)}{d\omega} \right]_{\omega_c}^{-1} = \left. \frac{c}{n_r(\omega) + \omega \frac{dn_r}{d\omega}} \right|_{\omega_c}.
\]  

(2.2)

Therefore, the leading edge would be advanced by 6.8 ns \( (v_g/c = -5.9 \times 10^{-4}) \) for the case when \( \alpha_{21}L = 0.41 \) and 17.0 ns for \( \alpha_{21}L = 1.03 \) \( (v_g/c = -3.9 \times 10^{-4}) \). I do not observe such advancement. Thus, my experiment is consistent with relativistically causal information propagation in a region of anomalous dispersion [11], where the pulse propagates at \( c \).

### 2.4.2 Transient transmission taken near the 4S\(_{1/2}\)(F = 1) \leftrightarrow 4P\(_{1/2}\) transition

Figure 2.7 shows the transmitted pulse intensities for the different carrier frequency \( \omega_c \neq \omega_{21} \) tuned near the 4S\(_{1/2}\)(F = 1) \leftrightarrow 4P\(_{1/2}\) transition and for the maximum absorption depth \( \alpha_{21}L = 1.03 \). The reference resonance frequency is \( \omega_{21} \) for the transition near the \( (\omega_{11}, \omega_{21}) \) pair; thereby \( \Delta_1 \equiv \omega_c - \omega_{21} \). Nine data sets are taken for different \( \Delta_1 \) and are labelled from (1) to (9) as shown in Figure 2.7(a).

For the case \( \Delta_1 \sim 5\delta_{21} \), Figure 2.7 (1) shows that the transient transmitted intensity immediately reaches \( \sim 90\% \) of the incident pulse height, oscillates with a modulation frequency of approximately \( \Delta_1/2\pi \), corresponding to the period \( \sim 40 \) ns. The oscillation amplitude then decays to the steady state obeying Beer’s law. Note that the steady-state values after the decay shown in Figure 2.7 (1)-(9) are equivalent to the steady-state transmission values shown in Figure 2.7(b) at each points of (1)-(9). The time scale for the transmitted intensity to reach steady state is similar to that observed for \( \Delta = 0 \), which implies that the precursor time scale only depends on \( \delta_{F'}F \). As discussed later, the oscillation of the envelope results from
Figure 2.7: Experimentally observed transient transmission taken near the $4S_{1/2}(F = 1) \leftrightarrow 4P_{1/2}$ transition.
frequency modulation between the precursors oscillating at $\omega_0$ and the main signal oscillating at $\omega_c$. For the moderately blue-shifted carrier frequency case ($\Delta \sim \delta_{21}$), as shown in Figure 2.7(2), the initial transmission also rises immediately to $\sim 90\%$, decays to the steady state with slower oscillation than the case when $\Delta \sim 5\delta_{21}$. Figure 2.7 (3) is taken at the resonance peak $\omega = \omega_{21}$ as in the previous section.

Figure 2.7 (4)-(9) shows the pulse intensity transmission for the case of red-shifted carrier frequencies from $\omega_{21}$ ($\Delta < 0$). Figure 2.7 (4) is taken at $\Delta \sim -\delta_{21}$, which shows similar behavior as in the moderate blue-shifted case of $\Delta \sim \delta_{21}$ in data (2). The data shown in (5), taken near $\Delta \sim -5\delta_{21}$, shows the same pattern as in the data shown in (1) for most blue-shifted case of $\Delta \sim 5\delta_{21}$. The data shown in (6)-(9) are taken near the weak absorption resonance at $\omega = \omega_{11}$. Relatively fast oscillation sitting on slow modulation has a time scale corresponding to the separation of two peaks $58 \text{ MHz} \sim 12\delta_{21}$. This separation is originated from the excited state splitting $\Delta_e/2\pi = 58 \text{ MHz}$ as shown in Figure 2.5. This fast oscillation resides on all nine data sets, but it is negligible near the strong absorption peak at $\omega_{21}$, while it is significant near the weak absorption peak at $\omega_{11}$.

For the case of $\Delta = 4.79\delta$ [the data sets shown in (1) and (5)], transient transmission intensity rises to $\sim 90\%$, oscillates until it reaches steady state value. During the oscillation, the intensity transmission shows $T > 1$. This phenomena can be explained as follows. When the medium polarization starts to react back to the incident light, the phase of the polarization field is delayed, thereby storing the energy of incident light for the delayed moments. The stored energy is emitted afterward sitting on top of the polarization field. The $T > 1$ is explained by both theories in later chapters.
2.4.3 Transient transmission taken near the $4S_{1/2}(F=2) \leftrightarrow 4P_{1/2}$ transition

I have discussed the transient transmission data associated with the $F=1$ (lower) ground state. In this section, I will discuss the other sets of data associated with the $F=2$ (upper) ground state. Figure 2.8(1)-(9) shows data taken at nine different carrier frequencies, but shifted down by $\Delta_g$ from the previous case associated with the $F=1$ state. In this case, the data shows similarities to that of the $F=1$ data. For example, the modulation depends on the detuning $\Delta_2$ as shown in Figure 2.7. The two types of modulations mentioned before are also seen: the relatively slow modulation at the frequency of the detuning $\Delta/2\pi$ and the fast modulation at the fixed frequency of $\Delta_e/2\pi$. Note that some data shows similarity among them, such as (1,5,9), or (2,4,6,8), or (3,7) for both types of modulation. As we will see in Ch.5, the fast modulation is larger than the data taken near strong absorption peak at $\omega_{21}$ in the previous section.

In this chapter, I present the experimental techniques used to measure the temporal evolution of the step-modulated field transmitted through a narrow-resonance Lorentz dielectric. The initial transient peak of the data shows essentially 100% transmission. To understand the transmitted field, I will discuss two theoretical approaches in Chs.3 and 4.
Figure 2.8: Experimentally observed transient transmission taken near the $4\text{S}_{1/2}(F = 2) \leftrightarrow 4\text{P}_{1/2}$ transition.
Chapter 3

Asymptotic Theory

In the previous chapter, I described the first experimental observation of optical precursor in a resonant medium. In this chapter, I evaluate the modern asymptotic theory of optical precursors put forth by Oughstun and Sherman in Ref. [14]. This theory allows me to identify each part of the transmitted field: the Sommerfeld and Brillouin precursors, and the main signal.

The modern asymptotic theory of Ref. [14] is very complicated, thereby making it difficult to obtain a physical understanding of the problem. Therefore, one important part of my thesis is to evaluate the asymptotic theory for my experimental conditions, and eventually obtain more transparent mathematical expressions. One notable aspect of my work is that it reveals several problems in the asymptotic theory, at least for my experimental conditions, as discussed later in this chapter.

To begin this chapter, I recall the inverse Fourier transform of the wave equation (Eq. (1.18)). When a step-modulated incident pulse propagates through single-Lorentz dielectric,¹ there is no analytic solution. The integral can be evaluated using asymptotic methods, which are valid in the limit when $z \to \infty$. This asymptotic methods are associated with “saddle points” methods. As discussed in Ref. [14], the asymptotic theory is valid when $z$ is greater than one optical penetration depth.

¹In Chapters 3-4, I only focus on the single-Lorentz dielectric case, then extend this to the case of double-Lorentz dielectric in Chapter 6. For the single-Lorentz model, I will use a general subscript “0” indicating on-resonance absorption peak. The absorption coefficient $\alpha_0$ is associated with the resonance peak.
$z_d \equiv \alpha_0^{-1}$. The inverse of absorption coefficient $\alpha_0^{-1}$ indicates the distance over which the incident electromagnetic field intensity decreases by $1/e$ of its initial value.

### 3.1 Saddle point method

To obtain a solution for the transmitted field in the asymptotic regime $z \gg z_d = \alpha_0^{-1}$, the so-called “saddle points method” is considered [13, 29]. In this section, I start with presenting the “stationary phase” approximation to understand some of the concepts underlying the saddle points method [13], and introduce the derivation of the saddle points and the results in Ref. [14].

In the stationary phase method, an integral can be evaluated at the stationary points if it has the form

$$I = \int A(\omega) e^{i q(\omega)} d\omega,$$

where $A(\omega)$ is a relatively slowly-varying function in terms of $\omega$, and $q(\omega)$ is a phase that has a large value, resulting in the rapid oscillation of $e^{i q(\omega)}$, as illustrated in Figure 3.1(a). Integration of the fast-oscillating sinusoidal wave over the whole range of $\omega$ is zero except when $q(\omega)$ is slowly varying near the stationary point $\omega_{sp}$, as shown in Figure 3.1(b). Here, the subscript ‘sp’ denotes the “stationary point,” but will indicate “saddle points” later. The stationary points are obtained through solution of the condition $\partial_\omega q(\omega)|_{\omega_{sp}} = 0$. At these stationary points $\omega_{sp}$, the phase has an extremum value $q(\omega_{sp})$, and the fast oscillation turns into a slow change. Consequently, $q(\omega_{sp})$ is the so-called “stationary phase” at the stationary point $\omega_{sp}$. The integral along the stationary point provides a non-zero contribution to the integral. The stationary-phase approximation is a special case (first leading-order term) of the “steepest-decent” method. The steepest-decent method is a subset
of the saddle points method for the case when \( q(\omega) \) is real-valued.

Now, let us extend our discussion to the saddle points. Equation (1.18) has the same structure as Eq. (3.1) if we take \( A(\omega) = 1/(\omega - \omega_c) \) and \( q(\omega) = z\phi(\omega, \theta)/c \). The phase \( z\phi(\omega, \theta)/c \) becomes large as \( z \to \infty \), and its saddle points are evaluated by \( \partial_{\omega} \phi(\omega, \theta)|_{\omega_{sp}} = 0 \). Therefore, Eq. (1.18) can be solved asymptotically along the saddle points. In the case of a single-resonance Lorentz medium, there are two types of saddle points at which the integral obtains a non-zero value. The two types of saddle points are related to the two types of transient responses: the Sommerfeld \( (E_S(z,t)) \) and Brillouin precursors \( (E_B(z,t)) \). On the other hand, the main signal \( (E_C(z,t)) \) is associated with the pole contribution to the integral.

The saddle points are not the only source of a non-zero contribution to the integral, as seen in Eq. (1.18). Note that the saddle points contribute the most when \( A(\omega) \) varies slowly compared to \( e^{i\eta(\omega)} \). For \( \omega = \omega_c \), however, the \( A(\omega) \) has a
singular point (pole) where it diverges rapidly. The pole contribution to the integral is related to the steady-state response of the medium to the incident field. The steady-state response is known as the main signal $E_C(z,t)$, as we will see later. Once the pole contribution dominates, the saddle-point contribution to the integral becomes negligible, although it exists. Therefore, the saddle-point method includes a pole contribution as well as the contribution from the saddle points.

To evaluate each part of the transmitted field asymptotically, Oughstun and Sherman have extended and refined Brillouin’s asymptotic analysis [12] in several aspects. They keep one more higher-order term in the saddle points equation [14], and apply modern mathematical methods to choose a convenient path of integration, the so-called “Olver-type path” [30]. They keep $\theta \equiv ct/z$ as a time-space coordinate in a moving frame, as defined by Brillouin.

Based on Oughstun and Sherman’s modern asymptotic analysis, we can find the saddle points $\omega_{sp}$ via $\partial_{\omega} \phi(\omega, \theta)|_{\omega_{sp}} = 0$. The phase of the integrand $\phi(\omega, \theta) = i\omega [n(\omega) - \theta]$ is obtained from the index of refraction $n(\omega)$ for the case of single-resonance Lorentz dielectric (equivalent to a collection of two level atoms). Therefore, the phase of integrand is given as

$$\phi(\omega, \theta) = i\omega \left( \sqrt{\frac{\omega^2 - \omega_1^2}{\omega^2 - \omega_0^2 + 2i\omega\delta}} - \theta \right), \quad (3.2)$$

where $\omega_1^2 \equiv \omega_0^2 + \omega_p^2$.

There are two types of saddle points $\omega_{sp}$, as shown in Figure 3.2. One class is the “distant” saddle points $\omega_{sp_D}^{\pm}$ [Eq. (7.2.2a) of Ref. [14]], and are equal to

$$\omega_{sp_D}^{\pm} \cong \pm \xi(\theta) - i\delta(1 + \eta(\theta)),$$
where

\[ \xi(\theta) \equiv \sqrt{\omega_0^2 - \delta^2 + \frac{\omega_p^2 \theta^2}{\theta^2 - 1}}, \tag{3.3} \]

\[ \eta(\theta) \equiv \frac{\delta^2/27 + \theta^2 \omega_p^2/(\theta^2 - 1)}{\xi^2(\theta)}. \tag{3.4} \]

The other class is the “near” saddle points \( \omega_{spN}^- \) \{Eq. (7.3.2) of Ref. [14]\}, and are equal to

\[ \omega_{spN}^\pm \approx \pm \psi(\theta) - \frac{2}{3} i \delta \zeta(\theta), \]

where

\[ \psi(\theta) \equiv \sqrt{\frac{\omega_0^2(\theta^2 - \theta_0^2)}{\theta^2 - \theta_0^2 + 3 \theta \omega_p^2/\omega_0^2} - \frac{4}{9} \delta^2 \zeta^2(\theta)}, \tag{3.5} \]

\[ \zeta(\theta) \equiv \frac{3 \theta^2 - \theta_0^2 + 2 \theta \omega_p^2/\omega_0^2}{2 \theta^2 - \theta_0^2 + 3 \theta \omega_p^2/\omega_0^2} \tag{3.6} \]

\[ \varrho \equiv 1 - \frac{\delta^2(4 \omega_0^2 + 5 \omega_p^2)}{3 \omega_0^2(\omega_0^2 + \omega_p^2)}, \tag{3.7} \]

\[ \theta_0 \equiv \sqrt{1 + \omega_p^2/\omega_0^2}. \tag{3.8} \]

Based on these saddle points, Oughstun and Sherman obtain complicated expressions for the Sommerfeld precursor \( E_S(z, t) \) \{Eq.(7.2.20) of Ref. [14]\}, and the Brillouin precursor \( E_B(z, t) \) \{Eq. (7.3.66) of Ref. [14]\}. These complicated expressions can be simplified for my experimental conditions, as will be discussed next.
3.2 Approximate form of the saddle points for a narrow-resonance dielectric

To come up with simplified analytic expressions for the saddle points, the first step is to investigate their behavior initially ($\theta \to 1$) and asymptotically ($\theta \to \infty$), as shown in Table 3.2.

From Eq. (3.3)-(3.8), the distant-saddle-point functions are given by

$$
\xi(\theta = 1) \sim \infty,
\eta(\theta = 1) \sim 1,
$$

at $\theta = 1$. From Eq. (3.8),

$$
\theta_0^2 = 1 + \omega_p^2/\omega_0^2. \tag{3.10}
$$

Using Eq. (3.10), the near-saddle-point functions [Eq. (3.5) and Eq. (3.6)] initially are

$$
\zeta(\theta = 1) = \frac{3}{2} \frac{1}{3\varrho - 1},
\psi(\theta = 1) = \sqrt{-\frac{\omega_0^2}{3\varrho - 1} - \frac{\delta^2}{(3\varrho - 1)^2}}. \tag{3.11}
$$

On the other hands, as $\theta \to \infty$, the distant saddle points functions are

$$
\xi(\infty) \sim \sqrt{\omega_0^2 + \omega_p^2 - \delta^2} = \sqrt{\omega_T^2 - \delta^2},
\eta(\infty) \sim \frac{\delta^2/27}{\omega_0^2 + \omega_p^2 - \delta^2}. \tag{3.12}
$$
Table 3.1: Parameters used in my experiment

<table>
<thead>
<tr>
<th>symbol</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_0$</td>
<td>$4 \times 10^{14} \text{ [s}^{-1}]$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$3 \times 10^7 \text{ [s}^{-1}]$</td>
</tr>
<tr>
<td>$\omega_p$</td>
<td>$3 \times 10^9 \text{ [s}^{-1}]$</td>
</tr>
<tr>
<td>$z$</td>
<td>0.2 [cm]</td>
</tr>
</tbody>
</table>

The near-saddle-point functions are

$$
\zeta(\infty) \sim \frac{3}{2},
$$

$$
\psi(\infty) \sim \sqrt{\omega_0^2 - \delta^2}.
$$

(3.13)

The saddle-point functions can take a more simplified form with my experimental conditions [Table 3.1]. As discussed in the Ch.2, I use a cold cloud of potassium atoms (medium length $z = 0.2$ cm) to obtain a resonance of width $\delta = 3 \times 10^7$ [s$^{-1}$], which is narrow in comparison to the resonance frequency of the medium $\omega_0 = 4 \times 10^{14}$ [s$^{-1}$], so that $\omega_0 \gg \delta$ is satisfied. The plasma frequency is $\omega_p = 3 \times 10^9$ [s$^{-1}$], so that $\omega_p \ll \sqrt{8\delta \omega_0}$ is satisfied. Using $\omega_0 \gg \omega_p, \delta$,

$$
\varrho \sim 1 - \frac{4\delta^2}{\omega_0^2 + \omega_p^2} \sim 1.
$$

Using $\varrho \sim 1$ and $\omega_0 \gg \delta$,

$$
\zeta(\theta = 1) \sim \frac{3}{4},
$$

$$
\psi(\theta = 1) \sim i\sqrt{\frac{\omega_0^2 - \delta^2}{2}} \sim i\frac{\omega_0}{\sqrt{2}}.
$$

(3.14)
By considering the fact that \( \omega_1 \sim \omega_0 \) for \( \omega_0 \gg \omega_p \),

\[
\xi(\infty) \sim \sqrt{\omega_1^2 - \delta^2} \sim \sqrt{\omega_0^2 - \delta^2}.
\tag{3.15}
\]

This is further simplified by using the fact that \( \omega_0 \gg \delta \) finally resulting in

\[
\xi(\infty) \sim \sqrt{\omega_0^2 - \delta^2} \sim \omega_0.
\tag{3.16}
\]

Also in this limit,

\[
\eta(\infty) \sim 0.
\tag{3.17}
\]

At \( \theta = 1 \) (\( t = z/c \)), when the transmitted pulse just emerges from the medium, the saddle points are given as

\[
\omega_{sp}^\pm(\theta = 1) \sim \pm \infty i 2 \delta
\tag{3.18}
\]

\[
\omega_{sp_N}^\pm(\theta = 1) \sim \pm i \omega_0 / \sqrt{2} - i \delta / 2.
\tag{3.19}
\]

As \( \theta \to \infty \), the saddle points approach fixed values given as

\[
\omega_{sp_D}^\pm(\infty) \sim \pm \omega_0 - i \delta,
\tag{3.20}
\]

\[
\omega_{sp_N}^\pm(\infty) \sim \pm \omega_0 - i \delta.
\tag{3.21}
\]

Therefore, the two class of saddle points approach each other approximately as
Table 3.2: The saddle points related functions at extreme $\theta$ considering $\omega \gg \omega_p$, $\delta$.

<table>
<thead>
<tr>
<th></th>
<th>$\theta = 1$</th>
<th>$\theta \to \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi(\theta)$</td>
<td>$\infty$</td>
<td>$\omega_0$</td>
</tr>
<tr>
<td>$\eta(\theta)$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\psi(\theta)$</td>
<td>$i\omega_0/\sqrt{2}$</td>
<td>$\omega_0$</td>
</tr>
<tr>
<td>$\zeta(\theta)$</td>
<td>$3/4$</td>
<td>$3/2$</td>
</tr>
</tbody>
</table>

$\theta \to \infty$, and both saddle points are given as

$$\omega_{sp^\pm_D} \simeq \omega_{sp^\pm_N} \simeq \pm \omega_0 - i\delta. \quad (3.22)$$

I have discussed the complex form of the saddle points at $\theta = 1$ and $\theta \to \infty$ for my experimental condition of a narrow resonance and small plasma frequency, $\omega_0 \gg \omega_p$, $\delta$. In the next section, I will discuss the real part of the saddle points as a function of $\theta$ to understand how the frequencies of each precursor change.

### 3.3 Frequency chirp: the real part of the saddle points

It is worthwhile to focus on the real part of the saddle points as a function of $\theta$, because it gives the frequency of each precursor; the real part of $\omega_{sp^\pm_D}$, denoted by $\xi(\theta)$, is the frequency of the Sommerfeld precursor, and the real part of $\omega_{sp^\pm_N}$, denoted by $\psi(\theta)$, is the frequency of the Brillouin precursor.

The simplified expressions of $\xi(\theta)$ [Eq. (3.3)] and $\psi(\theta)$ [Eq. (3.5)] are obtained by taking into account the narrow resonance and small plasma frequency condition.
Figure 3.2: Path of saddle points in a complex frequency plane. Note that I only consider Re(ω) > 0 plane, which is symmetric with other half plane.

(ω₀ ≫ ωₚ, δ). The real parts of the saddle points are given as

\[ ξ(θ) ≃ ω₀ + \frac{ωₚ^2 - δ^2}{2ω₀} + \frac{ωₚ^2}{2ω₀ (θ - 1)(θ + 1)}, \]  \hspace{1cm} (3.23) \]

\[ ψ(θ) ≃ ω₀ - \frac{δ^2}{2ω₀} - \frac{3ωₚ^2}{2ω₀ (θ - 1)(θ + 1)}. \]  \hspace{1cm} (3.24) \]

Equation (3.23) is valid for any θ. The frequency of the Sommerfeld precursor ξ(θ) starts off at infinity at θ = 1 and rapidly approaches ω₀ as θ → ∞ along the path (dashed line) shown in Figure 3.2. The original form of ψ(θ) [Eq.(3.5)] is pure imaginary at θ = 1, as shown in the previous section. The frequency of the Brillouin precursor ψ(θ) becomes zero at θ = θ₁, then turns into a pure real value. The turning
point $\theta_1$ \{(7.3.58b) of Ref. [14]\} is given as

\[
\theta_1 \simeq \theta_0 + \frac{2\delta^2 \omega_p^2}{\theta_0 \omega_0^2 (3\delta \omega_0^2 - 4 \omega_p^2)} \simeq \theta_0 + \vartheta (10^{-24}) \sim \theta_0 \sim 1 + 10^{-11}.
\] (3.25)

Therefore, Eq. (3.24) is valid for $\theta > \theta_1$ when $\psi(\theta)$ becomes pure a real value. The frequency of the Brillouin precursor has the value $\psi(\theta) = i\omega_0/\sqrt{2}$ at $\theta = 1$, then moves off the imaginary axis at $\theta = \theta_1$, and rapidly approaches $\omega_0$ within some time scale. It is required to quantify how rapidly these times correspond to.

I determine the time scale of the chirp $\tau_c = [z(\theta_c - 1)/c]$ when both $\xi(\theta)$ and $\psi(\theta)$ approach $\omega_0 \sim 10^{14} \text{ s}^{-1}$ within one linewidth $\delta \sim 10^7 \text{ s}^{-1}$. To do that, let us check each term of Eq. (3.23) and Eq. (3.24). The second term of $\xi(\theta)$ ($\psi(\theta)$) is on the order of $\omega_p^2/2\omega_0^2 \sim \vartheta (10^4)$ ($\delta^2/2\omega_0^2 \sim \vartheta (10^0)$), which is constant in time. Thus, the time dependence is only in the third term. The third term of Eq. (3.23) and Eq. (3.24) are identical except for the coefficients and the sign. Both are infinite at $\theta = 1$, then rapidly become zero as $\theta \to \infty$. To determine $\tau_c$, I consider that the difference between the frequency of the precursors and the resonance frequency is negligible when it is less than the narrow linewidth of the resonance $\delta \sim \vartheta (10^7)$ compared to $\omega_0 \sim \vartheta (10^{14})$, and let $\theta_c$ be a critical point where the third term is $\sim \delta$. To find $\theta_c$, I set the equation as

\[
\frac{\omega_p^2}{\omega_0 (\theta_c - 1)(\theta_c + 1)} = \delta = 3 \times 10^7 \text{ [s}^{-1}],
\] (3.26)

which provides $\theta_c \sim 1 + 10^{-3}$. This means that, for both $\xi(\theta)$ and $\psi(\theta)$, the absolute values of the third terms are less than $\delta \sim 10^7 \text{ s}^{-1}$ beyond $\theta_c = ct_c/z \sim 1 + 10^{-3}$. When I convert $\theta_c$ into $\tau_c$, the delayed time scale is expressed as $\tau_c = t_c - z/c = z(\theta_c - 1)/c \sim 10^{-3} z/c$. Therefore, the time it takes to approach $\omega_0$ depends linearly
on the penetration depth \( z = L \); for \( \alpha_0 L = 1.03, \tau_c \sim 10 \text{ fs} \), which is comparable to a few optical periods (2.6 fs). Beyond \( \tau_c \), the saddle point functions are \( \xi(\theta) \to \omega_0 \), and \( \psi(\theta) \to \omega_0 \), as in Table 3.2. Therefore, the frequencies of Sommerfeld and Brillouin precursor are equally \( \omega_0 \) for \( \tau > \tau_c \).

This analysis shows the extremely rapid frequency chirp of the Sommerfeld and Brillouin precursors. The frequencies of the Sommerfeld and Brillouin precursors start off at infinity and zero, respectively. It then takes less than 10 fs (a few optical period of 2.6 fs) for both to approach \( \omega_0 \) to within one linewidth \( \delta \). The time scale of chirp is essentially \textit{instantaneous} compared to the time scale in my experiment. For example, the rise time of the step-modulated incident pulse is \( \sim 100 \text{ ps} \). Therefore, the frequencies of the precursors in my data will be regarded as \( \omega_0 \) from now on. This result makes it much easier to obtain simple asymptotic forms of the precursors. In the next two sections, I will discuss the precursor field expressions.

### 3.4 The Sommerfeld precursor: \( E_S(z, t) \)

Based on the asymptotic form of saddle points in the previous section, I present how the Sommerfeld precursor \{See (7.2.20) of Ref. [14]\} can be simplified. Using the asymptotic frequency \( \omega_0 \), the Sommerfeld precursor is given as

\[
E_S(z, t) \simeq A_0 \Theta(\tau) \frac{\omega_0}{2\omega_p} \sqrt{\theta - 1} e^{-\delta \xi(\theta - 1)}
\times \left[ \frac{\delta}{2} \left( \frac{5\omega_0 + 3\omega_c}{(\omega_0 + \omega_c)^2 + \delta^2} - \frac{5\omega_0 - 3\omega_c}{(\omega_0 - \omega_c)^2 + \delta^2} \right) J_0(\zeta z \omega_0(\theta - 1)) \right.
\]
\[
+ \left. \left( \frac{\omega_0(\omega_0 - \omega_c) - 3\delta^2/2}{(\omega_0 - \omega_c)^2 + \delta^2} - \frac{\omega_0(\omega_0 + \omega_c) - 3\delta^2/2}{(\omega_0 + \omega_c)^2 + \delta^2} \right) J_1(\zeta z \omega_0(\theta - 1)) \right].
\]

(3.27)
The first and the fourth term containing \((ω_0 + ω_c)\) can be ignored for my experimental condition of \(|ω_c - ω_0| < 2δ\), i.e. \(|ω_0 - ω_c| \ll |ω_0 + ω_c| \approx 2ω_0\), and Eq. (3.27) is rewritten as

\[
E_S(z, t) \approx A_0 Θ(τ) \frac{ω_0}{2ω_p} \sqrt{\frac{ct}{z}} e^{-δτ} \left[ -\frac{δ}{2(ω_0 - ω_c)^2 + δ^2} J_0[ω_0τ] \right.
\]
\[
+ \frac{ω_0(ω_0 - ω_c) - 3δ^2/2}{(ω_0 - ω_c)^2 + δ^2} J_1[ω_0τ] \right],
\]

where \(θ - 1 = c(t - z/c)/z = ct/z\). The Bessel functions \(J_0[ω_0τ]\) and \(J_1[ω_0τ]\) are asymptotically harmonic functions as

\[
J_n(x) \sim \sqrt{\frac{2}{πx}} \cos(x - nπ/2 - π/4),
\]

for a large argument \(x\). With the definition \(Δ ≡ ω_c - ω_0\), Eq. (3.28) is rewritten as

\[
E_S(z, t) \approx A_0 Θ(τ) \frac{ω_0}{2ω_p} \sqrt{\frac{ct}{z}} \sqrt{\frac{2}{πω_0τ}} e^{-δτ} \left[ -\frac{δ}{2} \frac{2ω_0 - 3Δ}{Δ^2 + δ^2} \cos(ω_0τ - π/4) \right.
\]
\[
- \frac{ω_0Δ + 3δ^2/2}{Δ^2 + δ^2} \sin(ω_0τ - π/4) \right].
\]

Considering the narrow resonance condition \(ω_0 \gg δ\), and small detuning \(|Δ| < 2δ\), Eq. (3.30) is further simplified by the addition of trigonometric functions

\[
\frac{δ}{\sqrt{Δ^2 + δ^2}} \cos(ω_0τ - π/4) + \frac{Δ}{\sqrt{Δ^2 + δ^2}} \sin(ω_0τ - π/4)
\]
\[
= \frac{δ}{\sqrt{Δ^2 + δ^2}} \sin(ω_0τ + π/4) - \frac{Δ}{\sqrt{Δ^2 + δ^2}} \cos(ω_0τ + π/4),
\]

\[
= \sin(ω_0τ + π/4) \cos(Δ) - \cos(ω_0τ + π/4) \sin(Δ),
\]

\[
= \sin(ω_0τ + π/4 - φ(Δ)),
\]

65


where

$$
\varphi(\Delta) \equiv \tan^{-1}(\Delta/\delta).
$$

(3.32)

Equation (3.30) is rewritten as

$$
E_S(z, t) \simeq -A_0 \Theta(\tau) \frac{\omega_0^{3/2}}{\omega_p} \sqrt{\frac{c}{2\pi}} \frac{e^{-\delta\tau}}{\sqrt{\Delta^2 + \delta^2}} \sin(\omega_0 \tau + \pi/4 - \varphi(\Delta)).
$$

(3.33)

By letting

$$
\beta(z, \Delta) = \left( \frac{\omega_0^3/\delta}{[2(\pi \alpha_0 z)^{1/2}] \sqrt{\Delta^2 + \delta^2}] \right)^{1/2},
$$

the Sommerfeld precursor is given by

$$
E_S(z, t) \simeq -A_0 \Theta(\tau) \beta(z, \Delta) e^{-\delta\tau} \sin(\omega_0 \tau + \pi/4 - \varphi(\Delta)),
$$

(3.34)

for \( \tau > \tau_c \).

### 3.5 The Brillouin precursor: \( E_B(z, t) \)

As in the previous section, I will discuss how the expression for the Brillouin precursor \{See (7.3.66) of Ref. [14]\} can be simplified for time greater than \( \tau_c \). Using the asymptotic frequency \( \omega_0 \), the Brillouin precursor is given by

$$
E_B(z, t) \simeq -A_0 \Theta(\tau) \frac{\omega_0^2}{\omega_p} \sqrt{\frac{1}{6\omega_0} \left( \frac{c}{z} \right)^{1/3} e^{-\delta(\theta-1)}} \left[ -\frac{\delta |\alpha_1(\theta)|^{1/4}}{(\omega_0 - \omega_c)^2 + \delta^2} A_i [-|\alpha_1(\theta)| (\frac{z}{c})^{2/3}] - \frac{1}{|\alpha_1(\theta)|^{1/4} (\omega_0 - \omega_c)^2 + \delta^2} \left( \frac{c}{z} \right)^{1/3} A_i (1) [-|\alpha_1(\theta)| (\frac{z}{c})^{2/3}] \right].
$$

(3.35)
Using the asymptotic form of the Airy function \( \{ \text{pp. 176 of Ref. [14]} \} \) for \( \theta > \theta_c \), and \( |\alpha_1(\theta)|^{3/2} \sim 3\omega_0(\theta - 1)/2 \) \( \{ \text{Eq. (7.3.54b) of Ref. [14]} \} \)

\[
A_i[-|\alpha_1(\theta)|](\frac{z}{c})^{2/3} \simeq \frac{\sin(\frac{2}{3}|\alpha_1(\theta)|^{3/2}(\frac{z}{c}) + \pi/4)}{\sqrt{\pi}|\alpha_1(\theta)|^{1/4}(\frac{z}{c})^{1/6}},
\]

\[
\simeq \frac{\sin(\omega_0\frac{z}{c}(\theta - 1) + \pi/4)}{\sqrt{\pi}|\alpha_1(\theta)|^{1/4}(\frac{z}{c})^{1/6}},
\]

\[
A^{(1)}_i[-|\alpha_1(\theta)|](\frac{z}{c})^{2/3} \simeq -\frac{1}{\sqrt{\pi}}|\alpha_1(\theta)|^{1/4}(\frac{z}{c})^{1/6} \cos\left(\frac{2}{3}|\alpha_1(\theta)|^{3/2}(\frac{z}{c}) + \pi/4\right),
\]

\[
\simeq -\frac{1}{\sqrt{\pi}}|\alpha_1(\theta)|^{1/4}(\frac{z}{c})^{1/6} \cos\left(\omega_0\frac{z}{c}(\theta - 1) + \pi/4\right).
\]

Recalling the retarded time \( \tau = z(\theta - 1)/c = t - z/c \), the asymptotic form of Brillouin precursor is given as

\[
E_B(z, t) \simeq A_0\Theta(\tau)\omega_0^{3/2}\omega_p \sqrt{\frac{c}{6\pi z}} \frac{e^{-\delta\tau}}{\sqrt{\Delta^2 + \delta^2}} \left[\frac{\delta}{\sqrt{\Delta^2 + \delta^2}} \sin(\omega_0\tau + \pi/4) + \frac{\Delta}{\sqrt{\Delta^2 + \delta^2}} \cos(\omega_0\tau + \pi/4)\right].
\]

By considering

\[
\frac{\delta}{\sqrt{\Delta^2 + \delta^2}} \sin(\omega_0\tau + \pi/4) + \frac{\Delta}{\sqrt{\Delta^2 + \delta^2}} \cos(\omega_0\tau + \pi/4) = \sin(\omega_0\tau + \pi/4 + \varphi(\Delta)),
\]

and the definition of \( \beta(z, \Delta) \) in the previous section, the Brillouin precursor obtains the final asymptotic form for the on-resonance case as

\[
E_B(z, t) \sim A_0\Theta(\tau)\beta(z, \Delta)e^{-\delta\tau} \sin(\omega_0\tau + \pi/4 + \varphi(\Delta))/\sqrt{3}.
\]
According to Eq. (3.39), the Brillouin precursor has an exponentially decaying amplitude. The initial peak is determined by $\beta(z, \Delta)$ as in the Sommerfeld precursor Eq. (3.34). The Brillouin precursor, however, has the opposite sign and is smaller by $1/\sqrt{3}$ compared to the Sommerfeld precursor. Now we are only left with the main signal, which will be discussed next.

3.6 The main signal: $E_C(z, t)$

Let us recall that the analysis to obtain the main signal is different from that of precursors in the previous sections. It is not related to any saddle points, but to the simple pole $\omega = \omega_c$ of the integrand of the wave equation Eq. (1.18). The main signal is merely evaluated from the contour integration around the simple pole at $\omega_c$ [14]. In this section, I investigate the main signal for my experimental conditions.

The main signal expression is given by Eq. (8.3.4 a,b,c) of Ref. [14] for the case of $\sqrt{\omega_0^2 - \delta^2} < \omega_c < \sqrt{\omega_0^2 + \omega_p^2 - \delta^2}$, and is equal to

$$E_c(z, t) = -A_0 \Theta(\tau) e^{-z\alpha(\omega_c)} \sin(\beta(\omega_c)z - \omega_c t),$$  

where $\alpha(\omega_c) = \omega_c n_i(\omega_c)/c$, $\beta(\omega_c) = \omega_c n_r(\omega_c)/c$, and $n(\omega) = n_r(\omega) + in_i(\omega)$ is the complex index of refraction. Considering the fact that $\omega_c$ is pure real,
\[ n_r(\omega_c) \simeq \left( 1 + \frac{\omega_p^4 - 2\omega_p^2(\omega_0^2 - \omega_c^2)}{(\omega_c^2 - \omega_0^2)^2 + 4\delta^2\omega_c^2} \right)^{1/4} \cos \left( \frac{\zeta(\omega_c)}{2} \right) \]
\[ \simeq (1 + \vartheta(10^{-6}))^{1/4} \cos \left( \frac{\zeta(\omega_c)}{2} \right), \]  
(3.41)

\[ n_i(\omega_c) \simeq \left( 1 + \frac{\omega_p^4 - 2\omega_p^2(\omega_0^2 - \omega_c^2)}{(\omega_c^2 - \omega_0^2)^2 + 4\delta^2\omega_c^2} \right)^{1/4} \sin \left( \frac{\zeta(\omega_c)}{2} \right) \]
\[ \simeq (1 + \vartheta(10^{-6}))^{1/4} \sin \left( \frac{\zeta(\omega_c)}{2} \right), \]

where

\[ \zeta(\omega_c) = \tan^{-1} \left( \frac{2\omega_c\delta\omega_p^2}{(\omega_c^2 - \omega_0^2)^2 - \omega_p^2(\omega_c^2 - \omega_0^2) + 4\omega_c^2\delta^2} \right) \]
\[ \simeq \tan^{-1} \left( \frac{\omega_p^2\delta}{2\omega_c(\Delta^2 + \delta^2)} \right), \]  
(3.42)

\[ \simeq \frac{\omega_p^2\delta}{2\omega_c(\Delta^2 + \delta^2)}, \]

from (6.1.13) of Ref. [14]. By inserting Eq. (3.42) into (3.41), the real and imaginary part of the index of refraction becomes

\[ n_r(\omega_c) \simeq \cos \left( \frac{\omega_p^2\delta}{4\omega_c(\Delta^2 + \delta^2)} \right) \simeq 1, \]

\[ n_i(\omega_c) \simeq \sin \left( \frac{\omega_p^2\delta}{4\omega_c(\Delta^2 + \delta^2)} \right) \simeq \frac{\omega_p^2\delta}{4\omega_c(\Delta^2 + \delta^2)}. \]

Finally, the main signal is given by

\[ E_c(z, t) = -A_0 \Theta(\tau) \exp \left( -\frac{\omega_p^2z\delta}{4\omega_c(\Delta^2 + \delta^2)} \right) \sin(-\omega_c\tau) \]  
(3.43)

\[ = A_0 \Theta(\tau) \exp \left( -\frac{\alpha_0z\delta^2}{2\omega_c(\Delta^2 + \delta^2)} \right) \sin(\omega_c\tau), \]  
(3.44)
which is simply a reduced-amplitude step-modulated pulse.

The main signal has nothing to do with saddle points, but is associated with the pole \( \omega = \omega_c \), the carrier frequency of the incident pulse). Thus, there is no asymptotic form in deriving Eq. (3.44). The simple expression Eq. (3.44) is obtained merely from the experimental condition \( \omega_0 \gg \omega_p, \delta \).

In these sections, I discussed how Oughstun and Sherman’s modern asymptotic expression of each field components [14] applies to my case. It is seen that each component can be written as harmonic functions modulated by a slowly-varying envelope. In the next sections, each field components will be discussed in detail and compared to the experimental data for the case of an on- or off-resonance carrier frequency and a single-resonance Lorentz medium.

3.7 The transmitted field for an on-resonance carrier frequency

This section describes each part of the transmitted field for the case of an on-resonance carrier frequency \( \omega_c = \omega_0 \) (\( \Delta = 0 \)). The three distinct parts of the transmitted field are given by

\[
E_S(z, t) = -A_0 \Theta(\tau) \beta(z)e^{-\delta \tau} \sin(\omega_c \tau + \pi/4),
\]

\[
E_B(z, t) = A_0 \Theta(\tau) \beta(z)e^{-\delta \tau} \sin(\omega_c \tau + \pi/4)/\sqrt{3},
\]

\[
E_C(z, t) = A_0 \Theta(\tau)e^{-\alpha_0 z/2} \sin(\omega_c \tau),
\]
where \( \tau = t - z/c \) is the retarded time, and

\[
\beta(z) = \frac{1}{2} \sqrt{\frac{(\omega_0/\delta)^3}{\pi \alpha_0 z}}.
\] (3.48)

Figure 3.3 shows plots of the envelope of three parts of the transmitted field in Eqs.(3.45)-(3.47). The step function \( \Theta(\tau) \) implies that the precursors and the main signal arrive immediately after the front (\( \tau = 0 \)). The intensity of Sommerfeld and Brillouin precursors decay exponentially with the time constant of \( 1/2\delta = 16.6 \) ns. This result confirms my earlier statement that the persistence of the precursors is governed by the resonance width. The precursor amplitudes are also modulated by a \( z^{-1/2} \) spatial dependence and they are out of phase, resulting in a partial cancellation of their combined amplitudes. Furthermore, the main signal is just the incident step-modulated field reduced in amplitude by an amount expected from Beer’s Law. Based on these findings, I conclude that the transient spike observed in my experiments [Figure 3.4] is composed of both the Sommerfeld and Brillouin precursors, which sit on top of the main signal. Thus, Figure 3.4 constitutes the first direct measurement of precursors in the optical domain for a step-modulated field propagating through a medium characterized approximately as a single-resonance Lorentz dielectric.

A detailed analysis of Eqs. (3.45)-(3.47) reveals some surprises. One of them is the \( \pi/4 \) phase shift in both precursor expressions. For \( \omega_c = \omega_0 \), all components of the transmitted field are expected to be in phase with the incident field. This is easily understood if we imagine a mass attached to a spring driven by an external sinusoidal force. If the driving frequency is on resonance with the harmonic oscillator, the driving force and the mass are always in phase. Thus in the Lorentz model, I
Figure 3.3: Each part of transmitted field predicted by the asymptotic theory (a) $A_S(z, t)$, $A_B(z, t)$, $A_S(z, t) + A_B(z, t)$, and (b) $A_C(z, t)$ for $\Delta = 0$. 

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expect that the electric field and the medium polarization have to be in phase when the
carrier frequency of the electric field is on-resonance.

The most notable problem in the asymptotic expressions is the precursor amplitude
in Eqs. (3.45)-(3.46). In the case when \( \alpha_0 L = 1.03 \), I find that \( \beta(L) = 1.30 \times 10^{10} \), which is unphysical (I expect it should be of the order of 1 so that the
\( |E(z, t)| \lesssim A_0 \)). The origin of this problem can be seen by looking at the expression
for \( \beta(z) \). Equation (3.48) shows that the precursor amplitude is proportional to
\((\omega_0 / \delta)^{3/2} \sim 10^{10}\). Note that the asymptotic analysis is valid in the so-called mature
dispersion limit (\( \alpha_0 z \gg 1 \)); my experiment with the largest \( \alpha_0 L \) borders on this limit
and thus I expect that it will only give a qualitative understanding of my results.

One possibility is that \( \alpha_0 L \) needs to be larger because my experiment is conducted
on the theory’s boundary of validity. However, I would have to increase \( \alpha_0 L \) by a
factor of \( \sim 10^{100} \) to make \( |E(z, t)| \sim A_0 \), which seems unreasonable. Future inves-
tigations are needed to address these apparent problems. For now, these errors can
be tested using my experimental data as in the next subsection.

### 3.7.1 Modified Asymptotic Theory

While the asymptotic theory predicts a transient spike whose shape is consistant
with my data, it appears to have some problems, as discussed above. To empirically
correct some of the errors, I have set the \( \pi/4 \) phase shift in Eqs. (3.45) and (3.46) to
zero. To correct the erroneous value of \( \beta(z) \), a scale factor \( s(z) \) is introduced. Each
field components in Eqs. (3.45)-(3.47) are rewritten as

\[
E_{SB}(z, t) = -s(z)(1 - 1/\sqrt{3})A_0 \Theta(\tau) \beta(z)e^{-\delta \tau} \sin(\omega_c \tau), \quad (3.49)
\]

\[
E_{C}(z, t) = A_0 \Theta(\tau)e^{-\alpha_0 z/2} \sin(\omega_c \tau). \quad (3.50)
\]
To find an appropriate scale factor $s(z)$, I set the envelope of the total field $A(z, t) = A_0$ at $\tau = 0$. This is based on the experimental data, which shows essentially 100% transmission at the front $\tau = 0$. To obtain an expression for the field envelope from the field expression, I substitute Eqs. (3.49)-(3.50) into the expression for the total field as

$$E(z, t) = E_{SB}(z, t) + E_C(z, t),$$

$$= [s(z)A'_{SB}(z, t) + A_C(z, t)]\sin(\omega_c \tau), \quad (3.51)$$

where

$$A'_{SB}(z, t) = -(1-1/\sqrt{3})A_0\Theta(\tau)\beta(z)e^{-\delta \tau},$$

$$A_C(z, t) = A_0\Theta(\tau)e^{-\alpha_0 z/2}. \quad (3.52)$$

At $\tau = 0$ ($t = z/c$),

$$s(z)A'_{SB}(z, z/c) + A_C(z, z/c) = A_0[-s(z)(1-1/\sqrt{3})\beta(z) + e^{-\alpha_0 z/2}] = A_0. \quad (3.53)$$

Therefore,

$$s(z) = \frac{1 - e^{-\alpha_0 z/2}}{-(1-1/\sqrt{3})\beta(z)}. \quad (3.54)$$

Finally, the modified asymptotic expressions are

$$E_{SB}(z, t) = A_0\Theta(\tau)(1 - e^{-\alpha_0 z/2})e^{-\delta \tau}\sin(\omega_c \tau), \quad (3.55)$$

$$E_C(z, t) = A_0\Theta(\tau)e^{-\alpha_0 z/2}\sin(\omega_c \tau). \quad (3.56)$$
Figure 3.4: Comparison of the $T(z, t)$ predicted by the modified asymptotic theory (dots) to the on-resonance data (solid line) for the case of three different absorption depths $\alpha_0 L$.

The new envelope of the total precursor field is given as

$$A_{SB}(z, t) = A_0 \Theta(\tau)(1 - e^{-\alpha_0 z/2})e^{-\delta \tau},$$

(3.57)

which has a physically reasonable value in terms of the dependence on the absorption coefficient $\alpha_0$. It implies that the precursor amplitude decrease as the medium absorption is weak.

To compare the modified asymptotic theory Eq. (3.55)-(3.56) to the experimental
data, the normalized total transmission intensity $T(z, t)$ is evaluated as

\[ T(z, t) = \langle |E(z, t)|^2/A_0^2 \rangle = |A_{SB}(z, t) + A_C(z, t)|^2/A_0^2, \]

\[ = \Theta(\tau)[(1 - e^{-\alpha_0 z/2})^2 e^{-2\delta \tau} + e^{-\alpha_0 z} + 2e^{-\alpha_0 z/2-\delta \tau}(1 - e^{-\alpha_0 z/2})], \]  

(3.58)

where $\langle \cdot \rangle$ denotes time average. Eq. (3.58) reveal that $T(z, t)$ jumps to 100% immediately after the front. To make a direct comparison to the observations, I take into account the finite rise time of the step-modulated pulse and detection system by convolving the intensity transmission function $T(z, t)$ with a single-pole low-pass filter with a $\gamma_f = 206$-MHz 3-dB roll-off frequency using the expression

\[ \frac{dy(t)}{dt} = -\gamma_f[y(t) - T(z, t)], \]  

(3.59)

where $y(t)$ is the filtered transmission function. This differential equation is obtained from the equation of an RC (resistance and capacitor) series circuit. The filter reduces the transmission to $\sim 95\%$ immediately after the front. The dots in Figure 3.4 indicate the predicted low-pass-filtered intensity transmission function with no free parameters. The theory agrees reasonably well with the experimental observations, as shown in Figure 3.4.

Figure 3.4 shows that the modified asymptotic theory predicts the time scale and the transient peak height of the transmission intensity reasonably well. The time scale in the theory is always proportional to $1/2\delta$. However, the time scale for the case of $\alpha_0 L = 1.03$ disagrees with the data compared to the case of $\alpha_0 L = 0.41$. In Ch. 4, we will see that the time scale also depends on $\alpha_0 L$. According to the theory described in Ch. 4, based on the SVA, the time scale shortens as $\alpha_0 L$ increases, in agreement with my observations.
Figure 3.5: Each part of $A(z, t)$ predicted by the modified asymptotic theory (a) $A_S(z, t)$ (dots), $A_B(z, t)$ (dashed line), $A_S(z, t) + A_B(z, t)$ (solid line), and (b) $A_C(z, t)$ for $\Delta = 0$. 
So far, I have discussed the asymptotic theory for an on-resonance carrier frequency and its modifications. In the next section, I will discuss each field component for the case of an off-resonance carrier frequency.

### 3.8 Transmitted field for an off-resonance carrier frequency

In this section, I present the theory for the case of an off-resonance carrier frequency where $\omega_c \neq \omega_0$ ($\Delta \neq 0$). The three distinct transmitted fields are already given in Eq. (3.34)-(3.44), which I repeat here for clarity and given by

\[
E_S(z, t) = -A_0 \Theta(\tau) \beta(z, \Delta) e^{-\delta \tau} \sin(\omega_0 \tau + \pi/4 - \varphi(\Delta)), \tag{3.60}
\]

\[
E_B(z, t) = A_0 \Theta(\tau) \beta(z, \Delta) e^{-\delta \tau} \sin(\omega_0 \tau + \pi/4 + \varphi(\Delta)) / \sqrt{3}, \tag{3.61}
\]

\[
E_C(z, t) = A_0 \Theta(\tau) e^{-\alpha_0 L \delta^2 / [2(\Delta^2 + \delta^2)]} \sin(\omega_e \tau), \tag{3.62}
\]

where $\varphi(\Delta)$ is given in Eq. (3.32). From Eqs. (3.60)-(3.62), it can be seen that the precursors and the main signal arrive immediately after the front ($\tau = 0$), and the Sommerfeld and Brillouin precursors decay with an intensity time constant of $1/2 \delta = 16.6$ ns (as for the on-resonance case). Note that the detuning $\Delta$ does not effect the precursor decay time. The precursor amplitudes are also modulated by a $z^{-1/2}$ factor, and the main signal is just the reduced step-modulated field according to Beer’s law.

Despite the similarities to the on-resonance case, it is seen that there are several differences for the off-resonance case. One is the additional phase in the expressions for the Sommerfeld (Brillouin) precursor, appearing as $-\varphi(\Delta)$ in Eq. (3.60) $[\varphi(\Delta)$
Figure 3.6: Each part of transmitted field predicted by the modified asymptotic theory (a) $|A_S(z,t)|$ (dots), $|A_B(z,t)|$ (dashed line), $|A_{SB}(z,t)|$ (solid line), (b) $\text{Arg}[A_{SB}(z,t)]$ and (b) $A_C(z,t)$ for $\Delta = 1.25\delta$. 

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Figure 3.7: Each part of transmitted field predicted by the modified asymptotic theory (a) $|A_S(z,t)|$ (dots), $|A_B(z,t)|$ (dashed line), $|A_{SB}(z,t)|$ (solid line), (b) $\text{Arg}[A_{SB}(z,t)]$ and (b) $A_C(z,t)$ for $\Delta = 4.79\delta$. 
in Eq. (3.61)], which depends on the detuning. Another is that the amplitude of the
main signal increases, while the amplitudes of the precursors $\beta(z, \Delta)$ decrease as $\Delta$
increases. Finally, the main signal $E_C(z, t)$ oscillates at $\omega_c$, as seen in Eq. (3.62),
while the precursors still oscillate at the medium resonance $\omega_0$, as shown in Eq. (3.60)
and Eq. (3.61). Note that the frequency difference between the precursors and the
main signal $\Delta = \omega_c - \omega_0$ agrees with the modulation frequency in the experimental
data shown in the Figure 3.8.

A detailed analysis of Eqs.(3.60)-(3.62) for $\Delta \neq 0$ reveals unreasonable results
(as for the on-resonance case), such as enormous precursor amplitudes $\beta(z, \Delta)$ (of-the-order-of $10^{10}$), and the $\pi/4$ phase shift in the precursors. As will be discussed
next, the asymptotic theory fits the experimental data by correcting empirically
these errors.

3.8.1 Modified Asymptotic Theory

For the case of $\omega_c \neq \omega_0$, the problems with the asymptotic theory continue to appear
as for the case of on-resonance case: the $\pi/4$ phase shift and an enormous precursor
amplitude $\beta(z, \Delta)$. In addition to these errors, the sign in the additional phase
$(+\varphi(\Delta))$ of Eq. (3.61) need to be reset to `-' ($-\varphi(\Delta)$) for the theory to agree with
the experimental data, which points out another problem in the asymptotic analysis.
Another additional phase $\phi_{SB}$ is needed to correct the phase in the main signal $\phi_C$.
As will be discussed later, the SVA theory predicts that the main signal has a phase
that depends on the detuning $\Delta$. The correction to the asymptotic theory is made
by setting $\phi_C = \Delta \alpha_0 z\delta/[2(\Delta^2 + \delta^2)]$. By correcting the phases, the field components
for $\omega_c \neq \omega_0$ are given as

$$E_{SB}(z,t) = -s(z,\Delta)(1 - \frac{1}{\sqrt{3}})A_0\Theta(\tau)\beta(z,\Delta)e^{-\delta \tau}$$

$$\times \sin(\omega_0 \tau - \varphi(\Delta) + \phi'_{SB}), \quad (3.63)$$

$$E_C(z,t) = A_0\Theta(\tau)e^{-\alpha_0 L \delta^2/[2(\Delta^2 + \delta^2)]}\sin(\omega_c \tau + \phi_C). \quad (3.64)$$

where

$$\beta(z,\Delta) = \frac{(\omega_0/\delta)^{3/2}}{2(\pi \alpha_0 z)^{1/2} \sqrt{\Delta^2 + \delta^2}}. \quad (3.65)$$

To find the envelope of each field oscillating at $\omega_c$, I use $\omega_0 = \omega_c - \Delta$. The complex form of Eq. (3.63)-(3.64) are then written as

$$\tilde{E}_{asymp}^{SB}(z,t) = -s(z,\Delta)(1 - \frac{1}{\sqrt{3}})A_0\Theta(\tau)\beta(z,\Delta)e^{-\delta \tau}$$

$$\times e^{-i(\Delta \tau + \varphi(\Delta) - \phi'_{SB})}e^{i\omega_c \tau}, \quad (3.66)$$

$$\tilde{E}_{asymp}^{C}(z,t) = A_0\Theta(\tau)e^{-\alpha_0 L \delta^2/[2(\Delta^2 + \delta^2)]}e^{i\phi_C}e^{i\omega_c \tau}, \quad (3.67)$$

Each field in Eqs. (3.66)-(3.67) is related to the real field component as

$$E(z,t) = \text{Im}[\tilde{E}_{asymp}(z,t)] \quad (3.68)$$

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The total complex field is expressed as

\[
\tilde{E}_{\text{asymp}}(z, t) = \tilde{E}_{SB\text{asymp}}(z, t) + \tilde{E}_{C\text{asymp}}(z, t) = [\tilde{A}_{SB\text{asymp}}(z, t) + \tilde{A}_{C\text{asymp}}(z, t)] e^{i\omega_c \tau},
\]

\[
= \tilde{A}_{\text{asymp}}(z, t) e^{i\omega_c \tau}.
\]

where

\[
\tilde{A}_{SB\text{asymp}}(z, t) = -s(z, \Delta)(1 - \frac{1}{\sqrt{3}})A_0 \Theta(\tau) \beta(z, \Delta) e^{-\delta \tau} e^{-i(\Delta \tau + \varphi(\Delta) - \varphi'_{SB})},
\]

\[
\tilde{A}_{C\text{asymp}}(z, t) = A_0 \Theta(\tau) e^{-\alpha_0 L \delta^2 / [2(\Delta^2 + \delta^2)]} e^{i\phi_C}.
\]

Equation (3.63) is not complete until the scale factor \( s(z, \Delta) \) and \( \varphi'_{SB} \) is determined. To determine these two expressions, consider the complex envelope \( \tilde{A}_{\text{asymp}}(z, t) \) expressed in polar coordinates \( re^{i\phi} \) with amplitude \( r = |\tilde{A}_{\text{asymp}}(z, t)| \) and phase \( \phi = \text{Arg}[\tilde{A}_{\text{asymp}}(z, t)] \). At \( \tau = 0 \ (t = z/c) \), I insist that the field envelope is constrained to take the following value

\[
r = |\tilde{A}_{\text{asymp}}(z, z/c)| = A_0 \]

\[
\phi = \text{Arg}[\tilde{A}_{\text{asymp}}(z, z/c)] = 0.
\]

Therefore,

\[
\tilde{A}_{SB\text{asymp}}(z, z/c) = -s(z, \Delta)(1 - \frac{1}{\sqrt{3}})A_0 \Theta(\tau) \beta(z, \Delta) e^{-i(\varphi(\Delta) - \varphi'_{SB})} = r_{SB} e^{i\phi_{SB}}
\]

\[
\tilde{A}_{C\text{asymp}}(z, z/c) = A_0 \Theta(\tau) e^{-\alpha_0 L \delta^2 / [2(\Delta^2 + \delta^2)]} e^{i\phi_C} = r_{C} e^{i\phi_{C}}
\]

\[
\tilde{A}_{\text{asymp}}(z, z/c) = re^{i\phi},
\]

where \( r_{SB} = |\tilde{A}_{SB\text{asymp}}(z, t)|, r_{C} = |\tilde{A}_{C\text{asymp}}(z, t)|, \phi_{SB} = \text{Arg}[\tilde{A}_{SB\text{asymp}}(z, t)], \) and \( \phi_{C} = \text{Arg}[\tilde{A}_{C\text{asymp}}(z, t)] \).
As will be discussed in the SVA theory, the scale factor \( s(z, \Delta) \) and the correction of the phase \( \phi_{SB}' \) for the precursor envelope \( \tilde{A}_{SB}^\text{asymp}(z, z/c) \) are determined based on the fact that the main signal envelope \( \tilde{A}_{C}(z, z/c) \) for the two different theories should be identical. Therefore, we only need to compare the envelope of the precursors \( \tilde{A}_{SB}(z, z/c) \) between two theories is required. The SVA theory (Ch. 4) predicts that

\[
\tilde{A}_{SB}^{\text{SVA}}(z, z/c) = A_0(1 - e^{\alpha_0 z/[2(i\Delta - \delta)]}) = r_{SB} e^{-i\phi_{SB}} \\
\tilde{A}_{C}^{\text{SVA}}(z, z/c) = A_0 e^{\alpha_0 z/[2(i\Delta - \delta)]} = r_C e^{-i\phi_C} \\
\tilde{A}_{C}^{\text{SVA}}(z, z/c) = A_0 = r e^{-i\phi},
\]

at \( \tau = 0 \) (\( t = z/c \)). Note that there are unavoidable inconvenience in the notation for two different theories, which are originated from their own development.

\[
E(z, t) = \text{Im}[\tilde{E}^\text{asymp}(z, t)] = -\text{Im}[\tilde{E}^{\text{SVA}}(z, t)] \\
= \text{Im}[\tilde{A}^\text{asymp}(z, t)e^{i\omega_c \tau}] = -\text{Im}[\tilde{A}^{\text{SVA}}(z, t)e^{-i\omega_c \tau}]. \tag{3.75}
\]

This inconvenience in the opposite sign of the phase only appears when I consider each field component. In addition, regardless of the opposite sign of the phase, the real field and the total transmitted intensity \( T(z, t) \) are same for both cases. From Eq. (3.74) and Eq. (3.73),

\[
s(z, \Delta) = \sqrt{1 + e^{-\alpha_0 L\delta^2/(\Delta^2 + \delta^2)} - e^{-\alpha_0 L\delta^2/[2(\Delta^2 + \delta^2)]} \cos \left[ \frac{\Delta \alpha_0 L\delta}{2(\Delta^2 + \delta^2)} \right]} \\
- (1 - 1/\sqrt{3}) \beta(z, \Delta), \tag{3.76}
\]

\[
\phi_{SB} = -\text{Arg}[1 - e^{\alpha_0 z/[2(i\Delta - \delta)]}] = -\varphi(\Delta) + \phi_{SB}'.
\]

From Eq. (3.73) and Eq. (3.76), the modified complex envelope of the precursor
is given by

\[
\tilde{A}^{\text{asymp}}_{SB}(z, z/c) = A_0 \Theta(\tau) e^{-i \text{Arg}[(1 - e^{\alpha_0 z \delta/[2(\Delta - \delta)])}}] \\
\times \sqrt{1 + \frac{\alpha_0 L \delta^2}{(\Delta^2 + \delta^2)} - e^{-\frac{\alpha_0 L \delta^2}{2(\Delta^2 + \delta^2)}} \cos \left[\frac{\Delta \alpha_0 L \delta}{2(\Delta^2 + \delta^2)}\right]} = r_{SB} e^{i \phi_{SB}} \tag{3.77}
\]

\[
\tilde{A}^{\text{asymp}}_{C}(z, z/c) = A_0 \Theta(\tau) e^{-\alpha_0 L \delta^2/[2(\Delta^2 + \delta^2)]} e^{i \text{Arg}[(1 - e^{\alpha_0 z \delta/[2(\Delta - \delta)])}}] = r_{C} e^{i \phi_{C}}
\]

\[
\tilde{A}^{\text{asymp}}(z, z/c) = r e^{i \phi}.
\]

Finally, the modified asymptotic theory predicts each part of the transmitted field as

\[
E_{SB}(z, t) = A_0 \Theta(\tau) e^{-\delta \tau} \sqrt{1 + \frac{\alpha_0 L \delta^2}{(\Delta^2 + \delta^2)} - e^{-\frac{\alpha_0 L \delta^2}{2(\Delta^2 + \delta^2)}} \cos \left[\frac{\Delta \alpha_0 L \delta}{2(\Delta^2 + \delta^2)}\right]} \times \sin(\omega_0 \tau - \text{Arg}[(1 - e^{\alpha_0 z \delta/[2(\Delta - \delta)])}})], \tag{3.78}
\]

\[
E_{C}(z, t) = A_0 \Theta(\tau) e^{-\alpha_0 L \delta^2/[2(\Delta^2 + \delta^2)]} \sin(\omega_c \tau + \Delta \alpha_0 z \delta/[2(\Delta^2 + \delta^2)]). \tag{3.79}
\]

Equation (3.78) is equivalent to Eq. (3.55) for \(\Delta = 0\).

For the case of \(\Delta \neq 0\), I plot both the \(|A_{SB}(z, t)|\) and \(\text{Arg}[A_{SB}(z, t)]\) in Eq. (3.77), as shown in Figure 3.6 for \(\Delta = 1.25 \delta\), and Figure 3.7 for \(\Delta = 4.79 \delta\). In both graphs, the total precursor amplitude reduces as the detuning is increased. On the other hand, the main-signal amplitude increases because of the reduced absorption. Figure 3.6 (b) and Figure 3.7 (b) show the phase \(\text{Arg}[A_{SB}(z, t)]\) for \(\Delta = 1.25 \delta\) and \(\Delta = 4.79 \delta\), respectively. The phase is varied from \(+\pi\) to \(-\pi\) with a period of \(2\pi/\Delta\).

The intensity transmission function \(T(z, t)\) obtained from each part Eq. (3.78)-
(3.79) is given as

\[ T(z, t) = \frac{|\hat{A}_{\text{asymp}}^{SB}(z, t) + \hat{A}_{\text{asymp}}^{\text{asymp}}(z, t)|^2}{A_0^2}, \]

\[ = \left[ |\hat{A}_{\text{asymp}}^{\text{asymp}}(z, t)|^2 + |\hat{A}_{\text{asymp}}^{\text{asymp}}(z, t)|^2 \right. \]

\[ + 2|\hat{A}_{\text{asymp}}^{\text{asymp}}(z, t)||\hat{A}_{\text{asymp}}^{\text{asymp}}(z, t)| \cos(\phi_{SB} - \phi_C)|/A_0^2. \]

\[ (3.80) \]

\[ T(z, t) \] is fit to the experimental data, as shown in Figure 3.8. Dots indicate the prediction of the modified asymptotic theory taking into account the finite bandwidth of the electronics used in the experiment. For the case of \( \Delta = 4.79\delta \), \( T(z, t) \) rises to \( \sim 90\% \), then oscillates until it reaches a steady-state value. During the oscillation, it is seen that \( T > 1 \). As mentioned before, it indicates that the delayed response results in storing the energy of the incident light. For the case of \( \Delta = 1.25\delta \), the intensity reaches the same initial peak, and decays to its steady state with rather mild modulation.

3.9 Summary

In this chapter, I evaluated the modern asymptotic theory developed by Oughstun and Sherman [14], to identify each part of transmitted field. While it predicts the transient spikes and the time scales observed in the data, the asymptotic expressions give unreasonable prediction for the precursor amplitude and phase. To obtain agreement with the experimental conditions, several modifications were made based on an empirical analysis. With these modifications, the theory agrees reasonably well with my data. These modifications will be confirmed again by comparing the asymptotic theory to another theory, as discussed in the next chapter.
Figure 3.8: Comparison of the $T(z, t)$ predicted by the modified asymptotic theory (dots) to the off-resonance data (solid line) for the case of different carrier frequency detunings $\Delta$. 
Chapter 4

Slowly Varying Approximation Theory (SVA)

In this chapter, I will discuss another approximate theoretical approach to understand our experimental data, known as the slowly-varying amplitude (SVA) theory. In the previous chapter, I identified each part of the transmitted field using the asymptotic theory, then compared the total field to the experimental data. The asymptotic theory only predicts, qualitatively, the precursor part of the transmitted field. Hence, another theoretical approach (such as SVA theory) may give additional insight to the problem. The SVA theory assumes that the plasma frequency is small ($\omega_p \ll \sqrt{8 \delta \omega_0}$) material resonance is narrow ($\delta \ll \omega_0$), the field carrier frequency is nearly resonant with the material oscillators ($\omega_c \sim \omega_0$), and that the field varies slowly \([22]\). Under these assumptions, it is possible to obtain an simple analytic solution describing the propagation of the step-modulated field through the dielectric \([23,24]\).

The SVA theory is a well known approach by researchers in the quantum optics community, but has been barely connected to precursor phenomena. It is generally believed that precursor phenomena exists only in the ultra fast regime, so that the slowly varying amplitude theory is not appropriate. In this regard, Crisp \([22]\) states, without proof, that the SVA approximations rule out precursor phenomena because the envelope of the field must vary slowly in the theory. In the case that the
field envelope contains 1,000 optical cycles,\(^1\) however, the SVA should be reasonable approximation.

Opposed to Crip’s comments, several researchers have shown that the SVA theory is applicable to precursor phenomena. Varoquaux \textit{et al.} [23], for example, noticed that Rothenberg’s observation of a “weak 0\(\pi\) pulse” [31] could be a “0\(^{th}\) order precursor” through the use of a SVA analysis. Their statement, however, is only a conjecture because they never compare the analysis to the asymptotic theory. Another group of researchers, Aaviksoo \textit{et al.} [24], also applied the SVA theory to their own experiment [19]. They \textit{indirectly} observed optical precursors by propagating an ultra fast pulse through thin GaAs crystals, but he could not identify Sommerfeld and Brillouin precursors from their data. Both groups of researchers did not compare their analysis to the asymptotic analysis of precursors developed by Brillouin. In Chapter 5, I present the first comparison of the asymptotic and SVA theories to provide a better understanding of their behavior. I discuss the SVA theory in this chapter.

To begin, let us recall the integral form of the wave equation [Eq. (1.18)]. To obtain an approximate analytic expression to this equation, the key assumption is to obtain a simplified expression for the index of refraction \(n(\omega)\). Using the assumption of small plasma frequency (\(\omega_p \ll \sqrt{8\delta \omega_0}\)), which is valid for a weakly dispersive medium as in the experiment, the square root appearing in the exact expression for \(n(\omega)\) is eliminated as

\[
n(\omega) = \sqrt{1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + 2i\omega\delta}} \simeq 1 - \frac{1}{2} \frac{\omega_p^2}{\omega^2 - \omega_0^2 + 2i\omega\delta}. \tag{4.1}
\]

\(^1\)In my experimental condition, the rise time of the step-pulse is a few hundred ps, and one optical cycle is 2.6 fs.
Furthermore, assuming that the carrier frequency of the incident light is tuned near the oscillator’s resonance frequency \( \omega \simeq \omega_0 \) allows me to simplify the denominator in Eq. (4.1) as

\[
\omega^2 - \omega_0^2 + 2i\omega \delta = (\omega + \omega_0)(\omega - \omega_0) + 2i\delta \omega; \\
\simeq 2\omega(\omega - \omega_0) + 2i\delta \omega; \\
= 2\omega(\omega - \omega_0 + i\delta).
\]

Under these assumptions, Eq. (4.1) is simplified to

\[
n(\omega) \simeq 1 - \frac{1}{4} \frac{\omega_p^2}{\omega(\omega - \omega_0 + i\delta)}.
\]

(4.2)

This simplified form of \( n(\omega) \) allows me to obtain an analytic form of transmitted field.

When an incident step-modulated pulse\(^2\) of the form

\[
E(z = 0, t) = -\text{Im}[A_0 \Theta(t)e^{-i\omega_c t}]
\]

(4.3)

propagates through single-Lorentz dielectric, the transmitted field is given by

\[
E(z, t) = -\text{Im}\left[\frac{A_0}{2\pi i} \oint \frac{e^{-i\omega \tau}}{\omega - \omega_c} e^{-ip/\omega - \omega_0 + i\delta} d\omega\right],
\]

(4.4)

\[
= -\text{Im}\left[\frac{A_0}{2\pi i} \left(\oint_{\omega_c} Gd\omega + \oint_{\omega_0 - i\delta} Gd\omega\right)\right],
\]

(4.5)

\(^2\)In this section, I use an exponential representation of the sinusoidal wave in the form \( e^{-i\omega_c t} \) instead of \( \sin(\omega_c t) \) for mathematical convenience. Once I have the transmitted field in a complex form \( E(z, t) \), the real transmitted field can be obtained by the relation \( -\text{Im}[E(z, t)] \).
for $\tau > 0$, where $p \equiv \omega^2 z/4c = \alpha_0 \delta z/2$, and the integrand is equal to

$$G \equiv \frac{e^{-i\omega \tau}}{\omega - \omega_c} e^{-ip/(\omega - \omega_0 + i\delta)}.$$ 

The first term in Eq. (4.5) is the contour integral around the simple pole $\omega = \omega_c$, giving rise to the main signal $E_C(z, t)$, and the second term is the contour integral around the essential singularity $\omega = \omega_0 - i\delta$, which results in the total precursor field $E_{SB}(z, t) = E_S(z, t) + E_B(z, t)$. We only obtain the sum of the precursor fields from second term, not the individual Sommerfeld and Brillouin precursors. This inability to separate each precursor is explained from Eq.(4.1), in which two essential singularity points (discussed in the previous chapter) relevant to each precursors collapse onto each other, and remain as one essential singularity.

We can rewrite Eqs.(4.4)-(4.5) in terms of the total precursors and the main signal as

$$E(z, t) = E_C(z, t) + E_{SB}(z, t) = -\text{Im}[\tilde{E}_{SV A}^C(z, t) + \tilde{E}_{SV A}^{SB}(z, t)], \quad (4.6)$$

where

$$\tilde{E}_{SV A}^C(z, t) \equiv \frac{A_0}{2\pi i} \int_{\omega_c} G d\omega,$$

$$\tilde{E}_{SV A}^{SB}(z, t) \equiv \frac{A_0}{2\pi i} \int_{\omega_0 - i\delta} G d\omega. \quad (4.7)$$

Note that relativistic causality is preserved in evaluating Eq.(4.4) because the contour integral only has a value for $\tau = t - z/c > 0$. Derivation of each term will be presented in the next two sections.
4.1 The Main Signal: $E_C(z, t)$

In this section, I will find a solution to the first term in Eq. (4.5). The first term $\tilde{E}_C^{SV A}(z, t)$ is a contour integral around the carrier frequency $\omega_c$ of the incident pulse, thereby related to the main signal as $E_C(z, t) = -\text{Im}[\tilde{E}_C^{SV A}(z, t)]$. The first term $\tilde{E}_C^{SV A}(z, t)$ is given by

$$\tilde{E}_C^{SV A}(z, t) = \frac{A_0}{2\pi i} \oint_{\omega_c} Gd\omega = \frac{A_0}{2\pi i} \int_{\omega_c} \frac{e^{-i\omega\tau}}{\omega - \omega_c} e^{-ip/(\omega - \omega_0 + i\delta)} d\omega. \quad (4.8)$$

By letting the complex variable $z \equiv \omega - \omega_c$,

$$\tilde{E}_C^{SV A}(z, t) = \frac{A_0}{2\pi i} \oint_{z=0} \frac{dz}{z} e^{-i(z+\omega_c)\tau} e^{-ip/(z+\omega_c-\omega_0+i\delta)}. \quad (4.9)$$

A solution to Eq. (4.9) is obtained by the residue theorem [32], which states that

$$\oint_C f(z)dz = 2\pi i(\text{sum of enclosed residues}). \quad (4.10)$$

Therefore, the solution to Eq. (4.9) is written as

$$\tilde{E}_C^{SV A}(z, t) = A_0 \Theta(\tau)e^{p/(i\Delta-\delta)}e^{-i\omega_c\tau}. \quad (4.11)$$

Equation (4.11) is essentially the same expression for main signal as found from the asymptotic theory [See Eq. (3.62)] when I take $-\text{Im}[\tilde{E}_C^{SV A}(z, t)]$.

Equation (4.11) can be expressed in different way using the “generating Bessel
function,” given by

\[ e^{\frac{z}{2} (u - \frac{1}{u})} = \sum_{m=-\infty}^{\infty} u^m J_m(x). \] (4.12)

If \( u = iz\sqrt{\tau/p} \) and \( x = 2\sqrt{p\tau} \), then we have a different form of Eq. (4.12) as

\[ e^{i(\tau y - \frac{z}{y})} = \sum_{m=-\infty}^{\infty} i^m \left( \frac{\tau}{p} \right)^{m/2} y^m J_m(2\sqrt{p\tau}), \] (4.13)

where \( y = -\Delta - i\delta \). Therefore, Eq. (4.11) is given by

\[ \tilde{E}_{SV A}^C(z, t) = A_0 \Theta(\tau) e^{(-i\omega_0 - \delta)\tau} \sum_{m=-\infty}^{\infty} i^m \left( \frac{\tau}{p} \right)^{m/2} (-\Delta - i\delta)^m J_m(2\sqrt{p\tau}) e^{-i\omega_c\tau}, \]

\( = A_0 \Theta(\tau) e^{(i\Delta - \delta)\tau} \sum_{m=-\infty}^{\infty} (p\tau)^{m/2} \left( \frac{-i\Delta + \delta}{p} \right)^m J_m(2\sqrt{p\tau}) e^{-i\omega_c\tau}. \) (4.14)

By considering \( n = -m \), and \( \sum_{n=-\infty}^{\infty} = \sum_{n=-\infty}^{\infty} \), Eq. (4.14) is rewritten as

\[ \tilde{E}_{SV A}^C(z, t) = A_0 \Theta(\tau) e^{(i\Delta - \delta)\tau} \sum_{n=-\infty}^{\infty} (p\tau)^{-n/2} \left( \frac{p}{i\Delta - \delta} \right)^n J_n(2\sqrt{p\tau}) e^{-i\omega_c\tau}. \] (4.15)

Note that Eq. (4.15) is identical to Eq. (4.11).

The two equivalent expressions for \( \tilde{E}_{SV A}^C(z, t) \), Eq. (4.11) and Eq. (4.15), will be used in two different expressions of the total transmitted field later. To obtain the total transmitted field, I will derive the second term of Eq (4.5) in the next section.
4.2 Precursors : $E_{SB}(z, t)$

In this section, I will find a solution to the second term $\hat{E}^{SVA}_{SB}(z, t)$ in Eq. (4.5), which is the contour integral around $\omega_0 - i\delta$, and is related to the total precursors $E_{SB}(z, t) = -\text{Im}[\hat{E}^{SVA}_{SB}(z, t)]$.

By letting $z \equiv \omega - \omega_0 + i\delta$, the contour integral around the essential singularity is given by

$$\hat{E}^{SVA}_{SB}(z, t) = \frac{A_0}{2\pi i} e^{-i(\omega_0-i\delta)\tau} \oint_{z=0} dz \frac{e^{-iz\tau-ip/z}}{z + \omega_0 - \omega - i\delta}. \quad (4.16)$$

When the denominator of Eq. (4.16) is rewritten as

$$\frac{1}{z-a} = -\frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n = -\sum_{n=0}^{\infty} \frac{z^n}{a^{n+1}}, \quad (4.17)$$

where $a \equiv \omega_c - \omega_0 + i\delta$, and

$$e^{-iz\tau-ip/z} = \sum_{m=-\infty}^{\infty} z^m i^m (\frac{\tau}{p})^{m/2} (-1)^m J_m(2\sqrt{p\tau}), \quad (4.18)$$

we are left with

$$\hat{E}^{SVA}_{SB}(z, t) = -\frac{A_0 \Theta(\tau)}{2\pi i} e^{(-i\omega_0-i\delta)\tau} \oint_{z=0} dz \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{z^{m+n} i^m (\tau/p)^{m/2} (-1)^m J_m(2\sqrt{p\tau})}{(\omega_c - \omega_0 + i\delta)^{n+1}}. \quad (4.19)$$

Using the residue theorem $\oint_{z=0} dz z^{m+n} = 2\pi i \delta_{m,-(n+1)}$, and $J_{-n}(x) = (-1)^n J_n(x)$,

$$\hat{E}^{SVA}_{SB}(z, t) = -A_0 \Theta(\tau) e^{i(\Delta - \delta)\tau} \sum_{n=1}^{\infty} \left(\frac{p}{i\Delta - \delta}\right)^n (p\tau)^{-n/2} J_n(2\sqrt{p\tau}) e^{-i\omega_c\tau}, \quad (4.20)$$
for $\sqrt{p/\tau(\Delta^2 + \delta^2)} < 1$. Equation (4.20) contains an exponential decay term, $e^{-\delta\tau}$, which implies that the second term of Eq. (4.5) is the transient response to the medium. Note that the transient time scale is not only governed by $e^{-\delta\tau}$, but it is also effected by the Bessel functions $J_n(2\sqrt{p\tau})$.

I will derive the total transmitted field by summing the transient response [Eq. (4.20)] obtained in this section, and the steady-state response [reduced step-modulated field Eq. (4.11) or Eq. (4.15)] in the previous section.

### 4.3 Total transmitted field: $E(z, t)$

In this section, I will present two different ways to find the total integral [Eq. (4.5)] related to the total transmitted field $E(z, t) = -\Im[\tilde{E}_{SV}^{C}(z, t) + \tilde{E}_{SV}^{B}(z, t)]$. Since there are two alternate expressions of $\tilde{E}_{SV}^{C}(z, t)$ as seen in Eq. (4.11) and Eq. (4.15), the two alternate, but identical, forms of total transmitted fields $E(z, t) = E_{C}(z, t) + E_{SB}(z, t)$ are obtained. These two have different convergence regimes of their respective series, which, together, cover the entire parameter space. One of them, the sum of Eq. (4.11) and Eq. (4.20) result in

$$ E(z, t) = -\Im\left[A_0 \Theta(\tau) \left(e^{p/(i\Delta - \delta)} - e^{i(\Delta - \delta)\tau} \sum_{n=1}^{\infty} \left(\frac{p}{i\Delta - \delta}\right)^n (p\tau)^{-n/2} J_n(2\sqrt{p\tau}) e^{-i\omega_c\tau}\right]\right], \quad (4.21) $$

for $\sqrt{p/\tau(\Delta^2 + \delta^2)} < 1$.

The other expression of the total field is given by the sum of Eq. (4.15) and Eq.
\[(4.20)\] when we let
\[S_n \equiv \left( \frac{p}{i\Delta - \delta} \right)^n (p\tau)^{-n/2} J_n(2\sqrt{p\tau}), \quad (4.22)\]

and consider \(\sum_{n=1}^{\infty} S_n = \sum_{n=0}^{\infty} S_n - \sum_{n=0}^{\infty} \sum_{n=1}^{\infty} S_n.\) The total transmitted field is given as

\[E(z, t) = -\text{Im} \left[ A_0 \Theta(\tau) e^{(i\Delta-\delta)\tau} \left( - \sum_{n=-\infty}^{\infty} S_n + \sum_{n=-\infty}^{\infty} S_n - \sum_{n=-\infty}^{0} S_n \right) e^{-i\omega_c \tau} \right]. \quad (4.23)\]

By considering \(\sum_{n=0}^{\infty} S_n = \sum_{n=0}^{\infty} S_{-n},\) Eq. (4.23) is finally given as

\[E(z, t) = -\text{Im} \left[ A_0 \Theta(\tau) e^{(i\Delta-\delta)\tau} \sum_{n=0}^{\infty} \left( -\frac{i\Delta + \delta}{p} \right)^n (p\tau)^{n/2} J_n(2\sqrt{p\tau}) e^{-i\omega_c \tau} \right]. \quad (4.24)\]

for \(\sqrt{p/\tau(\Delta^2 + \delta^2)} > 1.\)

In this section, I obtained the solution for the total transmitted field analytically, as in Eq. (4.21) and Eq. (4.24), using the SVA method [33]. These results will be used to explain my data in the next sections.

### 4.4 Total transmitted field for an on-resonance carrier frequency

I evaluated integral Eq. (4.5) to obtain the transmitted field expressions using the slowly varying amplitude approximation [24, 33] in the previous section. In this section, I will compare the SVA theory for a single-resonance Lorentz medium to my data for an on-resonance carrier frequency.
For $\Delta = 0$, the total transmitted field derived in the previous section is given as

$$E(z, t) = A(z, t) \sin(\omega_0 \tau)$$

$$= A_0 \Theta(\tau) \left[ e^{-\alpha_0 z/2} - e^{-\delta \tau} \sum_{n=1}^{\infty} (-1)^n (\alpha_0 z/2 \delta \tau)^n/2 J_n \left( \sqrt{2\alpha_0 z \delta \tau} \right) \right] \sin(\omega_0 \tau),$$

(4.25)

$$E(z, t) = A(z, t) \sin(\omega_0 \tau)$$

$$= A_0 \Theta(\tau) e^{-\delta \tau} \sum_{n=0}^{\infty} (\alpha_0 z/2 \delta \tau)^{-n/2} J_n \left( \sqrt{2\alpha_0 z \delta \tau} \right) \sin(\omega_0 \tau).$$

(4.26)

The series in Eq. (4.25) [Eq. (4.26)] converges when $\sqrt{\alpha_0 z/2 \delta \tau} < 1$ [$\sqrt{\alpha_0 z/2 \delta \tau} > 1$]. The first term of Eq. (4.25) is merely a step-modulated pulse reduced by the medium absorption, as shown in Figure 4.1 (b). It is identical to $E_C(z, t)$ of the asymptotic theory [see Eq. (3.47)], which represents the main signal. The second term is the transient response of the propagated field, as shown in Figure 4.1 (a), which should be equal to the sum of the two precursors $E_S(z, t) + E_B(z, t)$. For the on-resonance case, the second term is indeed similar to $E_S(z, t) + E_B(z, t)$ in the asymptotic theory showing exponential decay.\(^3\) It is not easily separated into the individual Sommerfeld and Brillouin precursor fields by SVA theory itself, yet we can easily compare it to the total precursor field. Note that the total precursor field (second term) and the main signal (first term) in Eq. (4.25) are in phase. This justify why the $\pi/4$ phase shift is set to zero in the asymptotic theory.

From Eqs. (4.25)-(4.26), The total intensities $T(z, t)$ predicted by the SVA theory

\(^3\)The presence of the Bessel functions affect the compression of the transient decay time scale, but there are no significant oscillations for $\alpha_0 L < 2$. 

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Figure 4.1: Each part of transmitted field (a) $E_S(z, t) + E_B(z, t)$, and (b) $E_C(z, t)$ predicted by SVA theory for $\Delta = 0$. 

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are

\[
T(z, t) = \Theta(\tau) \left[ e^{-\alpha_0 z/2} - e^{-\delta \tau} \sum_{n=1}^{\infty} (-1)^n (\alpha_0 z/2\delta \tau)^n/2 J_n \left( \sqrt{2\alpha_0 z\delta \tau} \right) \right]^2, \tag{4.27}
\]

\[
T(z, t) = \Theta(\tau) e^{-2\delta \tau} \left[ \sum_{n=0}^{\infty} (\alpha_0 z/2\delta \tau)^{-n/2} J_n \left( \sqrt{2\alpha_0 z\delta \tau} \right) \right]^2. \tag{4.28}
\]

Eqs. (4.27)-(4.28) are denoted as dots in Figure 4.2 for different cases of absorption depth \(\alpha_0 L\). The 100% total intensity transmission peak predicted by Eq. (4.27)-(4.28) is smoothed out considering the total bandwidth 206 MHz of the electronics. These results agree with the experimental data denoted as solid line. The agreement between the SVA theory and the data is better than shown in Figure 3.4 in terms of the time scale. For the case \(\alpha_0 L = 0.41\), there is not much difference between the SVA theory and the asymptotic theory. For the most absorptive case of \(\alpha_0 L = 1.03\), the time scale of the transient part differs from each other. The data shows a shorter time scale in the SVA theory as the absorption depth increases. This compression in the time scale does not appear in the asymptotic theory, whose time scale only depends on \(\delta\). The difference in the time scale is explained by the Bessel function \(J_n \left( \sqrt{2\alpha_0 z\delta \tau} \right)\) in both Eqs. (4.27) and (4.28), where \(z = L\). The Bessel function causes a compression of the time scale as the absorption depth \(\alpha_0 L\) increases. Therefore, the SVA theory does a better job of predicting the transmitted field.

In this section, I have presented the SVA theory to understand the experimental data for the case of an on-resonance carrier frequency, and have shown that there is good agreement with the data.
Figure 4.2: Comparison of the $T(z,t)$ predicted by the SVA theory (dots) to the on-resonance data (solid line) for the case of three different absorption depths $\alpha_0L$. 

\[ \alpha_0L = 0.00 \]
\[ \alpha_0L = 0.41 \]
\[ \alpha_0L = 1.03 \]
4.5 Total transmitted field for an off-resonance carrier frequency

In this section, I will discuss the SVA theory for the case of an off-resonance carrier frequency where $\omega_c \neq \omega_0$ ($\Delta \neq 0$). For this case, the total transmitted field is given by Eq. (4.21) and Eq. (4.24),

\[ E(z, t) = -\text{Im} \left[ A_0 \Theta(\tau) \left( e^{\frac{p}{i(\Delta - \delta)}} e^{-i\omega_c \tau} \right. \right. \]
\[ \left. \left. - e^{-\delta \tau} \sum_{n=1}^{\infty} \left( \frac{p}{i(\Delta - \delta)} \right)^n (p\tau)^{-n/2} J_n \left( 2\sqrt{p\tau} e^{-i\omega_0 \tau} \right) \right] \right], \tag{4.29} \]

\[ E(z, t) = -\text{Im} \left[ A_0 \Theta(\tau) e^{-\delta \tau} \sum_{n=0}^{\infty} \left( \frac{-i\Delta + \delta}{p} \right)^n (p\tau)^{n/2} J_n \left( 2\sqrt{p\tau} e^{-i\omega_0 \tau} \right) \right], \tag{4.30} \]

Equation (4.29) converges when $\sqrt{p/(\tau(\Delta^2 + \delta^2))} < 1$, and Eq. (4.30) converges when $\sqrt{p/(\tau(\Delta^2 + \delta^2))} > 1$. The first term of Eq. (4.29) corresponding to the main signal $E_C(z, t)$ increases as the detuning $\Delta$ increases because there is less absorption for the off-resonance case, as in the asymptotic theory. The second term is the sum of Sommerfeld and Brillouin precursors $E_{SB}(z, t)$.

The complex envelopes of Eq. (4.29)-(4.30) are given as

\[ \tilde{A}^{SV A}(z, t) = A_0 \Theta(\tau) e^{(i\Delta - \delta)\tau} \sum_{n=1}^{\infty} \left( \frac{p}{i(\Delta - \delta)} \right)^n (p\tau)^{-n/2} J_n \left( 2\sqrt{p\tau} e^{-i\omega_0 \tau} \right), \tag{4.31} \]

\[ \tilde{A}^{SV A}(z, t) = A_0 \Theta(\tau) e^{(i\Delta - \delta)\tau} \sum_{n=0}^{\infty} \left( \frac{-i\Delta + \delta}{p} \right)^n (p\tau)^{n/2} J_n \left( 2\sqrt{p\tau} \right), \tag{4.32} \]

The first term of Eq. (4.31) is the envelope of the main signal $A_C(z, t)$, which can
Figure 4.3: Each part of transmitted field predicted by the SVA theory (a) $|A_{SB}(z,t)|$, (b) $\text{Arg}[A_{SB}(z,t)]$ and (c) $A_C(z,t)$ for $\Delta = 1.25\delta$. 
Figure 4.4: Each part of transmitted field predicted by the SVA theory (a) $|A_{SB}(z, t)|$, (b) $\text{Arg}[A_{SB}(z, t)]$ and (c) $A_C(z, t)$ for $\Delta = 4.79\delta$. 

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be rewritten as

\[ A_C(z, t) = A_0 \Theta(\tau) e^{-\alpha_0 L \delta^2 / [2(\Delta^2 + \delta^2)]} e^{-i \Delta \alpha_0 L \delta / [2(\Delta^2 + \delta^2)]}. \] (4.33)

The absolute value of Eq. (4.33) is identical to the expression predicted by the asymptotic theory Eq. (3.71), as shown in Figure 4.3 (c). The phase of Eq. (4.33) depends on the detuning \( \Delta \) and the absorption depth \( \alpha_0 L \). The phase does not depend on time. That is, the main signal predicted by the SVA theory has a constant phase shift for off-resonance cases. The phase of the main signal predicted by the SVA is used for correction of the phase of the main signal predicted by the asymptotic theory in Ch.3.

Figure 4.3 (a)-(b) shows the envelope of the total precursors. As Eq. (4.32) predicts, the precursor peak decreases as the detuning increases. Note that the total precursor field (the second term in (4.29)) always oscillates at the medium resonance frequency \( \omega_0 (= -\Delta + \omega_c) \), while the main signal (first term) oscillates at the carrier frequency of the incident pulse \( \omega_c \). This is also predicted by the asymptotic theory, as discussed in the previous chapter. Again, it is confirmed that the difference in frequencies between total precursors and main signal gives rise to frequency modulation of the total transmission field intensity. From Eqs. (4.29)-
The total intensities \( T(z, t) \) predicted by the SVA theory are

\[
T(z, t) = \Theta(\tau)e^{\frac{\pi}{2} + \frac{\pi}{4\tau^2}} - e^{-2\delta\tau}\left\{ \sum_{n=1}^{\infty} \left( \frac{p}{i\Delta - \delta} \right)^n (p\tau)^{-n/2} J_n(2\sqrt{p\tau}) \right\}
\]

\[
\times \left\{ \sum_{n'=1}^{\infty} \left( \frac{p}{-i\Delta - \delta} \right)^{n'} (p\tau)^{-n'/2} J_{n'}(2\sqrt{p\tau}) \right\}
\]

\[
+ e^{\frac{\pi}{2} - (i\Delta + \delta)\tau} \left\{ \sum_{n'=1}^{\infty} \left( \frac{p}{-i\Delta - \delta} \right)^{n'} (p\tau)^{-n'/2} J_{n'}(2\sqrt{p\tau}) \right\}
\]

\[
+ e^{-\frac{\pi}{2} + (i\Delta - \delta)\tau} \left\{ \sum_{n=1}^{\infty} \left( \frac{p}{i\Delta - \delta} \right)^n (p\tau)^{-n/2} J_n(2\sqrt{p\tau}) \right\},
\]

(4.34)

\[
T(z, t) = \Theta(\tau)e^{-2\delta\tau}\left\{ \sum_{n=0}^{\infty} \left( \frac{i\Delta + \delta}{p} \right)^n (p\tau)^{n/2} J_n(2\sqrt{p\tau}) \right\}
\]

\[
\times \left\{ \sum_{n'=0}^{\infty} \left( \frac{-i\Delta + \delta}{p} \right)^{n'} (p\tau)^{n'/2} J_{n'}(2\sqrt{p\tau}) \right\}.
\]

(4.35)

The dots in Figure 4.5 denote \( T(z, t) \). It agrees well with the experimental data denoted by the solid line Figure 4.5. The dots also show modulation depending on the detuning \( \Delta \).

To understand the modulation behavior in the data, I evaluate the cross term \( E_{SB}(z, t)^*E_C(z, t) \) of \( T(z, t) \) as

\[
T(z, t) \sim (|E_{SB}(z, t)|^2 + |E_C(z, t)|^2)/E_0^2
\]

\[
+ \frac{2p}{\sqrt{\Delta^2 + \delta^2}} \frac{J_1(2\sqrt{p\tau})}{\sqrt{p\tau}} e^{-\delta\tau - \frac{2p}{\Delta + \delta}\cos \left( \Delta\tau + \Delta\tau + \frac{\Delta p}{\Delta^2 + \delta^2} + \varphi(\Delta) \right)} + ...,
\]

(4.36)

where I have retained the dominant first term \( (n = 1) \) and dropped the higher order terms in Eq. (4.29). Approximately, \( T(z, t) \) with only the \( n = 1 \) term agrees with my data. The modulation frequency is represented as a frequency of a cosine function, \( \Delta/2\pi \), in Eq. (4.36).
Figure 4.5: Comparison of the $T(z,t)$ predicted by the SVA theory (dots) to the off-resonance data (solid line) for the case of different carrier frequency detunings $\Delta$. 
4.6 Summary

In this chapter, I have shown the reasonable agreement between the SVA theory and the experimental data without any modification. The SVA theory distinguishes between the total precursor field and the main signal, yet it cannot give predictions regarding the individual Sommerfeld and Brillouin precursor. This shortcoming, however, is overcome by the asymptotic theory discussed in the previous chapter. The details of comparison between two theory will be discussed in the next chapter.
Chapter 5

Comparison of the two theories

In the previous chapters, the asymptotic theory and the SVA theory are used to understand how a step-modulated pulse propagates through a single-Lorentz medium. The asymptotic theory identifies each part of the transmitted field: the Sommerfeld and Brillouin precursors, and the main signal. However, the asymptotic theory predicts unreasonably large amplitude and phase characteristics for the precursors. The asymptotic theory is modified empirically, and agrees reasonably with the experimental data. Another theory, the slowly varying amplitude (SVA) approximation, fits the experimental data reasonably well without modification. Although the SVA theory cannot identify the individual Sommerfeld and Brillouin precursors, it distinguishes between the transient part and the main signal. While the main signal predicted by both theories is identical, the total precursor fields differ. The difference is not obvious when the maximum propagation distance is only $z = 0.2$ cm, as in my experimental data. In this chapter, I compare the total precursor fields predicted by both theories for different absorption path lengths that extend well into the mature-dispersion regime (large $z$) where the asymptotic theory should be valid.
5.1 Total precursor for the case of on-resonance carrier frequency

In this section, the total precursor fields $E_{SB}(z, t)$ for the on-resonance case ($\Delta = 0$) predicted by both the modified asymptotic and SVA theories are compared. In the modified asymptotic theory, the total precursors $E_{SB}(z, t)$ is given by Eq. (3.55),

$$E_{SB}^{\text{asymp}}(z, t) = A_0 \Theta(\tau)(1 - e^{-\alpha_0 z/2})e^{-\delta \tau} \sin(\omega_0 \tau).$$

(5.1)

The SVA theory predicts that

$$E_{SB}^{\text{SVA}}(z, t) = A_0 \Theta(\tau) \left\{ e^{\delta \tau} \sum_{n=1}^{\infty} (-1)^n (\alpha_0 z/2 \delta \tau)^{n/2} J_n \left( \sqrt{2 \alpha_0 z \delta \tau} \right) \sin(\omega_0 \tau) \right\}$$

(5.2)

$$E_{SB}^{\text{SVA}}(z, t) = A_0 \Theta(\tau) \left\{ e^{-\delta \tau} \sum_{n=0}^{\infty} (\alpha_0 z/2 \delta \tau)^{-n/2} J_n \left( \sqrt{2 \alpha_0 z \delta \tau} \right) - e^{-\alpha_0 z/2} \right\} \sin(\omega_0 \tau).$$

(5.3)

The series in Eq. (5.2) [Eq. (5.3)] converges when $\sqrt{\alpha_0 z/2 \delta \tau} < 1$ [$\sqrt{\alpha_0 z/2 \delta \tau} > 1$].

Note that Eq. (5.2) is just the contour integral associated with the transient part of the total field. In Eq. (5.3), the first term is total transmitted field $E(z, t)$ and the second term is main signal $E_C(z, t)$.

Despite the similarity between the theories, such as an exponential decay term $e^{-\delta \tau}$, there are differences. The Bessel function in the SVA theory, for example, impresses a pattern on the total precursor field, which becomes more pronounced as $z$ increases.

To investigate the patterns, let us focus on the envelope of the precursor field
$A_{SB}(z, t)$ oscillating at $\omega_c$ as

$$E_{SB}(z, t) = A_{SB}(z, t) \sin(\omega_0 \tau) = A_{SB}(z, t) \sin(\omega_c \tau),$$

(5.4)

for $\Delta = \omega_c - \omega_0 = 0$. From Eq. (3.55), the asymptotic theory predicts the envelope of the precursor as

$$A_{SB}^{asypmp}(z, t) = A_0 \Theta(\tau)(1 - e^{-\alpha_0 z/2})e^{-\delta \tau}.$$

(5.5)

From Eq. (5.2)-(5.3), the envelopes predicted by the SVA theory are given as

$$A_{SB}^{SVA}(z, t) = -A_0 \Theta e^{-\delta \tau} \sum_{n=1}^{\infty} (-1)^n (\alpha_0 z/2\delta \tau)^{n/2} J_n \left( \sqrt{2\alpha_0 z \delta \tau} \right),$$

(5.6)

$$A_{SB}^{SVA}(z, t) = A_0 \Theta \left\{ e^{-\delta \tau} \sum_{n=0}^{\infty} (\alpha_0 z/2\delta \tau)^{-n/2} J_n \left( \sqrt{2\alpha_0 z \delta \tau} \right) - e^{-\alpha_0 z/2} \right\}.$$

(5.7)

Figure 5.1 shows the difference more clearly, where I compare the field envelope predicted by both the asymptotic theory, Eq.(5.5) (dots), and the SVA theory, Eq.(5.3), (solid lines) for different path lengths $z$. There is a good agreement between two theories in terms of the overall envelope for $z = 0.2$ cm, as seen in Figure 5.1 (a). For the other three cases, only the initial peak heights agree with each other. These peak heights increase as $z$ increases. This characteristic keeps the peak of the normalized total field equal to $A_0$ considering that the main signal decreases with increasing $z$, consistent with Beer’s law.

As the path length increases [Figure 5.1(b)-(d)], however, the SVA theory predicts oscillations in the field envelope that are not present in the asymptotic theory.

---

1For $\Delta = 0$, Eq.(5.3) is valid for all $z$ used in Figure 5.1.
Figure 5.1: Temporal evolution of the total precursor field envelope for the SVA theory (solid line) and the scaled asymptotic theory (dotted line) for $\Delta = 0$. Each case has different medium distances $z$, (a) 0.2 cm, (b) 2 cm, (c) 20 cm, and (d) 200 cm.
Figure 5.2: Temporal evolution of $T(z,t)$ for the SVA theory (solid line) and the scaled asymptotic theory (dotted line) for $\Delta = 0$. Each case has different medium distances $z$, (a) 0.2 cm, (b) 2 cm, (c) 20 cm, and (d) 200 cm.

Physically, these oscillations are due to the absorption of the central part of the pulse spectrum by the material resonance and subsequent beating between the remaining sidebands [22–24]. The beating between the sidebands is represented by the Bessel functions in the SVA theory. The compression of the time scale, especially as shown in Figure 5.1 (b)-(d), is also due to the Bessel functions.

The total transmitted pulse intensities predicted by the two theories are compared, as shown in Figure 5.2. Because the total precursors and the main signal has the same carrier frequency and phase, there is no frequency modulation between $E_{SB}(z,t)$ and $E_{C}(z,t)$. Thus, the oscillatory behavior seen for the SVA theory for
large absorption depth is only due to the beat note between the sidebands of the propagated field.

5.2 Total precursors for the case of off-resonance carrier frequency

In the previous section, I compared the two theories for the on-resonance case. In this section, I will compare them for the off-resonance case. For $\Delta \neq 0$, the total precursor field predicted by the modified asymptotic expression is given by Eq. (3.78) as

\[
E_{SB}^{asymp}(z, t) = A_0 \Theta(\tau) e^{-\delta \tau} \sqrt{1 + e^{-\alpha_0 L z^2/(2(\Delta^2 + \delta^2))} - e^{-\alpha_0 L z^2/(2(\Delta^2 + \delta^2))} \cos \left[ \frac{\Delta \alpha_0 L \delta}{2(\Delta^2 + \delta^2)} \right]} \times \sin(\omega_0 \tau - \text{Arg}(1 - e^{\alpha_0 z^2/(2(\Delta - \delta))}))
\]

The SVA theory predicts a total precursor field as in Eq. (4.29) and Eq. (4.30) as

\[
E_{SB}^{SVA}(z, t) = -\text{Im} \left[ -A_0 \Theta(\tau) e^{-\delta \tau} \sum_{n=0}^{\infty} \left( \frac{\frac{p}{\Delta^2}}{\gamma} \right)^n (p\tau)^{-n/2} J_n(2\sqrt{p\tau}) e^{-i\omega_0 \tau} \right] + e^{\frac{\alpha_0 L \delta}{2(\Delta^2 + \delta^2)}} e^{-i\omega_0 \tau}
\]

where $p \equiv \omega_0^2 z^2/4c = \alpha_0 \delta z/2$. The series in Eq. (5.9) [Eq. (5.10)] converges when $\sqrt{p/(\tau(\Delta^2 + \delta^2))} < 1$ [\sqrt{p/(\tau(\Delta^2 + \delta^2))} > 1]. Again, both expressions are required
for the SVA theory to cover whole parameter space \((\alpha_0 z, \tau, \delta, \Delta)\).

From Eq. (5.8), the asymptotic theory predicts the complex envelope of the total precursor field \(A(z, t)\) as

\[
\tilde{A}_{\text{asymp}}(z, z/c) = A_0 \Theta(\tau) e^{-\delta \tau} \sqrt{1 + e^{-\frac{\alpha_0 L \delta^2}{(\Delta^2 + \delta^2)}} - e^{-\frac{\alpha_0 L \delta^2}{(\Delta^2 + \delta^2)}} \cos \left[ \frac{\Delta \alpha_0 L \delta^2 (\Delta^2 + \delta^2)}{2(\Delta^2 + \delta^2)} \right]} e^{i \phi_{SB}},
\]

(5.11)

using \(\omega_0 = \omega_c - \Delta\). From Eqs. (5.9)-(5.10), the SVA theory predicts \(A(z, t)\) as

\[
\tilde{A}_{\text{SVA}}(z, t) = -A_0 \Theta(\tau) e^{-\delta \tau + i \Delta \tau} \sum_{n=0}^{\infty} \left( \frac{p}{i \Delta - \delta} \right)^n (p \tau)^{-n/2} J_n(2 \sqrt{p \tau}),
\]

(5.12)

\[
\tilde{A}_{\text{SVA}}(z, t) = A_0 \Theta(\tau) \left\{ e^{-\delta \tau + i \Delta \tau} \sum_{n=0}^{\infty} \left( \frac{-i \Delta + \delta}{p} \right)^n (p \tau)^{n/2} J_n(2 \sqrt{p \tau}) \right. \\
- e^{\frac{-p}{p \tau}} \left. \right\},
\]

(5.13)

For the case of \(\Delta = 0\) in the previous section, the field components from either theory are pure real so that the envelopes are also pure real, as in Figure 5.1. For the case of \(\Delta \neq 0\), I plot the \(|A(z, t)|\) and \(\text{Arg}[A(z, t)]\) to compare both theories, as shown in Figure 5.3-Figure 5.6.

Figure 5.3 and Figure 5.4 show the envelopes of the total precursor field predicted by the modified asymptotic theory (dots) and SVA theory (solid lines) for \(\Delta = 1.25 \delta\). In Figure 5.3, the overall amplitudes of \(|A(z, t)|\) shows agreement between both theories. The phases from the both theories agree each other for \(z = 0.2\) cm case, as shown in Figure 5.4 (a). Figure 5.4 (b)-(d), however, shows more complicated phase
pattern in the SVA (solid lines). The period of the phase corresponds to $2\pi/\Delta$ as well as for $\Delta = 4.79\delta$ case in Figure 5.6. Figure 5.7 shows $T(z,t)$ for the case of $\Delta = 1.25\delta$.

For $\Delta = 4.79\delta$, Figure 5.5 shows the absolute values of the precursor envelope. The associated phase is shown in Figure 5.6. The modified asymptotic theory agrees well with the SVA theory (solid lines) for $z = 0.2$. In this case, the transient spike decreases as $\Delta$ increases. However, for large $z$, as shown in Figure 5.5 (b)-(d), the main signal is absorbed and the peak heights increase up to 1 regardless of the detuning. As $z$ increases, the compression in the time scale appears in the SVA theory. This is due the Bessel functions, which also provide oscillations.

Figure 5.8 shows $T(z,t)$ for $\Delta = 4.79\delta$ predicted by the both theories. Both predict frequency modulation due to the detuning $\Delta$. For example, the oscillation pattern appeared in the modified asymptotic theory (dots in Figure 5.8) indicates the remaining main signal. This type of modulation can be used to test the remnant of the main signal at certain $z$ as seen in Figure 5.8 (c). For $z = 200$ cm, the modulation disappears (See dots of Figure 5.8 (d)), thereby indicating that the transmission intensity only contains the total precursor. This type of modulation is not obvious in the solid lines (the SVA theory) because the Bessel functions impress the time scale at large $z$. The Bessel functions also play a role in the irregular pattern in the SVA theory (solid lines of Figure 5.8 (b)). In this case, in addition to the frequency modulation, it has the amplitude modulation by Bessel function.
Figure 5.3: The envelope of the precursor amplitude for $\Delta = 1.25\delta$ case. Temporal evolution of the total precursor field envelope for the SVA theory (solid line) and the scaled asymptotic theory (dotted line). Each case has different medium distances $z$, (a) 0.2 cm, (b) 2 cm, (c) 20 cm, and (d) 200 cm.
Figure 5.4: The phase of the precursor envelope for $\Delta = 1.25\delta$ case. Temporal evolution of the total precursor field envelope for the SVA theory (solid line) and the scaled asymptotic theory (dotted line). Each case has different medium distances $z$, (a) 0.2 cm, (b) 2 cm, (c) 20 cm, and (d) 200 cm.
Figure 5.5: The absolute value of the precursor envelope for $\Delta = 4.79\delta$ case. Temporal evolution of the total precursor field envelope for the SVA theory (solid line) and the scaled asymptotic theory (dotted line). Each case has different medium distances $z$, (a) 0.2 cm, (b) 2 cm, (c) 20 cm, and (d) 200 cm. (The insets zoom in graphs near the front.)
Figure 5.6: The phase of the precursor envelope for $\Delta = 4.79\delta$ case. Temporal evolution of the total precursor field envelope for the SVA theory (solid line) and the scaled asymptotic theory (dotted line). Each case has different medium distances $z$, (a) 0.2 cm, (b) 2 cm, (c) 20 cm, and (d) 200 cm. (The insets zoom in graphs near the front.)
Figure 5.7: Temporal evolution of $T(z, t)$ for the SVA theory (solid line) and the scaled asymptotic theory (dotted line) for $\Delta = 1.25\delta$. Each case has different medium distances $z$, (a) 0.2 cm, (b) 2 cm, (c) 20 cm, and (d) 200 cm.
Figure 5.8: Temporal evolution of $T(z, t)$ for the SVA theory (solid line) and the scaled asymptotic theory (dotted line) for $\Delta = 4.79\delta$. Each case has different medium distances $z$, (a) 0.2 cm, (b) 2 cm, (c) 20 cm, and (d) 200 cm.
5.3 Summary

I have compared the total precursor fields predicted by the two theories for different absorption path lengths. Even for large $z$, where it was previously believed that the asymptotic theory should be valid, the asymptotic theory requires empirical modification such as a scale factor $s(z, \Delta)$, removal of a $\pi/4$ phase shift, and a change in sign to the frequency dependent phase shift $\varphi(\Delta)$ to agree with the SVA theory.

The modified asymptotic theory and the SVA theory agree well for $z = 0.2$ cm. As $z$ increases, the SVA theory shows oscillation due to the beating of the absorption-created sidebands, while their overall amplitudes are the same order of magnitude as that of the modified asymptotic theory. The beating is characterized by the Bessel functions appearing in Eq. (5.2), which results in more complicated amplitude modulation of the total intensity transmission.
Chapter 6
Double-Lorentz Dielectric

So far, I have discussed how a step-modulated pulse propagates through the single-Lorentz medium to explain the experimental data taken near the $4S_{1/2}(F = 1) \leftrightarrow 4P_{1/2}(F = 2)$ transition. By comparison to the experimental data, the single-Lorentz model agrees with the data reasonably well. On the other hand, the data set taken near $4S_{1/2}(F = 1) \leftrightarrow 4P_{1/2}(F = 2)$ and $4S_{1/2}(F = 2) \leftrightarrow 4P_{1/2}(F = 1, 2)$ should be treated using a double-resonance Lorentz model. Developing the analytic double-resonance Lorentz theory for the precursor is a real task compared to the single-resonance Lorentz model because the integral form of the wave equation does not have a solution even for the SVA theory. Fortunately, the double-resonance Lorentz model can be established just by the extension of the single-resonance Lorentz model in a special case of a narrow-resonance system, which is one of the characteristics of a cold atomic gas. In this chapter, I will discuss how I establish the method, thereby extending the single-resonance model to the double-resonance case and compare the theories to the experimental data.

6.1 General approach to double-Lorentz medium

A major difference between a double-resonance model and the single-resonance arises from the index of refraction $n(\omega)$ in the integral form of $E(z, t)$ Eq. (4.4). The index
of refraction for the double-resonance case is given by

\[ n(\omega) = \sqrt{\varepsilon(\omega)} = \sqrt{\sum_{F'F} \left(1 - \frac{\omega_{p_{F'F}}^2}{\omega^2 - \omega_{F'F}^2 + 2i\omega\delta_{F'F}}\right)}, \]  

(6.1)

where \( \omega_{p_{F'F}} \) is the plasma frequency, \( \delta_{F'F} \) is the atomic decay constant, and \( \omega_{F'F} \) is the atomic resonance frequency for each peak. There is no analytic solution for the double-resonance case.

The double-resonance problem was discussed a little in the original literature of Brilouin (pp. 128-130 of Ref. [12]). He mentioned the existence of a “third groups of forerunners,” which arises from the frequency \( \omega_M \) between the two absorption peak, as shown in Figure 6.1. Oughstun developed Brillouin’s idea of the “third groups of forerunners” for a double- or multi-resonance Lorentz dielectric. Oughstun used a term “middle precursor” instead of the “third groups of forerunners,” and investigated the dynamics of saddle points associated with the middle precursor. According to his analysis of the saddle points, the middle precursor arrives after the Sommerfeld precursors, and is followed by the Brillouin precursor. He evaluated the transmitted field when a step-pulse propagates through double-resonance Lorentz medium numerically [14, 34, 35]. But, none of analytic expression has been done yet. All of these previous studies assumed a “strongly dispersive broad resonance media.”

### 6.2 Weakly dispersive narrow resonance case

In Ch.4, I took advantage of my experimental condition of a weakly-dispersive narrow-resonance dielectric medium, \( \omega_{F'F} \gg \omega_{p_{F'F}}, \delta_{F'F} \), to obtain the single-resonance
Figure 6.1: The location of the middle precursor frequency $\omega_M$ in the double-resonance absorption lines. $\omega_S$ and $\omega_B$ denotes the frequency of Sommerfeld and Brillouin precursors. The middle precursor is denoted as $\omega_M$.

model. Following the procedure used in Ch. 4, I obtain a simplified expression of $n(\omega)$ for the case of double-resonance, which I find is given as,

$$n(\omega) \simeq \sum_{F'F} \left( 1 - \frac{\omega_{p_{F'F}}^2}{4 \omega(\omega - \omega_{p_{F'F}} + i \delta_{F'F})} \right).$$  \hspace{1cm} (6.2)

Using Eq.(6.2), the main signal is obtained by the contour integral around $\omega_c$. Unfortunately, to my knowledge, it is impossible to obtain an analytic solution for the total precursor field even with the simplifications that lead to Eq. (6.2).

To get around this problem, I postulate that each resonance acts as a independent single-resonance for the precursor evolution if each absorption resonance is narrow and well separated from each other. I postulate that the approximation is valid when $\Delta_{\text{separation}} \gg \delta_{F'F}$, for which the separation of the two peaks is greater than the absorption resonance linewidth. For example, $\Delta_{\text{separation}} = \Delta_e \simeq 12 \delta_{F'F}$ in my case. In this case, which I call the “linear superposition regime”, I assume that the
total precursor is simply the sum of two precursors arising from each absorption peak.

In the next sections, I apply this postulate, linear superposition method, for the double-resonance model to both the modified asymptotic theory and the SVA theory. I then compare these double resonance models to my data for both the \(4S_{1/2}(F = 1) \leftrightarrow 4P_{1/2}\) and \(4S_{1/2}(F = 2) \leftrightarrow 4P_{1/2}\) transitions.

### 6.3 Asymptotic theory for a double-resonance Lorentz dielectric

In this section, the postulate is applied to the modified asymptotic theory. With the postulate, the precursor fields are simply the superposition of a precursor field arising from each single-resonance line. The main signal only has a change in the absorption coefficient \(\alpha_{F'}\) for the double-resonance case in comparison to the single-resonance case. From Eq. (3.78), the total precursor is given as

\[
E_{SB}(z, t) = \text{Im}[E_{SB}^{\text{asymp}}(z, t)] = \text{Im}[\{\tilde{A}_{SB}^{\text{asymp}, F'}(z, t) + \tilde{A}_{SB}^{\text{asymp}, F'}(z, t)e^{i\omega_{c}t}\}],
\]

(6.3)
\[
\tilde{A}_{SB}^{\text{asympt}}(z, t) = A_0 \Theta(\tau) e^{-\delta_{2F}^g} \sqrt{1 + e^{-\frac{\alpha_{2F} L \delta_{2F}^g}{2(\Delta_F^2 + \delta_{2F}^g)}} - e^{-\frac{\alpha_{2F} L \delta_{2F}^g}{2(\Delta_F^2 + \delta_{2F}^g)}} \cos \left[ \frac{\Delta_F (\alpha_{2F} L \delta_{2F}^g)}{2(\Delta_F^2 + \delta_{2F}^g)} \right] e^{i\phi_{SB}^{\prime \prime}}},
\]

where

\[
\tilde{A}_{SB}^{\text{asympt}}(z, t) = A_0 \Theta(\tau) e^{-\delta_{1F}^g} e^{i\phi_{SB}^{\prime \prime}}\sqrt{1 + e^{-\frac{\alpha_{1F} L \delta_{1F}^g}{2(\Delta_F^2 + \delta_{1F}^g)}} - e^{-\frac{\alpha_{1F} L \delta_{1F}^g}{2(\Delta_F^2 + \delta_{1F}^g)}} \cos \left[ \frac{(\Delta_F + \Delta_e) \alpha_{1F} L \delta_{1F}^g}{2((\Delta_F + \Delta_e)^2 + \delta_{1F}^g)} \right] e^{i\phi_{SB}^{\prime \prime}}},
\]

\[
\phi_{SB}^{\prime \prime} = -\text{Arg}[1 - e^{\alpha_{2F} z \delta_{2F}/[2(\Delta_F - \delta_{2F})]}],
\]

\[
\phi_{SB}^{\prime \prime} = -\text{Arg}[1 - e^{\alpha_{1F} z \delta_{1F}/[2(\Delta_F + \Delta_e - \delta_{1F})]}].
\]

From Eq. (3.79), the main signal is given as

\[
E_C(z, t) = \text{Im}[\tilde{A}_{C}^{\text{asympt}}(z, t) e^{i\omega c \tau}],
\]

where

\[
\tilde{A}_{C}^{\text{asympt}}(z, t) = A_0 \Theta(\tau) e^{-\frac{\alpha_{2F} L \delta_{2F}^g}{2(\Delta_F^2 + \delta_{2F}^g)}} - e^{-\frac{\alpha_{1F} L \delta_{1F}^g}{2(\Delta_F^2 + \delta_{1F}^g)}} e^{i\phi_C},
\]

\[
\phi_C = \frac{\Delta_F \alpha_{2F} z \delta_{2F}^g}{2(\Delta_F^2 + \delta_{2F}^g)} + \frac{(\Delta_F + \Delta_e) \alpha_{1F} z \delta_{1F}^g}{2((\Delta_F + \Delta_e)^2 + \delta_{1F}^g)}.
\]

The asymptotic theory of the double-resonance model [Eqs. (6.3)-(6.5)] is compared to the data taken near the 4S_{1/2}(F=1) ↔ 4P_{1/2} transition, as shown in Figure 6.3. The 4S_{1/2}(F=1) ↔ 4P_{1/2} transition has a pair of resonance peaks at \[\omega_{F,F} = (\omega_{11}, \omega_{21})\], as shown in Figure 6.3(b). Each resonance has a different absorption coefficient so that the one peak is weak and the other is strong. The main signal shown in Figure 6.2(b) obeys Beer’s law.

Figure 6.2(a) shows the envelope of the total precursor, given in Eq. (6.4),
Figure 6.2: The envelope of the total precursor and main signal in a double-resonance, $4S_{1/2}(F = 1) \leftrightarrow 4P_{1/2}$ transition predicted by asymptotic theory. (a) Total precursors and (b) main signal for $\Delta = 0$ (solid line), $\Delta = 1.25\delta$ (dots), and $\Delta = 4.79\delta$ (dashed line).
for different carrier frequency detunings $\Delta_1 = \omega_c - \omega_{21}$. The time scale of the modulation on the envelope does not depend on $\Delta_1$. Instead, it is fixed to $\Delta_e \equiv \omega_{21} - \omega_{11} = \omega_{22} - \omega_{12}$, which is the frequency separation between the two resonances arising from the exited-state hyperfine splitting $\Delta_e \sim 12\delta_{21}$. This fixed modulation results in a fast oscillation on the transmitted intensity compared to the relatively slow oscillation of the frequency modulation due to the carrier-frequency detuning. Therefore, the relatively fast oscillation with period $2\pi/\Delta_e$ indicates the double-resonance effect in the experimental data.

Figure 6.3 shows $T(z,t)$ compared to the experimental data. It shows reasonable agreement between the modified asymptotic theory and the experimental data. Note that the first five data sets (1-5) near the strong absorption peak at $\omega_{21}$ do not show the obvious modulation due to $\Delta_e$, while data sets (6-9) taken near the weak absorption peak $\omega_{11}$ show an obvious modulation pattern. This modulation appears as well in the other data set taken near $(\omega_{21}, \omega_{22})$. Therefore, the fixed fast modulation pattern in the data might serve as a reference to test whether a dielectric medium can be treated as a single-resonance or double-resonance system. For example, data set (3) has mild modulation, while data set (7) has strong modulation. This is because the relatively strong absorption peak at $\omega_{21}$ seriously affects the weak one, while weak one barely affects strong one. Therefore, this is an evidence that strong absorption line near $\omega_{21}$ can be treated as a single-resonance model, the primary assumption of Chs.3, 4, and 5. Related to this, as will be discussed in the next section, the other pair is treated as double-resonance model because of the similar fixed-modulation pattern due to $\Delta_e$.

The detuning parameters are defined with respect to the reference resonance absorption peak $\omega_{21}$ for the $F = 1$ transition, $\Delta_1 = \omega_c - \omega_{21}$ and given as in
Figure 6.3: Temporal evolution of transmitted intensity through the cloud of atoms. Solid lines denote experimental data (a) taken at nine different frequencies near $4S_{1/2}(F = 1) \leftrightarrow 4P_{1/2}$ transition indicated on (b). Dots are predicted by asymptotic theory for double-resonance.
Table 6.1. The detuning $\Delta_1$ for each data set is chosen to minimize the error between the experimental data and the theory ($\chi^2$ fit [36]), as shown in the first column of Table 6.2. The theoretically-determined detuning parameters are almost identical to those measured in the experiment (Table 2.2 in Ch.2). I use the same method to chose the parameters for the other transition pair associated with the $F = 2$ ground state.

The asymptotic theory [Eqs. (6.3)-(6.5)] is next compared to the data obtained for the $4S_{1/2}(F=2) \leftrightarrow 4P_{1/2}$ transition. The double-resonance pair $\omega_{F'F} = (\omega_{12}, \omega_{22})$ has balanced absorption coefficients, as shown in Figure 6.5(b). Figure 6.4(a) shows the fast oscillation due to $\Delta_e$ on the envelopes of the total precursors regardless of the different $\Delta_2 = \omega_c - \omega_{22}$. The total transmission intensities predicted by Eq. (6.3)-(6.5) for the nine different carrier frequencies agree reasonably well with the experimental data, as shown in Figure 6.5 (a). The detuning parameter $\Delta_2 = \omega_c - \omega_{12}$ defined from reference resonance absorption peak $\omega_{12}$ for $F = 2$ transition are used in the fit and are given as in the second column of Table 6.1.

So far, I have shown how the postulate works for modified asymptotic theory for double-resonance model. The simple extension of the single-resonance model to the double-resonance model explain the data taken in the well separated double-resonance absorption peaks. The fast modulation in the data is explained by asymptotic double-resonance model.

6.4 SVA theory for a double-Lorentz dielectric

In this section, I present how the postulate works for the (SVA) theory to extend the single-resonance Lorentz model to double-resonance Lorentz model. For the case of
Figure 6.4: The envelope of the total precursor and main signal in a double-resonance, $4S_{1/2}(F = 2) \leftrightarrow 4P_{1/2}$ transition predicted by the asymptotic theory. (a) Total precursors and (b) main signal for $\Delta = 0$ (solid line), $\Delta = 1.25\delta$ (dots), and $\Delta = 4.79\delta$ (dashed line).
Figure 6.5: Temporal evolution of transmitted intensity through the cloud of atoms. Solid lines denote experimental data (a) taken at nine different frequencies near \(4S_{1/2}(F = 2) \rightarrow 4P_{1/2}\) transition indicated on (b). Dots are double-resonance Lorentz model predicted by the asymptotic theory.
Table 6.1: Detuning parameters (the asymptotic theory) for the double resonance pairs, $\omega_{F'F} = (\omega_{12}, \omega_{22})$ and $\omega_{F'F} = (\omega_{11}, \omega_{21})$. The “data” denotes experimentally determined detuning $\Delta$, as shown in Table 2.2.

<table>
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<tr>
<th>point ID</th>
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<th>$\Delta_2(F = 2 \leftrightarrow F')$ [MHz]</th>
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<td>-29.2$^{+7.8}_{-6.5}$</td>
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<td>-49.5$^{+0.1}_{-0.1}$</td>
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<tr>
<td>9</td>
<td>-83.2</td>
<td>-82.8$^{+3.4}_{-10.0}$</td>
</tr>
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</table>

Table 6.2: $\chi^2$ for each detuning parameters (the asymptotic theory) for the double resonance pairs, $\omega_{F'F} = (\omega_{12}, \omega_{22})$ and $\omega_{F'F} = (\omega_{11}, \omega_{21})$.

<table>
<thead>
<tr>
<th>point ID</th>
<th>$\chi^2(F = 1 \leftrightarrow F')$ [$10^{-3}$]</th>
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<td>1.14</td>
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<tr>
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<td>0.69</td>
<td>0.53</td>
</tr>
</tbody>
</table>
a single-resonance, the SVA theory predicts our data reasonably well.

Using the hypothesis, the total precursor fields for double-resonance model are equal to the sum of each precursor field arising from each resonance. The main signal is evaluated directly from the contour integral around $\omega_c$ by considering the absorption coefficients of the double-resonance case. Therefore, each part of the field is given as

$$E_{SB}(z, t) = \text{Im} \left[ A_0 \Theta(\tau) \left\{ e^{\frac{p_{2F}}{i(\Delta_F - \delta_{2F})}} \sum_{n=1}^{\infty} \left( \frac{p_{2F}}{i(\Delta_F - \delta_{2F})} \right)^n (p_{2F} \tau)^{-n/2} J_n(2\sqrt{p_{2F} \tau}) \right\} e^{-\delta_{2F} \tau} \right] - e^{-\delta_{1F} \tau} \sum_{n=1}^{\infty} \left( \frac{p_{1F}}{i(\Delta_F + \Delta_e) - \delta_{1F}} \right)^n (p_{1F} \tau)^{-n/2} J_n(2\sqrt{p_{1F} \tau}) \right] e^{-i\omega_0 \tau} \right],$$

(6.7)

$$E_C(z, t) = -\text{Im} \left[ A_0 \Theta(\tau)e^{\frac{p_{2F}}{i(\Delta_F - \delta_{2F})} + \frac{p_{1F}}{i(\Delta_F + \Delta_e) - \delta_{1F}}} e^{-i\omega_0 \tau} \right],$$

(6.8)

where $p_{F'} F' \equiv \omega_{F'}^2 z/4c = \alpha_{F'} F' \delta_{F'} F'/2$. Note that the transient part of the field, Eq. (6.7), shows a dependence on $p_{F'} F'$, which means that the precursor depends on the absorption depth $\alpha_{F'} F' z$. The envelopes are given as

$$\tilde{A}_{SB}(z, t) = -A_0 \Theta(\tau) \left\{ e^{i(\delta_{F} - \delta_{2F}) \tau} \sum_{n=1}^{\infty} \left( \frac{p_{2F}}{i(\Delta_F - \delta_{2F})} \right)^n (p_{2F} \tau)^{-n/2} J_n(2\sqrt{p_{2F} \tau}) \right\} e^{-\delta_{2F} \tau} \right] - e^{i(\Delta_{F} + \Delta_e) - \delta_{1F} \tau} \sum_{n=1}^{\infty} \left( \frac{p_{1F}}{i(\Delta_F + \Delta_e) - \delta_{1F}} \right)^n (p_{1F} \tau)^{-n/2} J_n(2\sqrt{p_{1F} \tau}) \right],$$

(6.9)

$$\tilde{A}_C(z, t) = A_0 \Theta(\tau)e^{\frac{p_{2F}}{i(\Delta_F - \delta_{2F})} + \frac{p_{1F}}{i(\Delta_F + \Delta_e) - \delta_{1F}}}.$$

(6.10)

The SVA theory [Eq. (6.7)-(6.8)] is compared to the data sets obtained for the
The envelopes of each field [Eq. (6.9)-(6.10)] are shown in Figure 6.6 for different $\Delta_1$. The amplitudes of the each fields in Figure 6.6 are similar to the asymptotic theory case in Figure 6.2. For example, the fixed modulation of the precursor envelope Figure 6.6 (a) is rather mild compared to that of the other pair shown in Figure 6.8a. This implies that the $4S_{1/2}(F=1) \leftrightarrow 4P_{1/2}$ transition is essentially a single-resonance in the SVA theory, as discussed in the previous section of the asymptotic theory. This is supported by the transmitted intensity shown in Figure 6.7. Data set (3) barely shows the fixed modulation due to the separation of the two resonances, while data set (7) shows an obvious modulation patterns. In this regard, the other pair is treated as a double-resonance model because of the similar pattern in the fixed modulation due to $\Delta_e$.

For the pair near $4S_{1/2}(F=2) \leftrightarrow 4P_{1/2}$ transition, the fast modulation in Figure 6.8 (a) appears throughout all the nine data set. This indicates the double-resonance pair at $(\omega_{21}, \omega_{22})$ need to be treated as a double-resonance model, while the relatively strong peak in the other pair can be regarded as a single-resonance.

The detuning parameters used in the SVA analysis are shown in Table 6.3. These parameters are similar to the prediction of the asymptotic theory in the previous section in Table 6.1.

### 6.5 Summary

In the case of well separated resonances, It is found that the transmitted field consists of the Sommerfeld and Brillouin precursor pairs arising from each resonance. The main signal, however, is obtained by the absorption coefficient for the double-resonance. In other words, each absorption peak acts as two independent single
Figure 6.6: The envelope of the total precursor and main signal in a double-resonance, $4S_{1/2}(F = 1) \leftrightarrow 4P_{1/2}$ transition predicted by SVA theory. (a) Total precursors and (b) main signal for $\Delta = 0$ (solid line), $\Delta = 1.25\delta$ (dots), and $\Delta = 4.79\delta$ (dashed line).
Figure 6.7: Temporal evolution of transmitted intensity through the cloud of atoms. Solid lines denote experimental data (a) taken at nine different frequencies near $4S_{1/2}(F = 1) \leftrightarrow 4P_{1/2}$ transition indicated on (b). Dots are predicted by SVA theory for double-resonance.
Figure 6.8: The envelope of the total precursor and main signal in a double-resonance, $4S_{1/2}(F = 2) \leftrightarrow 4P_{1/2}$ transition predicted by the SVA theory. (a) Total precursors and (b) main signal for $\Delta = 0$ (solid line), $\Delta = 1.25\delta$ (dots), and $\Delta = 4.79\delta$ (dashed line).
Figure 6.9: Temporal evolution of transmitted intensity through the cloud of atoms. Solid lines denote experimental data (a) taken at nine different frequencies near $4S_{1/2}(F = 2) \leftrightarrow 4P_{1/2}$ transition indicated on (b). Dots are predicted by SVA theory for double-resonance.
Table 6.3: Detuning parameters (the SVA theory) for the double resonance pairs, \( \omega_{F'F} = (\omega_{12}, \omega_{22}) \) and \( \omega_{F'F} = (\omega_{11}, \omega_{21}) \). The “data” denotes experimentally determined detuning \( \Delta \), as shown in Table 2.2.

<table>
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<tr>
<th>point ID</th>
<th>( \Delta_1(F = 1 \leftrightarrow F') ) [MHz]</th>
<th>( \Delta_2(F = 2 \leftrightarrow F') ) [MHz]</th>
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<td>fit (( \chi^2 ))</td>
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<tr>
<td>9</td>
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<td>-82.8(^{+3.4}_{-10.0})</td>
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</tbody>
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resonances for the precursors, but acts as a combined resonance for the main signal. Furthermore, using the theory, the experimental data is categorized by either the single- or double-resonance models by measuring the size of the fixed-frequency fast modulation due to \( \Delta_e/2\pi \).
Table 6.4: $\chi^2$ for each detuning parameters (the SVA theory) for the double resonance pairs, $\omega_{F'F} = (\omega_{12}, \omega_{22})$ and $\omega_{F'F} = (\omega_{11}, \omega_{21})$.

<table>
<thead>
<tr>
<th>point ID</th>
<th>$\chi^2(F = 1 \leftrightarrow F')$ [$10^{-3}$]</th>
<th>$\chi^2(F = 2 \leftrightarrow F')$ [$10^{-3}$]</th>
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Chapter 7

Conclusion

I have discussed the first direct observation of optical precursor, its experimental techniques, and theoretical analysis. The experiments and the theoretical analysis provide many insights into the optical precursor phenomena arising in electromagnetic pulse propagation. Yet, it is only a beginning. In this chapter, I will describe the important achievements, current status of precursor research and the future direction of the precursor research.

7.1 The important achievements

In the beginning of this conclusion chapter, I would like to put an emphasis on the notable achievements of this research. In the experimental aspect, it is the first direct observation of the optical precursor, which is realized by extending the precursor time scale up to several tens of nanoseconds. This longer time scale is achieved using a narrow resonance of a cold potassium atomic gas. Attention to the relation between the time scale and the resonance linewidth is the most important idea to realize the direct observation.

In the theoretical aspect, my research provides the first experimental test of the modern asymptotic theory developed by Oughstun and Sherman [14]. When the modern asymptotic theory is applied to my experimental data, a noble nature of
the optical precursors is found. The Sommerfeld and Brillouin precursor arrive right
after the front and sit on top of the main signal so that the initial transmission of
the pulse is $100\%$. The frequencies of each precursor chirp rapidly and approach
the medium resonance frequency.

Some errors that appeared in the modern asymptotic theory for my experimental
conditions are modified empirically based on a comparison of the theory to the data,
which shows $100\%$ transmission right after the front. To further test these modifi-
cations, the asymptotic theory is compared to the SVA theory. By this comparison,
it is confirmed that the SVA theory predicts optical precursors as the transient part
of the transmitted field, as discussed by several researchers [23, 24]. This is the first
direct comparison between the asymptotic theory and the SVA theory ever under-
taken.

Another theoretical achievement is the simple extension of the single-resonance
Lorentz model to the double-resonance case, which appears to be valid for my experi-
mental conditions. This is based on the case of weakly-dispersive, narrow-resonance
medium. Such a medium acts as two independent single resonances for the pre-
cursor field while it acts as a regular double-resonance system for the main signal.
Therefore, the optical precursor is the linear superposition of the precursors origi-
nating from each line. The postulate is confirmed by comparing both theories to my
experimental data.
7.2 The current status and the future direction of the optical precursor research

The precursor research presented in this thesis have been performed based on the knowledge obtained from previous precursor research. As introduced in the Ch. 1, despite a long history of precursor theory [12, 14, 24], there are a few experimental demonstrations [18–21, 37].

The lack of the experimental research is partly because of the conventional bias about the nature of the precursors. Since Sommerfeld and Brillouin studied precursor, it has been usually believed that optical precursors are extremely small and persists only a few femto-seconds. Although the followup experiments demonstrated detectable precursor amplitudes, an indirect measurement technique was still required to detect ultra-fast phenomena because they used broad resonance linewidth, the conventional parameter used in Brillouin’s analysis. By using a cold atomic gas, however, I use parameters which have never been tested in optical-precursor research, thereby easily detecting optical precursors. I hope this research provides useful knowledge to perform optical precursor research in a more accessible way. One can detect optical precursors through any medium in which a mechanism of the narrow linewidth is realizable.

Performing studies in the parameter range of my experiment not only provides the possibility of easy detection, but also gives a deeper conceptual understanding of precursor. I have compared two independently developed theories and suggested the empirical modification on the asymptotic theory. I hope that this research will give some guidance for the development of a general theory of precursors, including the weakly dispersive narrow-resonance regime. Finally, I believe this research provides
a hint to solve several controversies surrounding the observability of the precursor [38–40] and opens up a new era of the precursor research.
Appendix A

Potassium Magneto-Optical Trap

It is important to achieve narrow atomic resonance to enhance the precursor time scale up to tens of nano-seconds, as discussed in Ch. 2. To achieve a narrow resonance, I have cooled and trapped a neutral atomic gas, which is used as a Lorentz medium in my experiments. In this Appendix, I discuss the operation of a magneto-optical trap (MOT) for potassium atoms ($^{39}$K).

A.1 Fundamentals of magneto-optical trap

The mechanisms needed to realize a MOT are to slow down the atomic motion and trap the atoms in a confined region of space using a restoring force [41, 42]. As shown schematically in Figure A.1 (a), I use six counter-propagating circularly-polarized laser beams, which are red detuned from the $D_2$ transition (the $4S_{1/2} \leftrightarrow 4P_{3/2}$ transition). The six beams intersect in a vacuum chamber containing a dilute gas of potassium ($^{39}$K) atoms in the presence of an inhomogeneous magnetic field generated by an anti-Helmholtz current-carrying coil. Figure A.1(b) shows the cooling and trapping forces in one-dimension.

The cooling force is a radiation pressure force. When an atom moves against the direction of the laser beam, the red detuned laser beam is tuned to the atomic resonance when it is travelling from left-to-right due to the Doppler shift. Hence it
Figure A.1: Experimental setup of magneto-optical trap. (a) Six counter-propagating beam and the magnetic field produced by anti-Helmholtz magnet coils; (b) diagram explaining the restoring force responsible for creating MOT; and (c) picture of $^{39}$K MOT fluorescence taken by CCD.
is more readily excited to the higher energy level by absorption, which subsequently
decays to a lower energy state via spontaneous emission. Because the spontaneous
emission process is isotropic, the atomic momentum is dissipated. This process
transfers the momentum from the laser beam to the atoms in the propagation direc-
tion of the beam via atomic recoil. On average, atoms lose their momentum in the
direction of their movement in the viscous medium by radiation pressure from the
laser. The force on the atoms by radiation pressure can be described as a viscous
damping force \( F_{\text{viscous}} = -\alpha_{\text{opt}} \dot{z} \), where \( \alpha_{\text{opt}} \equiv \hbar k^2 / 2 \) is the optimum damping coef-
ficient, \( k \) is the optical wave vector, and \( \dot{z} \) is the atomic velocity in the \( z \)-direction.
This is a so-called “optical molasses.”

While the radiation pressure force (optical molasses) provides cooling, it does
not confine the atoms. The cold atoms need to be trapped within a restricted region
of space using a spatially-dependent force, such as the restoring force of a spring. In
Figure A.1(b), the inhomogeneous magnetic field along the \( z \)-axis causes a Zeeman
shift of the atomic sub-levels. Due to the splitting, atoms located to the left of the
origin are closer to resonance with the red-detuned right-circularly-polarized light
(\( \sigma^+ \)) and hence experience a radiation pressure force that pushes the atom back
to the center of the trap. On the other hand, atoms located to the right interact
with the red-detuned left-circularly-polarized light (\( \sigma^- \)) and hence are also pushed
back to the center of the trap. Overall, atoms located away from the origin are
pushed back towards the center by a force of magnitude \( F_{\text{restore}} = -\kappa_{\text{opt}} z \), where
\( \kappa_{\text{opt}} \equiv \mu' B \partial_z B / \hbar k \) is the optimum spring constant and \( \mu'(g_e m_e - g_g m_g) \mu_B \) is the
effective magnetic moment.\(^1\) Finally, the net force due to the cooling and trapping

\(^1\)The Landé \( g \)-factor is denoted by \( g_{e,g} \), where \( e \) and \( g \) denotes the excited and ground state,
respectively, \( m_{e,g} \) is the magnetic quantum number, and \( \mu_B \) is the Bohr magneton [42].
Figure A.2: The viscous damping force $F_{\text{viscous}}$ (the cooling force or the radiation pressure force) as a function of the velocity $v_z = \dot{z}$. The two dotted lines denote the force from each laser beam. The solid line is the sum of the two forces, and the approximate pure damping force over a small velocity range is indicated as dashed straight line. (The graph in Figure A.2 from Ref. [42].)
Figure A.3: Energy level diagram of the D\textsubscript{2} transition in 39K. (a) The hyperfine energy levels $F^\prime F$, and (b) their Zeeman sub-levels $m_F m_{F^\prime}$.

effects on the atom is then given by

$$F = m_F \ddot{z} = -\kappa_{opt} z - \alpha_{opt} \dot{z}.$$  \hspace{1cm} (A.1)

See Ref. [41,42] for more details of Eq. (A.1).

The atomic sub-levels associated with the restoring (confinement) forces are chosen as the Zemman energy sublevels in the $4S_{1/2}(F = 2) \leftrightarrow 4P_{3/2}(F^\prime = 3)$ transition (D\textsubscript{2}), as shown in Figure A.3 (b), due to the presence of the $4S_{1/2}(F = 2, m = 2) \leftrightarrow 4P_{3/2}(F^\prime = 3, m^\prime = 3)$ cycling transition [42]. The trapping beams (frequency $\omega_{\text{trap}}$)
are circularly polarized and red-detuned from the energy levels. The repumping beams are required to optically pump the atoms into the $4S_{1/2}(F = 2)$ state, so that the atoms continuously interact with the trapping field.

### A.2 Experimental apparatus

To build the potassium magneto-optical trap, the first step is to prepare the potassium vapor contained in a glass cell. The next step is to apply the MOT force [Eq. (A.1)] by sending the six way MOT laser beams through the vapor cell and running a current in the pair of coils (anti-Helmholtz magnet) located in the vertical-axis. In this section, I will present the experimental procedures for the $^{39}$K MOT.

#### A.2.1 $^{39}$K vapor cell operation

To build $^{39}$K vapor cell, I prepared an ultra-high vacuum chamber consisting of a glass cell, metal parts, a glass-ampule containing 1g of solid-state potassium (Aldrich Chemical Co., 24485-6), and vacuum pumps, as shown in Figure A.4. The glass (pyrex) cell with 8 windows needs to be cleaned using a mixture of methanol and water to remove potassium residue inside. The cleaning begins by flowing a small amount of the methanol slowly to avoid explosion; the ratio of the methanol to water decreases gradually. Once most of the potassium residue is removed, the glass cell is finally cleaned only using methanol. Next, the potassium ampule needs to be cleaned with immersing it in a combination of a nitric acid (30∼40%) and water (60∼70%) for about 3 hours. The ampule is then cleaned with detergent (Alconox) and put in the ultrasonic tub filled with methanol for 20 min. The clean vacuum parts are assembled including the potassium ampule. Once the chamber is tightly sealed, the
Figure A.4: Potassium vapor cell attached to vacuum systems. (a) Diagram showing the apparatus of the vapor cell. Dashed box indicates the cold finger. (Top view except for the Dashed box.) (b) Photo of the vapor cell and associated optics.
rough pump operates to pump out the air, resulting in the pressure of \(1 \times 10^{-4}\) torr. Immediately after turning on the rough pump, the turbo pump (Varian, Turbo-V70LP) is turned on to prevent the back stream of oil into the chamber. After \(\sim\)2 hours, once the turbo pump lower the pressure below \(1 \times 10^{-7}\) torr, heat is applied to the entire chamber evenly using heat tapes. The heat tapes are covered by a insulator (glass-fiber) with aluminum foil. The temperature of the heat tape is adjusted gradually using a “variable auto transformer” (VARIAC). Underneath the heat tapes, the temperature of the chamber is measured by thermocouples (type K). When the heat is applied, the pressure increases. Once the pressure reaches \(1 \times 10^{-6}\) torr again, the heat tapes are turned off. The pressure of the chamber then decreases and reaches \(1 \times 10^{-8}\) torr. At this pressure, the potassium ampule is ready to be broken. Before breaking the potassium ampule contained in the metal tube, I close the small stainless steel valve (Nupro, 6LV-DAVR4-P) above the thin metal tube containing the ampule. I also partially close the stainless steel right-angle valve (Varian) between the rest of the chamber and the turbo pump to avoid sudden change in the pressure of the entire chamber. The ampule contained in the metal tube then be broken by clamping the tube from outside and the breaking is confirmed by the crash noise. After breaking the ampule, the valve attached to the tube is slowly opened and the pressure goes up to \(1 \times 10^{-5}\) torr. Once the pressure goes down below \(1 \times 10^{-7}\) torr, the ion pump (Duniway, “Varian style” 8 L/S) is turned on to pump out gas inside the chamber to reach \(1 \times 10^{-9}\) torr. At this pressure, the mean free path of the gas is greater than the size of the entire ultra-high vacuum chamber.

\(^2\)It is made of stainless steel (SS) cylindrical tube (outer diameter is 1/2 inch and the thickness is 0.020 inch) and a blank flange(1 1/3 inch) with a hole fit to the outer diameter of the SS tube. One end of the tube is squeezed and welded, and the other end is attached to the flange and welded. I asked one of the technician in the FEL for drilling the hole in the blank flange and to clean the welding.
Once the vacuum chamber with the broken ampule is prepared at the pressure of \(1 \times 10^{-9}\) torr, the solid-state potassium is ready to sublime and migrate into the glass cell by increasing the pressure gradually. These procedures are achieved by heating the metal tube and raising the temperature of the potassium gas.\(^3\) The region containing the crushed ampule is the so-called “cold finger” because it contains the solid-state potassium while the rest of the chamber, including the glass cell, is filled with potassium vapor once the migration is completed. At the beginning of raising the temperature of the cold finger (metal tube), the atoms in the surface of the potassium chunk easily react with possible impurities nearby and form a skin. It is important to tap the metal tube to avoid forming the skin because the skin prevents sublimation of the potassium. Once the potassium sublimes as the temperature goes up, the skin is not formed anymore.

The migration of the potassium gas to the glass cell can be checked by the fluorescence or absorption of light when I pass a laser beam through the chamber. The frequency of the laser is scanned near the potassium \(D_1\) transition. (Scanning near the potassium \(D_2\) transition is also fine if the laser frequency is adjusted to that transition.) After heating for 3~4 hours, the temperature of the cold finger is about 170°C and the rest of the cell is about 130°C. As an evidence of the migration, bright fluorescence is seen from the glass cell using a CCD camera. It is also seen that the 5% absorption of the vapor (\(\sim 800\) MHz linewidth) by monitoring the transmission of the weak beam through the cell. Once bright fluorescence is seen, the heating is turned off. The ultra-high vacuum chamber is then cooled down to the room temperature. For normal operation, the number of atoms in the vapor cell is kept nearly constant at the room temperature for several months. Whenever the number

\(^3\)The details of the potassium characteristics are described in William Brown’s thesis, section 4.2.1 [43].

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density of the vapor is dropped after several months, I heat the metal parts for 30 min and the number density increases again.

A.2.2 $^{39}\text{K MOT operation}$

To achieve the $^{39}\text{K MOT}$ from the vapor cell, the six-way MOT laser beams need to be prepared. The preparation of the six-way trapping and repumping beams begins with the Coherent 899-21 titanium-doped sapphire ring laser (Ti:Sapp), as shown in Figure A.5. The laser beam is sent to a glass plate and about 4% of the total power is split off and sent to saturated absorber, which provides a reference frequency. Most of the remaining power ($\sim 300 \text{ mW}$), is passed through an adjustable attenuator and separated into two orthogonally linear polarized beams by the combination of a half-wave ($\lambda/2$) plate and polarizing beam-splitter (PBS). Each beam is sent to an acousto-optic (AO) modulator/deflector (IntraAction, ATM-2301A2).

The acousto-optic modulators have two functions. One is to shift the frequency of the incoming light. Upon passing through the AO, one (the other) of the beams achieves a lower (higher) frequency, and becomes the trapping (repumping) beams as in Figure A.3. Each frequency is up- or down-shifted by $\sim 230 \text{ MHz}$ with respect to the original frequency of the Ti:Sapp laser, which is equal to the half of the ground state splitting (462 MHz). To optimize the trap in terms of atomic number density, the trapping laser beam is actually tuned close to the $4P_{3/2}$ manifolds, as shown in Figure A.3 (a). That is, the trapping beam is up-shifted by 226 MHz instead of 230 MHz. The condition of the optimization is chosen when the frequency-dependent steady-state transmission indicates maximum absorption as I vary the frequency shifting of the AO. The other function of the AO is to modulate power driven by a pulse generator (Stanford Research Systems model DS345). Using the power
modulation, the laser beams are turned on and off to have trap-on-off interval as discussed in the Ch. 2.

After passing through the each AO, the trapping (repumping) beams are enlarged by telescope and are approximately Gaussian shaped with a $1/e^2$ intensity diameter of 1 cm. This MOT beam size determines the region over which atoms can be captured, which is much larger than the size of the eventual cloud of cooled and trapped atoms, which is $\sim$1-2 mm. In the capture region, the atoms in the dilute room-temperature vapor experiences optical molasses and the restoring force. After the telescope, each beam split into orthogonal linear polarized beams using a half-wave plate and PBS 2, as shown in Figure A.5. These beams are sent to the two orthogonal radial direction denoted as 1 or 2, and the vertical direction denoted as 3. The trapping beams are combined with the repumping beam at 50/50 beam splitter, and sent to the radial plane. There is only a trapping beam in the direction 3. Finally, the trapping (repumping) beams are sent to three orthogonal directions (the two orthogonal radial directions) and the power of each beam is adjusted by half-wave plates to equally balance the number of atoms in two ground states.

In the inset of the Figure A.5, the mirrors and the quater-wave plates for the six-way MOT laser beams and the coils for the inhomogeneous magnet are shown. The six-way beams are linearly polarized before the vapor cell and are converted into circularly-polarized beams via quarter-wave ($\lambda/4$) plates. The circularly-polarized beams are again linearly polarized when they pass through the other quarter-wave ($\lambda/4$) plates in the opposite side of the vapor cell. After being reflected by a mirror and passing again through the quarter-wave ($\lambda/4$) plates, the counter propagating light is circularly-polarized orthogonal to the incoming circularly-polarized beam. The total power of the trapping (repumping) beams is $\sim$20 mW ($\sim$13 mW) for
Figure A.5: Optical layout for MOT

$\alpha_0 L = 1.03$ case. For $\alpha_0 L = 0.42$ case, the total power of the trapping (repumping) beams is $\sim 4$ mW ($\sim 3$ mW).

The next stage is to prepare the inhomogeneous magnetic field, which gives rise to Zeeman splitting in the magnetic sublevels, as shown in Figure A.1 (b). The anti-Helmholtz magnet is located in the vertical-axis, 3. The anti-Helmholtz magnet consists of two coils\(^4\) separated apart from each other by 20 cm. The two coils are identical and have 210\~220 turns with an average diameter of 11 cm (inner diameter is 7.5 cm and outer diameter is 14.5 cm). The current run in the coil is 10 A. This

\(^4\)I am indebted to Prof. John Thomas’ group for lending me the coils.
A.3 Characterization of $^{39}$K MOT

A.3.1 Number Density measured by Absorption

The atomic number density in the potassium ($^{39}$K) MOT is an important parameter because it determines the absorption depth $\alpha_0 L$. One method to measure the number density $N$ is to measure the absorption $A$ of weak laser beam passing through the MOT. Because the probe laser beam is weak, most of the atoms are not excited and stay in the ground states. Only a small portion of the input beam intensity is absorbed, and I detect the transmission $T = I_{\text{out}}/I_{\text{in}}$ of the reduced-output beam intensity. The transmission is also proportional to the path length of the medium $L$.

Figure A.6 (a) shows the measured transmission of the probe beam as a function of the frequency. The four dips in Figure A.6 (a) correspond to the allowed four possible transitions in the $D_1$ ($4S_{1/2} \leftrightarrow 4P_{1/2}$) manifold, as shown in Figure A.6 (b). The absorption $A$ is directly related to the transmission through Beer’s Law: $A = 1 - T = 1 - \exp[-\alpha(\Delta)L]$, where $\alpha(\Delta)$ is the absorption coefficient. Here, the macroscopic absorption coefficient is related to a microscopic total absorption cross section $\sigma(\Delta)$ as $\alpha(\Delta) = N\sigma(\Delta)$. The total absorption cross section $\sigma(\Delta)$ can
Figure A.6: Transmission of weak probe for measuring number density
Table A.1: Probability of population being in $F$ state in the ideal case

<table>
<thead>
<tr>
<th>Case</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Both beams on (thermal equilibrium case)</td>
<td>1/8</td>
<td>1/8</td>
</tr>
<tr>
<td>Trapping beam on ($F = 1$ is optically pumped)</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>Repumping beam on ($F = 2$ is optically pumped)</td>
<td>0</td>
<td>1/5</td>
</tr>
</tbody>
</table>

be calculated using the dipole matrix element $|\vec{\mu}_{F'm'Fm}|$, which is the expectation value of the dipole moment operator $\vec{\mu} = -e\vec{r}$ between the Zeeman sub-levels, and $F'm'Fm$ denotes the $4S_{1/2}|F, m\rangle \leftrightarrow 4P_{1/2}|F', m'\rangle$ transition. The relation between the $\sigma(\Delta)$ and $|\vec{\mu}_{F'm'Fm}|$ is given as

$$
\sigma(\Delta) = \sum_{F'F} \sigma_{F'F}(\Delta),
$$

$$
\sigma_{F'F}(\Delta) = \sum_{m'm} \sigma_{F'm'Fm}(\Delta),
$$

$$
= \sum_{m'm} \rho_{Fm} \frac{4\pi k \Gamma \delta_{F'm'Fm} |\vec{\mu}_{F'm'Fm}|^2}{\hbar \Delta_{F'm'Fm}^2 + \delta_{F'm'Fm}^2},
$$

where $\rho_{Fm}$ is probability for an atom to be in $|F, m\rangle$ state, $^5\Gamma = 1/2\tau_{sp}$ is natural linewidth, and $\tau_{sp} = 25.8$ ns [44] is the spontaneous decay time, the index $m'$ denotes the magnetic sub-levels of excited state and $m$ represents that of the ground state associated with the transition. For the $D_1$ transition, there are $\sum_{F=1}^2 (2F + 1) = 8$ Zeeman sub-levels in the ground state, and $\sum_{F'=1}^2 (2F' + 1) = 8$ in the excited state. By assuming negligible Zeeman splitting and degenerate hyperfine levels $|F'F\rangle$, there are only 4 possible transitions, as shown in Figure A.6 (b), thereby simplifying the

$^5$For example, in thermal equilibrium, $\rho_{Fm} = 1/\sum_{F}(2F + 1)=1/8$. See also Table A.1.
expression for the cross section as

\[ \sigma_{F'F}(\Delta) = \rho_F \frac{4\pi k}{\hbar} \frac{\delta_{F'F} \sum_{m'm} |\tilde{\mu}_{F'm'Fm}|^2}{\Delta_{F'F}^2 + \delta_{F'F}^2}. \]  
(A.2)

To evaluate Eq. (A.2), the linewidth \( \delta_{F'F} \), detuning \( \Delta_{F'F} \), and \( \rho_F \) are determined by the experiment.

The dipole matrix element \( |\tilde{\mu}_{F'm'Fm}| \) is determined theoretically. The expectation value of the dipole moment is evaluated in the angular momentum states \( |(JI)Fm\rangle \) as

\[ |\tilde{\mu}_{F'm'Fm}| = \langle (J'I')F'm'|\mu^{l=1}_{q}\rangle |(JI)Fm\rangle, \]  
(A.3)

where \( J' = 1/2 \) (\( J = 1/2 \)) is the total angular momentum for the excited (ground) state, \( I = 3/2 \) is the nuclear spin, \( l = 1 \) is the angular momentum of light and \( q = 0 \) is its projection on the quantum axis for the case of the linearly polarized probe beam (\( \pi \) transition) at the D\(_1\) transition. The dipole matrix elements appearing in Eq. (A.3) are obtained by angular-momentum theory: \( F = I \oplus J \) via 3-j \([\left]\) or 6-j \([\left\{\right]\) symbols [45, 46]. As a result,

\[ \sum_{m'm} |\tilde{\mu}_{F'm'Fm}|^2 = \frac{3\hbar \hat{\lambda}^3 \Gamma}{16\pi^3} \times \sum_{m'm} f_{F'm'Fm} \]  
(A.4)

\[ = \frac{3\hbar \hat{\lambda}^3 \Gamma}{16\pi^3} \times f_{F'F} = \frac{3\hbar \hat{\lambda}^3 \Gamma}{16\pi^3} \times f_{F'F}, \]  
(A.5)
where

\[
f_{F'm'Fm} \equiv 2\left| (-1)^{F'-m'} \begin{pmatrix} F' & 1 & F \\ -m' & q & m \end{pmatrix} \right| (-1)^{J'+I+F+1}
\times \sqrt{(2F'+1)(2F+1)} \left\{ \begin{pmatrix} J' & I & F' \\ F & 1 & J \end{pmatrix} \right\}^2.
\]

Therefore, Eq. (A.2) is given as

\[
\sigma_{F'F}(\Delta) = \rho_F \sigma_0 f_{F'F} \frac{\Gamma \delta_{F'F}}{\Delta_{F'F}^2 + \delta_{F'F}^2}, \tag{A.6}
\]

where \( \sigma_0 = 3\lambda^2/2\pi \). Finally, the total absorption cross section is given as

\[
\sigma(\Delta) = \sum_{F'F} \rho_F \sigma_0 f_{F'F} \frac{\Gamma \delta_{F'F}}{\Delta_{F'F}^2 + \delta_{F'F}^2}. \tag{A.7}
\]

Once the relation between the absorption coefficient \( \alpha(\Delta) \) and the cross section \( \sigma(\Delta) \) is obtained, the number density is determined by

\[
N = \alpha_{F'F}/\sigma_{F'F}. \tag{A.8}
\]

From Eq. (A.7), for example, \( \sigma_{21} \equiv \sigma(\Delta_{21} = 0) = \rho_1 \sigma_0 f_{21} \Gamma/\delta_{21} \) is obtained for the \( 4S_{1/2}(F = 1) \leftrightarrow 4P_{1/2}(F' = 2) \) transition. Here \( \sigma_0, \ f_{21}, \) and \( \Gamma \) are known, and \( \delta_4 \) is determined by the data. Therefore, \( \sigma_{21} \) is known when the probability for an atom to be in \( F = 1, \rho_{F=1} = \rho_1 \) is determined.

Even when I attempt to obtain an equal balance of the ground state populations, \( \rho_F = 1/8 \ (F = 1, 2) \), there are always slight imbalances in the real situation. One of the ground states, for example, can have slightly more population than the others.
because of small differences in the intensities between the trapping and repumping beam. In my case, $\rho_1$ is larger than $\rho_2$ because the intensity of the trapping beams are slightly stronger than the intensity of the repumping beams. For the near-to-thermal-equilibrium case, I find $\rho_1 = 0.172$ and $\rho_2 = 0.097$ by fitting the absorption profile, as shown in Figure A.6 (a) with Eq. (A.7). In this case, with $\sigma_{21} = 2.60 \times 10^{-10}\text{cm}^2$ and $\alpha_{21} = 3.12\text{cm}^{-1}$ for the case of Figure A.6, I can estimate the number density as $N = \alpha_{21}/\sigma_{21} = 1.20 \times 10^{10}\text{cm}^{-3}$.

### A.3.2 Temperature: Release and Recapture Method

One method to measure the temperature of the cooled and trapped atoms is the release and recapture method, which takes advantage of the fact that the gas is characterized by a Maxwell-Boltzmann distribution for the atomic speed [28]. By turning off and on the MOT laser beam, the trapped atoms are released and recaptured. When the atoms are released, they expand ballistically. When the MOT laser beams are turned on again, the fast atoms have escaped from the capture region, which is dictated by the size of the trap laser beams; the slow ones remain within this region and are recaptured. For a given off-time interval, the number of remaining atoms depends on their velocity, which depends on the MOT temperature $T_{\text{MOT}}$. The temperature is related to the velocity distribution function $f(v)$ as

$$f(v) = \frac{4v^2}{\sqrt{\pi}\hat{v}^3}e^{-v^2/\hat{v}^2},$$

(A.9)

where $\hat{v} = \sqrt{2k_B T_{\text{MOT}}/M_K}$ is the most probable speed in the Maxwell-Boltzmann distribution, $k_B$ is the Boltzmann’s constant, and $M_K$ is the mass of potassium. The Maxwell-Boltzmann speed distribution [47] can be mapped onto the radial distance.
from the center of the trap $r$ and time $t$ as

$$f(v) = f(r; t) = \frac{4r^2}{\sqrt{\pi} \tilde{\nu}^3 t^2} e^{-r^2/\tilde{\nu}^2 t^2},$$

(A.10)

where $v = r/t$. By integrating Eq. (A.10) over the capture region (radius $R$) [27], the number of atoms remained in the trap is evaluated as a function of the off-time $t_{off}$,

$$U(t_{off}) = \int_0^R U_0 f(r; t_{off}) dr = \frac{4U_0}{\sqrt{\pi} \tilde{\nu}^3 t_{off}^2} \int_0^R r^2 e^{-r^2/\tilde{\nu}^2 t_{off}^2} dr,$$

(A.11)

where $U_0$ is initial number of trapped atoms. If I rewrite Eq. (A.11)

$$U(t_{off})/U_0 = \text{erf} Q - \frac{2Q}{\sqrt{\pi}} e^{-Q^2},$$

(A.12)

where \( Q = R\sqrt{M_K/2k_BT_{MOT}/t_{off}} \).

The ratio of the remaining number $U(t_{off})/U_0$ in Eq. (A.12) is obtained (dots in Figure A.7 (b)) by measuring the fluorescence of MOT (Figure A.7 (a)). I have varied the off-time from 0.5 ms to 10 ms. The data of $U(t_{off})/U_0$ is fit to Eq. (A.12) to find the temperature of the MOT, \( \sim400\mu K \), as shown in Figure A.7 (b). For \( \sim400\mu K \), the most probable speed $\tilde{\nu} \sim 41$ [cm/sec].
**Figure A.7**: The measurement of the MOT temperature. (a) The ratio of remaining number of atoms. (b) The fitting of the experimental data (dots) to Eq. (A.12) (Solid lines).
Bibliography


Biography

Heejeong Jeong was born in Seoul, Korea on April 13, 1973. In 1992, she graduated from Seoul Sejong High School. She entered Soongsil University in Seoul, Korea, in 1993. There she studied physics and received her B.S. with top honor in 1997. Immediately after graduation, she entered the graduate school of physics at Seoul National University where she majored in high energy physics. She completed her M.S degree in 1999. The following year, she moved to Durham (NC), United States to begin the graduate physics program at Duke University. There, under the supervision of Dr. Daniel J. Gauthier, she received her M.A. degree in 2003 and her Ph.D. degree in 2006 for her study of the optical precursors using cold potassium atoms.

Publications


Presentations
