I. INTRODUCTION

Nonlinear systems are fascinating because seemingly simple “textbook” devices, such as a strongly driven damped pendulum, can show exceedingly erratic, noise-like behavior that is a manifestation of deterministic chaos. Deterministic refers to the idea that the future behavior of the system can be predicted using a mathematical model that does not include random or stochastic influences. Chaos refers to the idea that the system displays extreme sensitivity to initial conditions so that arbitrary small errors in measuring the initial state of the system grow large exponentially and hence practical, long-term predictability of the future state of the system is lost (often called the “butterfly effect”).

Early nonlinear dynamics research in the 1980s focused on identifying systems that display chaos, developing mathematical models to describe them, developing new nonlinear statistical methods for characterizing chaos, and identifying the way in which a nonlinear system goes from simple to chaotic behavior as a parameter is varied. One outcome of this research was the understanding that the behavior of nonlinear systems falls into just a few universal categories. For example, the route to chaos for a pendulum, a nonlinear electronic circuit, and a piece of paced heart tissue are all identical under appropriate conditions as revealed once the data have been normalized properly. This observation is very exciting since experiments conducted with an optical device can be used to understand some aspects of the behavior of a fibrillating heart, for example. Such universality has fueled a large increase in research on nonlinear systems that transcends disciplinary boundaries and often involves interdisciplinary or multidisciplinary research teams. (Throughout this Resource Letter, I assume that the reader is familiar with the general concept of chaos and the general behavior of nonlinear dynamical systems. For those unfamiliar with these topics, Ref. 35 provides a good entry to this fascinating field.)

A dramatic shift in the focus of research occurred around 1990 when scientists went beyond just characterizing chaos: They suggested that it may be possible to overcome the butterfly effect and control chaotic systems. The idea is to apply appropriately designed minute perturbations to an accessible system parameter (a “knob” that affects the state of the system) that forces it to follow a desired behavior rather than the erratic, noise-like behavior indicative of chaos. The general concept of controlling chaos has captured the imagination of researchers from a wide variety of disciplines, resulting in well over a thousand papers published on the topic in peer-reviewed journals.

In greater detail, the key idea underlying most controlling-chaos schemes is to take advantage of the unstable steady states (USSs) and unstable periodic orbits (UPOs) of the system (infinite in number) that are embedded in the chaotic attractor characterizing the dynamics in phase space. Figure 1 shows an example of chaotic oscillations in which the presence of UPOs is clearly evident with the appearance of nearly periodic oscillations during short intervals. (This figure illustrates the dynamical evolution of current flowing through an electronic diode resonator circuit described in Ref. 78.) Many of the control protocols attempt to stabilize...
one such UPO by making small adjustments to an accessible parameter when the system is in a neighborhood of the state. Techniques for stabilizing unstable states in nonlinear dynamical systems using small perturbations fall into three general categories: feedback, nonfeedback schemes, and a combination of feedback and nonfeedback. In nonfeedback (open-loop) schemes (see Sec. V B below), an orbit similar to the desired unstable state is entrained by adjusting an accessible system parameter about its nominal value by a weak periodic signal, usually in the form of a continuous sinusoidal modulation. This is somewhat simpler than feedback schemes because it does not require real-time measurement of the state of the system and processing of a feedback signal. Unfortunately, periodic modulation fails in many cases to entrain the UPO; its success or failure is highly dependent on the specific form of the dynamical system.

The possibility that chaos and instabilities can be controlled efficiently using feedback (closed-loop) schemes to stabilize UPOs was described by Ott, Grebogi, and Yorke (OGY) in 1990 (Ref. 52). The basic building blocks of a generic feedback scheme consist of the chaotic system that is to be controlled, a device to sense the dynamical state of the system, a processor to generate the feedback signal, and an actuator that adjusts the accessible system parameter, as shown schematically in Fig. 2.

In their original conceptualization of the control scheme, OGY suggested the use of discrete proportional feedback because of its simplicity and because the control parameters can be determined straightforwardly from experimental observations. In this particular form of feedback control, the state of the system is sensed and adjustments are made to the accessible system parameter as the system passes through a surface of section. Figure 3 illustrates a portion of a trajectory in a three-dimensional phase space and one possible surface of section that is oriented so that all trajectories pass through it. The dots on the plane indicate the locations where the trajectory pierces the surface.

In the OGY control algorithm, the size of the adjustments is proportional to the difference between the current and desired states of the system. Specifically, consider a system whose dynamics on a surface of section is governed by the $m$-dimensional map $z_{i+1} = F(z_i, p_i)$, where $z_i$ is its location on the $i$th piercing of the surface and $p_i$ is the value of an externally accessible control parameter that can be adjusted about a nominal value $p_o$. The map $F$ is a nonlinear vector function that transforms a point on the plane with position vector $z_i$ to a new point with position vector $z_{i+1}$. Feedback control of the desired UPO (characterized by the location $z_o(p_o)$ of its piercing through the section) is achieved by adjusting the accessible parameter by an amount $\delta p_i = p_i$
\(-p_o = -\gamma n \cdot (z_i - z_n(p_o))\) on each piercing of the section when \(z_i\) is in a small neighborhood of \(z_n(p_o)\), where \(\gamma\) is the feedback gain and \(n\) is an \(m\)-dimensional unit vector that is directed along the measurement direction. The location of the unstable fixed-point \(z_n(p_o)\) must be determined before control is initiated; fortunately, it can be determined from experimental observations of \(z_i\) in the absence of control (a learning phase). The feedback gain \(\gamma\) and the measurement direction \(n\) necessary to obtain control is determined from the local linear dynamics of the system about \(z_n(p_o)\) using the standard techniques of modern control engineering (see Refs. 34 and 55), and it is chosen so that the adjustments \(\delta p_i\) force the system onto the local stable manifold of the fixed point on the next piercing of the section. Successive iterations of the map in the presence of control direct the system to \(z_n(p_o)\). It is important to note that \(\delta p_i\) vanishes when the system is stabilized; the control only has to counteract the destabilizing effects of noise.

As a simple example, consider control of the one-dimensional logistic map defined as

\[ x_{n+1} = f(x_n, r) = rx_n(1 - x_n). \tag{1} \]

This map can display chaotic behavior when the “bifurcation parameter” \(r\) is greater than \(\sim 3.57\). Figure 4 shows \(x_n\) (closed circles) as a function of the iterate number \(n\) for \(r = 3.9\). The nontrivial period-1 fixed point of the map, denoted by \(x^*\), satisfies the condition \(x_{n+1} = x^* = x^*\) and hence can be determined through the relation

\[ x^* = f(x^*, r). \tag{2} \]

Using the function given in Eq. (1), it can be shown that

\[ x^* = 1 - 1/r. \tag{3} \]

A linear stability analysis reveals that the fixed point is unstable when \(r > 3\). For \(r = 3.9\), \(x^* = 0.744\), which is indicated by the thin horizontal line in Fig. 4. It is seen that the trajectory naturally visits a neighborhood of this point when \(n \sim 32\), \(n = 64\), and again when \(n = 98\) as it explores phase space in a chaotic fashion.

Surprisingly, it is possible to stabilize this unstable fixed point by making only slight adjustments to the bifurcation parameter of the form

\[ r_n = r_o + \delta r_n, \tag{4} \]

where

\[ \delta r_n = -\gamma(x_n - x^*). \tag{5} \]

When the system is in a neighborhood of the fixed point (i.e., when \(x_n\) is close to \(x^*\)), the dynamics can be approximated by a locally linear map given by

\[ x_{n+1} = x^* + \alpha(x_n - x^*) + \beta \delta r_n. \tag{6} \]

The Floquet multiplier of the uncontrolled map is given by

\[ \alpha = \frac{\partial f(x, r)}{\partial x} \bigg|_{x=x^*} = r(1 - 2x^*), \tag{7} \]

and the perturbation sensitivity by

\[ \beta = \frac{\partial f(x, r)}{\partial r} \bigg|_{x=x^*} = x^*(1 - x^*), \tag{8} \]

where I have used the result that \(\delta r_n = 0\) when \(x = x^*\). For future reference, \(\alpha = -1.9\) and \(\beta = 0.191\) when \(r = 3.9\) (the value used to generate Fig. 4). Defining the deviation from the fixed point as

\[ y_n = x_n - x^*, \tag{9} \]

the behavior of the controlled system in a neighborhood of the fixed point is governed by

\[ y_{n+1} = (\alpha + \beta \gamma)y_n, \tag{10} \]

where the size of the perturbations is given by

\[ \delta r_n = \beta \gamma y_n. \tag{11} \]

In the absence of control (\(\gamma = 0\)), \(y_{n+1} = \alpha y_n\) so that a perturbation to the system will grow (i.e., the fixed point is unstable) when \(|\alpha| > 1\).

With control, it is seen from Eq. (10) that an initial perturbation will shrink when

\[ |\alpha + \beta \gamma| < 0 \tag{12} \]

and hence control stabilizes successfully the fixed point when condition (12) is satisfied. Any value of \(\gamma\) satisfying condition (12) will control chaos, but the time to achieve control and the sensitivity of the system to noise will be affected by the specific choice. For the proportional feedback scheme (5) considered in this simple example, the optimum choice for the control gain is when \(\gamma = -\alpha/\beta\). In this situation, a single control perturbation is sufficient to direct the trajectory to the fixed point and no other control perturbations are required if control is applied when the trajectory is close to the fixed point and there is no noise in the system. If control is applied when the \(y_n\) is not small, nonlinear effects become important and additional control perturbations are required to stabilize the fixed point.

Figure 5 shows the behavior of the controlled logistic map for \(r = 3.9\) and the same initial condition used to generate Fig. 4. Control is turned on suddenly as soon as the trajectory is somewhat close to the fixed point near \(n \sim 32\) with the control gain is set to \(\gamma = -\alpha/\beta = 9.965\). It is seen that only a few control perturbations are required to drive the system to the fixed point. Also, the size of the perturbations vanish as \(n\) becomes large since they are proportional to \(y_n\) [see Eq.

Fig. 4. Chaotic evolution of the logistic map for \(r = 3.9\). The circles denote the value of \(x_n\) on each iterate of the map. The solid line connecting the circles is a guide to the eye. The horizontal line indicates the location of the period-1 fixed point.
When random noise is added to the map on each iterate, the control perturbations remain finite to counteract the effects of noise, as shown in Fig. 6. This simple example illustrates the basic features of control of an unstable fixed point in a nonlinear dynamical system using small perturbations. Over the past decade since the early work on controlling chaos, researchers have devised many techniques for controlling chaos that go beyond the closed-loop proportional method described above. For example, it is now possible to control long period orbits that exist in higher dimensional phase spaces. In addition, researchers have found that it is possible to control spatiotemporal chaos, targeting trajectories of nonlinear dynamical systems, synchronizing chaos, communicating with chaos, and using controlling-chaos methods for a wide range of applications in the physical sciences and engineering as well as in biological systems. In this Resource Letter, I highlight some of this work, noting those that are of a more pedagogical nature or involve experiments that could be conducted by undergraduate physics majors. That said, most of the cited literature assumes a reasonable background in nonlinear dynamics.

II. JOURNALS

As mentioned above, research on controlling chaos is highly interdisciplinary and multi-disciplinary so that it is possible to find papers published in a wide variety of journals. I give only a partial list of journals in which papers that are written by physicists on the topic of controlling chaos or where other seminal results can be found.

- American Journal of Physics
- Chaos
- Chaos, Solitons and Fractals
- IEEE Transactions on Circuits and Systems
- International Journal of Bifurcation and Chaos in Applied Sciences and Engineering
- Nature
- Nonlinear Dynamics
- Physica D
- Physical Review Letters
- Physical Review E (prior to 1993, Physical Review A)
- Physics Letters A
- Science

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III. CONFERENCES

There are several conferences on nonlinear dynamics held regularly that often feature sessions on controlling chaos.

1. The American Physical Society March Meeting
2. Dynamics Days
3. Dynamics Days Europe
4. Gordon Conference on Nonlinear Science (every other year).
5. SIAM Conference on Applications of Dynamics Systems (every other year).

IV. CONFERENCES PROCEEDINGS

The following proceedings give an excellent snapshot of the state of research at the time of the associated conference.


Some conferences have elected to have papers appear as a special issue in a journal rather than in an independent book. See, for example, the following thematic issues.


V. TEXTBOOKS AND EXPOSITIONS

There are several texts on control and synchronization of chaos, but most are monographs or collections of articles around a basic theme. The intended audience is researchers or advanced graduate students for most of these books.


VI. INTERNET RESOURCES

36. Many nonlinear dynamics papers, including those on controlling chaos, are often posted on the Nonlinear Science preprint archive (http://arxiv.org/archive/nnlin). (A)
37. Nonlinear FAQ maintained by J.D. Meiss, with a section on controlling chaos (http://amath.colorado.edu/faculty/jdm/faq-%5B3%5D.html#Heading27). (E)
There have been many special issues in journals devoted to controlling chaos and its offshoots. They give a glimpse of the field without spending the day in the library searching out individual articles. A few of these collections are given below.


42. “Special Issue on Controlling Chaos,” Chaos Solitons Fractals 8 (9), 1413–1586 (1997). (A)


The following special issues all have at least one tutorial or review article.


A. Controlling chaos


50. “The Entrainment and Migration Controls of Multiple-Attractor Systems,” E.A. Jackson, Phys. Lett. A 151, 478–484 (1990). (A) Nonlinear control methods to direct a trajectory to a desired location in phase space. Usually requires a mathematical model of the system and may require the application of large perturbations to the system.


52. “Controlling Chaos,” E. Ott, C. Grebogi, and J.A. Yorke, Phys. Rev. Lett. 64, 1196–1199 (1990). (A) One of the early papers to suggest that chaos can be controlled using closed-loop feedback methods. The method does not require a mathematical model of the system and stabilizes the dynamics about one of an infinite number of unstable periodic orbits that are embedded in the chaotic attractor. It was very accessible to researchers in the field and triggered many future studies. While not fully appreciated by the authors at the time, the control method was very similar to elementary control techniques known from the control engineering community. See Ref. 50 where the authors make a connection to and distinguish their work from previous control engineering methods.


54. “Controlling Chaos Using Time Delay Coordinates,” U. Dressler and G. Nitsche, Phys. Rev. Lett. 68, 1–4 (1992). (A) Time-delay coordinates are often used to reconstruct an attractor in experiments so it is important to appreciate how the use of such coordinates affects the feedback signal.

55. “Controlling Chaotic Dynamical Systems,” F.J. Romeiras, C. Grebogi, E. Ott, and W.P. Dayawansa, Physica D 58, 165–192 (1992). (I) A detailed exposition of one method for closed-loop control of chaos using a proportional error signal. Extends the work in Ref. 52. Uses techniques developed in the controlling engineering field and makes a nice connection between the research of the two communities. As suggested by the authors, it is important to read one of the many introductions to modern control engineering used in many undergraduate electrical engineering programs; following their suggestion, see Ref. 34. Read and work through this paper if you want to have a solid foundation for understanding the principles of chaos control. Useful for both theorists and experimentalists.


59. “Continuous Control of Chaos by Self-Controlled Feedback,” K. Pyragas, Phys. Lett. A 170, 421–428 (1992). (A) Describes a technique for controlling chaos that does not require a priori knowledge of the desired unstable periodic orbit and uses an error signal that compares the current state of the system to its state one period in the past.


71. “Control of the Chaotic Driven Pendulum.” G.L. Baker, Am. J. Phys. 63, 832–838 (1995). (E) Describes how to use the methods described in Ref. 73 for controlling the dynamics of a chaotic pendulum that could be found in an undergraduate laboratory. Gives more details about the method than usually found in most journal articles on controlling chaos.


B. Controlling Chaos using weak periodic perturbations

C. Controlling chaos in electronic circuits

D. Controlling spatiotemporal chaos
F. Synchronizing chaos


139. “Synchronization in Chaotic Systems,” L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. 64, 821–824 (1990). (A) One of the early papers on synchronizing chaotic systems that was very popular and triggered significant interest in the problem.


150. “Phase Synchronization of Chaotic Oscillators,” M.G. Rosenblum, A.S. Pikovsky, and J. Kurths, Phys. Rev. Lett. 78, 1804–1806 (1996). (A) Phase synchronization refers to the case where the relative phase of the two oscillators is locked but the amplitudes are not. Most instants of naturally occurring synchronization (e.g., in biological systems) are probably of this form.


G. Communicating with chaos


“Communicating with Chaos,” S. Hayes, C. Grebogi, and E. Ott, Phys. Rev. Lett. 70, 3031–3034 (1993). (A) The technique described in this paper does not use synchronization for decoding the message. Rather, it encodes and decodes the information by targeting a specific symbolic sequence.


H. Applications of chaos control


I. Applications of chaos control in biological systems

“Controlling Cardiac Chaos,” Science 257, 1230–1235 (1992). (A) First demonstration that closed-loop feedback methods can be used to stabilize a chaotic cardiac rhythm.


“Controlling chaos in the brain,” S.J. Schiff, K. Jerger, D.H. Duong, T. Chang, M.L. Spano, and W.L. Ditto, Nature 370, 615–620 (1994). (I) Somewhat controversial result. They concluded that there was chaos in the brain since the controlling-chaos method appeared to stabilize the dynamics. Other researchers have found that the controlling-chaos methods work just fine in nonchaotic systems (see Refs. 202 and 204, for example).


