Experimental investigation of high-quality synchronization of coupled oscillators

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We describe two experiments in which we investigate the synchronization of coupled periodic oscillators. Each experimental system consists of two identical coupled electronic periodic oscillators that display bursts of desynchronization events similar to those observed previously in coupled chaotic systems. We measure the degree of synchronization as a function of coupling strength. In the first experiment, high-quality synchronization is achieved for all coupling strengths above a critical value. In the second experiment, no high-quality synchronization is observed. We compare our results to the predictions of the several proposed criteria for synchronization. We find that none of the criteria accurately predict the range of coupling strengths over which high-quality synchronization is observed. © 2000 American Institute of Physics. [S1054-1500(00)01203-9]

The once surprising fact that the irregular oscillations of two chaotic oscillators can be synchronized is now well established. However, deterministic chaos is not the only source of irregular oscillations. Recent research shows that stable periodic systems may oscillate irregularly when subject to small random noise [F. Ali and M. Menzinger, Chaos 9, 348 (1999); Trefethen et al., Science 261, 578 (1993)]. Can a high degree of synchronization be achieved between two systems undergoing irregular oscillations due to small noise rather than deterministic chaos? We address this question here through two experiments on coupled periodic electronic circuits. In each experiment, we observe the degree of synchronization between a pair of coupled oscillators as the coupling strength is increased from zero. In the first experiment, we observe a sudden transition to high-quality synchronization at a critical coupling strength. In the second experiment, no high-quality synchronization is observed. We apply to our experimental systems several proposed criteria for high-quality synchronization developed in studies of synchronized chaos. We find that none of these criteria accurately predict the behavior observed in the experiments. These results may provide some guidance in the development of practical applications of synchronized chaos such as secure communication schemes where a criterion for high-quality synchronization is needed.

I. INTRODUCTION

It is now well established that the dynamics of a nonlinear system can become highly irregular when small random noise is injected into the system. For example, Ali and Menzinger recently showed that a globally stable limit cycle oscillator subject to small amounts of noise can display an explosive divergence of trajectories away from the limit cycle. The origin of this disproportionate response to small perturbations is the fact that the limit cycle is composed of segments of varying local stability, most of which are stable but some of which are highly unstable as shown schematically in Fig. 1. In regions of pronounced local instability, a perturbation may undergo transient growth before decaying asymptotically. A similar behavior is displayed by non-normal linear systems as shown schematically in Fig. 2(a). Non-normal systems are characterized by nonorthogonal eigenvectors. A small perturbation to such a system expressed as a linear superposition of such vectors may have large coefficients but a small norm due to cancellation as depicted in Fig. 2(b). As shown in Fig. 2(c), when the system evolves in time, the coefficients of the superposition may decay at different exponential rates so the cancellation is lost, causing the norm to increase even though the individual eigenvector components are decaying asymptotically. Again, the result is a transient amplification of a perturbation by a stable dynamical system. These two examples do not exhaust the possible scenarios in which highly irregular oscillations are generated by large noise amplification in a dynamical system. An interesting question is whether the irregular oscillations occurring in two identical noise-amplifying dynamical systems can be synchronized.

The primary objective of this paper is to present the results of an experimental investigation of synchronization of noise-amplifying dynamical systems consisting of nonlinear electronic periodic oscillators. In our experiments, the dynamical behavior of the coupled oscillators is described by a set of nonlinear differential equations given by
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A couple of periodic oscillators, we couple two periodic oscillators. We find that these oscillators display high-quality synchronization. In the second experiment, the oscillators evolve on a limit cycle along which the Jacobian is non-normal at all points. The non-normality gives rise to transient amplification of perturbations as described previously. When two such oscillators are coupled and the coupling strength is increased from zero, we do not observe high-quality synchronization at any coupling strength in the range of experimentally accessible coupling strengths.

A second objective of this paper is to explore the possible implications of our experiments with noise-amplifying systems on recent experimental studies of synchronized chaotic oscillators. Several researchers have reported experiments where, in a regime where synchronization was expected, intervals of synchronization are interrupted irregularly by large, brief desynchronization events (or bursts). This bursting is attributed to the interaction of very small noise or parameter mismatch between oscillators and local variations in the stability of the synchronized state. Specifically, the synchronized state may be locally unstable near unstable periodic orbits embedded in the chaotic attractor on which the oscillators evolve. A perturbation to the synchronized motion in the neighborhood of such an orbit may be amplified before decaying asymptotically. With this scenario in mind, several researchers have attempted to develop criteria for “high-quality,” “robust,” or “burst-free” synchronization. Since none of these criteria explicitly require the oscillators to be chaotic for applicability, we apply them to our experimental systems and compare their predictions with our observations. We find that none of these criteria accurately predicts the behavior observed in our experiments.

In the next two sections, we describe our experimental systems. Section II describes a pair of coupled driven oscillators. Section III reports new experimental results.
II. EXPERIMENT 1: 1D DRIVEN OSCILLATORS

The choice of oscillators used in our first experiment is motivated by the noise-induced bursting observed in coupled “double scroll” chaotic oscillators in a previous study. The “double scroll” chaotic attractor has a saddle point at the origin of the phase space, as discussed previously.5 Two measures of the degree of synchronization as a function of the coupling strength.

The dynamical behavior of a single oscillator in the absence of noise is shown in Fig. 4(a) for \( I_0 = -186 \, \mu A \) (solid line), generated by a numerical simulation. A brief, sufficiently large perturbation to the system when the trajectory is in the vicinity of \( V = -V_0/G \) (dashed line) causes it to undergo a large excursion away from the orbit before returning, as shown by the dotted line in Fig. 4(a). Once the trajectory crosses the threshold, the growth rate of the perturbation is very large: \( V(t) \) increases from \(-0.23 \) V to \(-1 \) V in \~0.1 ms while the period of the driving signal is 10 ms. Thus, a perturbation during the brief interval when the trajectory is in the neighborhood of the threshold can be amplified significantly. This behavior resembles the bursting observed in coupled chaotic double scroll oscillators evolving near the saddle point at the origin of their phase space, as discussed previously.

In the experiment, the slave oscillator is coupled to the master by injecting a current \( I_{sync} = \gamma C(V_m - V_s) \) into the slave circuit at the same node as the drive signal. We bias both oscillators very close to the threshold by setting \( I_0 = -186 \, \mu A \). For this value of \( I_0 \) and the inherent level of noise in the system, the master oscillator never crosses the threshold and remains on the trajectory shown as the solid line in Fig. 4(a). A small Gaussian white noise current (bandwidth from 10 Hz to 1 kHz, rms current \~0.5% of \( I_d \) is injected into the slave oscillator. When there is no coupling (\( \gamma = 0 \)), the slave occasionally crosses the threshold and bursts away from the periodic orbit. For the oscillators to be synchronized, the coupling has to be chosen so that the slave never undergoes a burst.

For each of several different values of the coupling strength, we record a long time series of the Euclidean norm \( |x_\perp| = |V_m - V_s| \). To quantify the degree of synchronization, we determine from these time series the average distance from the synchronization manifold \( |x_\perp|_{rms} \) and the maximum observed value of the distance from the manifold \( |x_\perp|_{max} \) (Ref. 5) for each coupling strength, as shown in Fig. 4(b). For coupling strengths between 0.6 and \~0.8 \times 10^4 \, s^{-1}, \( |x_\perp|_{max} \) is on the order of the size of the orbit (\~2 V) even though \( |x_\perp|_{rms} \) is very small (\~1% of the orbit size), implying that there exist large, brief, occasional desynchronization events even when the oscillators are synchronized on average. From the figure, it is seen that the large desynchroniza-
tion events only cease for $\gamma \geq 1.3 \times 10^4 \text{s}^{-1}$, as indicated by the large drop in $|x_1|_{\text{max}}$.

To interpret these results, we must keep in mind that there is no precise way to define synchronization in an experiment. Gauthier and Bienfang referred to synchronization as high quality when $|x_1|_{\text{max}} < \epsilon$ where $\epsilon$ is a small length scale (typically within a few percent of the characteristic dimensions of the attractor). This condition inherently depends on the choice of a metric and can be violated simply by making the noise level large enough even when the coupled system does not amplify perturbations. In light of this trivial case, this condition should not be taken as a formal definition but rather as an attempt to quantify the idea that small noise or mismatch in the systems should only give rise to small deviations between the master and slave. Without making an explicit choice for $\epsilon$, we see from Fig. 4(b) that the behavior makes a transition from poor synchronization, where small noise gives rise to large separations between master and slave, to burst free synchronization around $\gamma \approx 1.3 \times 10^4 \text{s}^{-1}$ where small noise causes only small separations. Note that the coupling strength at which this transition occurs increases with increasing level of noise injected into the slave oscillator.

III. EXPERIMENT 2: NON-NORMAL LIMIT CYCLE OSCILLATORS

The choice of oscillators used in our second experiment is motivated by the increasing evidence that noise amplification due to non-normality plays an important role in many chaotic physical systems. For example, non-normality may be responsible for turbulence in some fluid flows and chaotic behavior in mode-locked lasers. The non-normality associated with homoclinic tangencies in nonhyperbolic chaotic attractors gives rise to noise-induced attractor deformation, a phenomenon recently observed experimentally in an electronic circuit. Almost all chaotic systems of physical interest are nonhyperbolic and therefore may have regions of phase space characterized by non-normality. The periodic oscillator used in our second experiment evolves on a stable limit cycle on which the Jacobian is highly non-normal. The experimental apparatus for several coupling strengths as shown in Fig. 6. To demonstrate the role of non-normality in amplifying noise, the experiment is performed once with very small (cot $\alpha = 1$) and once with rather large (cot $\alpha = 100$) non-normality. Gaussian white noise (30 mV rms, dc to 15 MHz) is added to the $z$ component of the slave circuit. The amount of noise added to the slave oscillator is the same in both situations.

In the experiment, the two oscillators are $'xy'$ coupled ($K_{11}=K_{22}=1$, $K_{ij}=0$ otherwise). As before, we quantify the degree of synchronization by measuring $|x_1|_{\text{max}}$ and $|x_1|_{\text{rms}}$ for several coupling strengths as shown in Fig. 6. To demonstrate the role of non-normality in amplifying noise, the experiment is performed once with very small (cot $\alpha = 1$) and once with rather large (cot $\alpha = 100$) non-normality. Gaussian white noise (30 mV rms, dc to 15 MHz) is added to the $z$ component of the slave circuit. The amount of noise added to the slave oscillator is the same in both situations.

For small non-normality (cot $\alpha = 1$), it is seen in Fig. 6(a) that the observed distance from the synchronization manifold $|x_1|_{\text{max}} < 0.1 \text{V}$, or 2.5% of the limit cycle radius, for $\gamma > 0.2 \times 10^3 \text{s}^{-1}$. Apparently, the perturbations are not amplified significantly when cot $\alpha = 1$ since the eigenvectors are almost orthogonal. We find dramatically different results when the non-normality is increased (cot $\alpha = 100$), as seen in Fig. 6(b). Over the range of experimentally attainable cou-
Lyapunov exponent is negative. The exponents are determinants is asymptotically stable when the largest transverse synchronized state of noise free, identical coupled chaotic oscillators is due to transversely unstable invariant sets embedded in the transversely stable chaotic attractor on the synchronization manifold. When the system is in the neighborhood of such a set, small noise can push it off the manifold resulting in a brief desynchronization event. Ashwin, Buescu, and Stewart named this behavior attractor bubbling. To avoid attractor bubbling, the largest transverse Lyapunov exponent characterizing the most unstable invariant set must be negative for synchronization in the presence of noise.

Criterion (3). Based on the same idea that the stability of unstable sets governs the region of high-quality synchronization, Brown and Rulkov\cite{Brown1993} suggest an alternative method for determining the transverse stability of these sets. They developed a sufficient, but not necessary, condition for the asymptotic stability of the synchronized state using Gronwald’s theorem. Briefly, they decompose the matrix $J$ into a time-independent $A=(DF)-\gamma K$ and a time-dependent $B(x_m;t)=DF[x_m(t)]-(DF)$ parts, where $\langle \cdot \rangle$ denotes a time average over the driving trajectory. A trajectory is transversely stable when

$$\Re\{\Lambda_1\}>\langle\|P^{-1}[B(x_m;t)]P]\rangle,$$

(8)

where $\Lambda_1$ is the largest eigenvalue of $A$ and $P$ is a matrix of eigenvectors of $A$. The notation $\|\|$ denotes a norm whose choice is arbitrary. The predictions of this criterion depend both on the choice of norm and metric. Following Brown and Rulkov, we use the Frobenius norm. As for the previous criterion, Brown and Rulkov suggest evaluating this criterion along every invariant set embedded in the attractor in order to determine the region of high-quality synchronization.

Criterion (4). Pecora, Carroll, and Heagy\cite{Pecora1990} and Johnson et al.\cite{Johnson1993} attempt to ensure high-quality synchronization by requiring all eigenvalues of the matrix $J$ have negative real parts at all points along the driving trajectory $x_m(t)$. As for the previous case, this criterion depends on the choice of metric.

Criterion (5). Gauthier and Bienfang\cite{Gauthier1993} introduce the Lyapunov function $L=|\delta x(t)|^2$ to obtain an estimate of the regime of high-quality synchronization. They suggest that high-quality synchronization occurs for coupling schemes where

FIG. 6. Synchronization of the “non-normal limit cycle” electronic circuit. Two measures of the degree of synchronization for (a) small ($\cot \alpha=1$) and (b) large ($\cot \alpha=100$) non-normality.
\[
\frac{dL}{dt} = 2 \delta x_-(t) \cdot J \delta x_+(t) < 0
\] (9)

for all times. An equivalent statement is that all eigenvalues of the matrix \((J + J^T)\) have negative real parts at all points along the driving trajectory. In effect, this criterion requires that all perturbations transverse to the synchronization manifold must decay to the manifold without transient growth.\(^{21}\)

As in the two previous cases, this criterion depends on the choice of metric.

Applying the criteria to our first experiment, we find that each predicts synchronization for all coupling strengths greater than a critical value. Although we do see a sharp transition to high-quality synchronization in the experiment, none of the criteria accurately predict the coupling strength \(\gamma \approx 1.3 \times 10^4 \text{ s}^{-1}\) at which this transition is observed. We denote the critical coupling strength for criteria 1–5 as \(\gamma_{FY}, \gamma_A, \gamma_{BR}, \gamma_T\), and \(\gamma_{GB}\), respectively. Our analysis of the circuit model reveals that \(\gamma_{FY} = -1.1 \times 10^4 \text{ s}^{-1}\). We attribute the negativity of \(\gamma_{FY}\) to the fact that, without the driving currents, the nonlinear RC circuits do not self-oscillate. Since the master and slave receive identical driving currents, the circuits would behave identically even with zero coupling if no noise were present. Criterion (2) is equivalent to criterion (1) in this case since the only invariant set in the attractor is the orbit; thus, \(\gamma_A = -1.1 \times 10^4 \text{ s}^{-1}\). In addition, for the criterion proposed by Brown and Rulkov (3), we find \(\gamma_{BR} = -1.1 \times 10^4 \text{ s}^{-1}\). From Fig. 4(b), it is clear that these three criteria fail to provide even a “sufficient” condition for high-quality synchronization since \(|x_\perp|_{\text{max}}\) and \(|x_\perp|_{\text{rms}}\) are large for a range of positive coupling strengths below \(\gamma = 1.3 \times 10^4 \text{ s}^{-1}\). On the other hand, criteria (4) and (5) predict \(\gamma_T = \gamma_{GB} = 2.2 \times 10^4 \text{ s}^{-1}\), overestimating the coupling strength needed to obtain high quality synchronization in the experiment for the particular level of noise injected into the slave oscillator.

Applied to our second experiment, each criterion again predicts a critical coupling strength above which high-quality synchronization should be observed. We first consider the case of small non-normality where experimentally we observe high-quality synchronization at coupling strengths greater than \(0.2 \times 10^5 \text{ s}^{-1}\). For the first four criteria, we find that \(\gamma_{FY} = \gamma_A = 0\), \(\gamma_{BR} = 0.13 \times 10^3 \text{ s}^{-1}\), and \(\gamma_T = 0.11 \times 10^3 \text{ s}^{-1}\). The fifth criterion predicts a much larger critical coupling strength of \(\gamma_{GB} = 0.9 \times 10^3 \text{ s}^{-1}\). Thus, the first four criteria reasonably predict the range of coupling strengths over which a high degree of synchronization is observed while the fifth criterion significantly overestimates the required coupling strength. More interesting is the case of large non-normality where we observe experimentally no transition to high-quality synchronization. Applying the criteria in this case, we find that the first four criteria are independent of the non-normality and hence predict high-quality synchronization at the same critical coupling strengths as in the previous case despite the fact that the observed degree of synchronization is degraded significantly. On the other hand, we find the fifth criterion predicts \(\gamma_{GB} = 6.9 \times 10^8 \text{ s}^{-1}\), a value much too large to implement using our experimental apparatus. We attribute the sensitivity to non-normality in the system of criterion (5) to the above-mentioned fact that this criterion restricts transient amplification of perturbations.

V. DISCUSSION

From these experiments we draw two conclusions. First, any physical system that contains some mechanism for transient growth of perturbations may show bursting behavior when coupled to an identical system because of the inevitable presence of noise. In this paper, we have presented two such mechanisms for transient growth: (1) a sharp threshold in phase space that can separate the master and slave, and (2) noise amplification due to non-normality. Note that these are not the only possible mechanisms for transient growth. Other examples are attractor bubbling in coupled chaotic oscillators\(^3\) and local stability variations in limit cycles.\(^1\)

Therefore, a general criterion for synchronization of physical systems must have some sensitivity to transient behavior. In our experiments, only criterion (5) shows any sensitivity to the transient growth displayed by the oscillators, but it appears to be overly conservative in its estimate of the coupling strength needed for high-quality synchronization. Second, the details of the bursts, their size and frequency, may depend intimately on the details of the noise. For example, in our first experiment, the frequency of the bursts is directly determined by the frequency of perturbations that are both large enough and time appropriately. We speculate that further progress in predicting high-quality synchronization will require explicitly taking into account the details of the noise in the system.

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