Green’s Function for the Poisson Equation

1. The nonhomogeneous problem for the Poisson Equation

\[ \nabla^2 \Phi(x) = -\frac{\rho(x)}{\varepsilon_0} \quad \text{in a domain } V \]

Closed boundary conditions on \( \Phi(x) \) at the surface \( S \)

2. The Green’s function \( G(x, x') \)

\[ \nabla^2 G(x, x') = -4\pi \delta(x - x') \]

3. The magic rule

\[ \Phi(x) = \frac{1}{4\pi \varepsilon_0} \int_V G(x', x) \rho(x') \, d^3x' - \frac{1}{4\pi} \oint_S \left( \Phi(x') \frac{\partial G}{\partial n'}(x', x) - G(x', x) \frac{\partial \Phi}{\partial n'}(x') \right) da' \]

4. Homogeneous boundary conditions for \( G(x, x') \)

We can choose homogeneous boundary conditions for \( G(x, x') \) on \( S \). Then

\[ G(x, x') = G(x', x)^* \]

Often we use the Dirichlet BC’s (\( G(x, x') = 0 \) for \( x \) on \( S \)), giving

\[ \Phi(x) = \frac{1}{4\pi \varepsilon_0} \int_V G_D(x, x') \rho(x') \, d^3x' - \frac{1}{4\pi} \oint_S G_D(x, x') \frac{\partial \Phi}{\partial n'}(x') \, da' \] 

(Dirichlet Green’s Fn)

or the Neumann BC’s (\( \partial G(x, x')/\partial n = -4\pi/S \) for \( x \) on \( S \)), giving

\[ \Phi(x) = \langle \Phi \rangle_S + \frac{1}{4\pi \varepsilon_0} \int_V G_N(x, x') \rho(x') \, d^3x' + \frac{1}{4\pi} \oint_S G_N(x, x') \frac{\partial \Phi}{\partial n'}(x') \, da' \] 

(Neumann Green’s Fn)

5. Techniques for constructing Green’s function

a. Method of images. \([J2.6]\)

b. Fundamental solution \( (1/|x - x'|) \) + solutions for Laplace’s equation. \([J1.10]\)

c. Eigenfunction expansions for two dimensions and using the matching method for the third. \([J3.9, J3.11]\)

d. Eigenfunction expansions for three dimensions (see below). \([J3.12]\)

6. Eigenfunction expansions

\[ \nabla^2 \phi_{lmn}(x) + \lambda_{lnm} \phi_{lmn}(x) = 0 \quad \text{(eigenfunctions)} \]

\[ \int_V \phi_{lmn}(x') \phi_{lmn}(x) \, d^3x = \delta_{ij} \delta_{mn} \delta_{l'n'} \quad \text{(orthonormal)} \]

\[ \nabla^2 \phi(x) + \mu \phi(x) = -\frac{\rho(x)}{\varepsilon_0} \quad \text{(nonhomogeneous problem)} \]

Use the same boundary conditions on \( \phi_{lmn}(x) \) as for \( G(x, x') \)

\[ G(x, x') = 4\pi \sum_l \sum_m \sum_n \frac{\phi_{lmn}(x) \phi_{lmn}^*(x')}{\lambda_{lnm} - \mu} \]