The Helmholtz equation is
\[(\nabla^2 + k^2)\psi = 0\]

- If \(k \neq 0\), then is called the Helmholtz equation. \(k^2\) is usually positive in most applications.
- If \(k = 0\), then is called the Laplace equation.
- The wave equation and heat/diffusion equations give rise to the Helmholtz equation when time is separated out, with time dependence \(e^{\pm ikvt}\) for the wave equation (with speed \(v\)) and \(e^{-\alpha^2k^2t}\) for the heat/diffusion equation (with thermal diffusivity or diffusion constant \(\alpha^2\)).
- In the following summary, any linear combination of the functions listed together in braces \{\} is allowed.
- Whenever I write cos \(x\) and sin \(x\), you could equally well have \(e^{\pm ix}\) or any linear combination thereof. Similarly cosh \(x\) and sinh \(x\) should be taken to include any linear combination of \(e^\pm x\). If there is a zero separation constant (i.e. \(\alpha = 0\)) for cosh \(\alpha x\)/sinh \(\alpha x\)/cos \(\alpha x\)/sin \(\alpha x\), then the functions change to \(c_1 + c_2x\) (or \(y/z/\phi\))
- \(m\) must be an integer if \(\phi\) is unrestricted (i.e. \(\phi \rightarrow \phi + 2\pi\)).
- \(l\) must be an integer if regular behavior is required at \(\theta = 0\) and \(\theta = \pi\).
- If \(m\) and \(l\) are both integers, then \(-l \leq m \leq l\) for non-vanishing \(J_m\).
- If regular behavior is required at the origin, then only the upper alternative is allowed in the braces \{\} below for the \(r\) or \(\rho\) dependence.

### Cartesian Coordinates
\[
\psi(x, y, z) = \left\{ \begin{array}{l}
\cos \alpha x \\
\sin \alpha x
\end{array} \right\} \left\{ \begin{array}{l}
\cos \beta y \\
\sin \beta y
\end{array} \right\} \left\{ \begin{array}{l}
\cos \gamma z \\
\sin \gamma z
\end{array} \right\} \quad (k^2 > 0) \quad \alpha^2 + \beta^2 + \gamma^2 = k^2
\]
\[
\psi(x, y, z) = \left\{ \begin{array}{l}
\cosh \alpha x \\
\sinh \alpha x
\end{array} \right\} \left\{ \begin{array}{l}
\cos \beta y \\
\sin \beta y
\end{array} \right\} \left\{ \begin{array}{l}
\cos \gamma z \\
\sin \gamma z
\end{array} \right\} \quad -\alpha^2 + \beta^2 + \gamma^2 = k^2
\]
\[
\psi(x, y, z) = \left\{ \begin{array}{l}
\cosh \alpha x \\
\sinh \alpha x
\end{array} \right\} \left\{ \begin{array}{l}
\cosh \beta y \\
\sin \beta y
\end{array} \right\} \left\{ \begin{array}{l}
\cos \gamma z \\
\sin \gamma z
\end{array} \right\} \quad -\alpha^2 - \beta^2 + \gamma^2 = k^2
\]
\[
\psi(x, y, z) = \left\{ \begin{array}{l}
\cosh \alpha x \\
\sinh \alpha x
\end{array} \right\} \left\{ \begin{array}{l}
\cosh \beta y \\
\sin \beta y
\end{array} \right\} \left\{ \begin{array}{l}
\cosh \gamma z \\
\sin \gamma z
\end{array} \right\} \quad (k^2 < 0) \quad -\alpha^2 - \beta^2 - \gamma^2 = k^2
\]
\[
\psi(x, y, z) = \text{other permutations of } (x, y, z)
\]

### Cylindrical Coordinates
\[
\psi(\rho, \phi, z) = \left\{ \begin{array}{l}
J_m(K\rho) \\
N_m(K\rho)
\end{array} \right\} \left\{ \begin{array}{l}
\cos m\phi \\
\sin m\phi
\end{array} \right\} \left\{ \begin{array}{l}
\cos \gamma z \\
\sin \gamma z
\end{array} \right\} \quad (k^2 > 0) \quad K^2 + \gamma^2 = k^2
\]
\[
\psi(\rho, \phi, z) = \left\{ \begin{array}{l}
J_m(K\rho) \\
N_m(K\rho)
\end{array} \right\} \left\{ \begin{array}{l}
\cos m\phi \\
\sin m\phi
\end{array} \right\} \left\{ \begin{array}{l}
\cosh \gamma z \\
\sinh \gamma z
\end{array} \right\} \quad K^2 - \gamma^2 = k^2
\]
\[
\psi(\rho, \phi, z) = \left\{ \begin{array}{l}
I_m(K\rho) \\
K_m(K\rho)
\end{array} \right\} \left\{ \begin{array}{l}
\cos m\phi \\
\sin m\phi
\end{array} \right\} \left\{ \begin{array}{l}
\cos \gamma z \\
\sin \gamma z
\end{array} \right\} \quad -K^2 + \gamma^2 = k^2
\]
\[
\psi(\rho, \phi, z) = \left\{ \begin{array}{l}
I_m(K\rho) \\
K_m(K\rho)
\end{array} \right\} \left\{ \begin{array}{l}
\cos m\phi \\
\sin m\phi
\end{array} \right\} \left\{ \begin{array}{l}
\cosh \gamma z \\
\sinh \gamma z
\end{array} \right\} \quad (k^2 < 0) \quad -K^2 - \gamma^2 = k^2
\]
\[
\psi(\rho, \phi, z) = \left\{ \begin{array}{l}
\rho^m \\
\rho^{-m}
\end{array} \right\} \left\{ \begin{array}{l}
\cos m\phi \\
\sin m\phi
\end{array} \right\} \left\{ \begin{array}{l}
\cos kz \\
\sin kz
\end{array} \right\} \quad (k^2 \geq 0) \quad (\rho^\pm m \rightarrow c_1 + c_2 \ln \rho \text{ for } m = 0)
\]
Spherical Coordinates

\[
\psi(r, \theta, \phi) = \left\{ \begin{array}{l}
\frac{j_l(kr)}{m_l(kr)} \left\{ \begin{array}{l}
P_l^m(\cos \theta) \\
Q_l^m(\cos \theta)
\end{array} \right\} \left\{ \begin{array}{l}
\cos m\phi \\
\sin m\phi
\end{array} \right\} \\
(l^2 > 0)
\end{array} \right.
\]

\[
\psi(r, \theta, \phi) = \left\{ \begin{array}{l}
\frac{i_l(Kr)}{k_l(Kr)} \left\{ \begin{array}{l}
P_l^m(\cos \theta) \\
Q_l^m(\cos \theta)
\end{array} \right\} \left\{ \begin{array}{l}
\cos m\phi \\
\sin m\phi
\end{array} \right\} \\
(l^2 < 0) -K^2 = k^2
\end{array} \right.
\]

\[
\psi(r, \theta, \phi) = \left\{ \begin{array}{l}
\frac{r^l}{r^{-l-1}} \left\{ \begin{array}{l}
P_l^m(\cos \theta) \\
Q_l^m(\cos \theta)
\end{array} \right\} \left\{ \begin{array}{l}
\cos m\phi \\
\sin m\phi
\end{array} \right\} \\
(l^2 = 0) \text{ (Laplace)}
\end{array} \right.
\]

Note that for integer \(l\) and \(m\), \(P_l^m(\cos \theta)e^{im\phi}\) may also be written \(Y_l^m(\theta, \phi)\) besides a numerical prefactor. For the case \(m = 0\), \(P_l^m(\cos \theta)\) reduces to \(P_l(\cos \theta)\), a Legendre polynomial (or Legendre function for non-integer \(l\)).