Test 1

1. Prove the following identities. You should just assume the fundamental theorems for grad, div, and curl (Stokes’s theorem).
   
   a. \[ \oint_C u \nabla v \cdot dl = - \oint_C v \nabla u \cdot dl \]
   
   b. \[ \int_V \nabla \times A \, d^3x = \oint_S n \times A \, da \]
   
   c. \[ \int_S n \times \nabla \psi \, da = \oint_C \psi \, dl \]

2. a. In its own rest frame (in a frame $K'$), the potential of a point charge is

   \[ \Phi' = \frac{q}{4\pi \varepsilon_0 r'} \]

   \[ A' = 0 \]

   Find the 4-vector potential, $A^\alpha = (\Phi/c, A)$, of a moving charge by placing the charge at the origin of a frame $K'$ moving at velocity $v$ in the $\hat{x}$ direction with respect to frame $K$ and applying a Lorentz transformation. Show that the components are given by the formulas

   \[ \Phi = \frac{\gamma q}{4\pi \varepsilon_0 \sqrt{\gamma^2 (x-vt)^2 + y^2 + z^2}} \]

   \[ A_x = \frac{v}{c^2} \Phi \]

   \[ A_y = A_z = 0 \]

   b. In its own rest frame (in a frame $K'$), the potential of a point electric dipole $p$ is

   \[ \Phi' = \frac{p \cdot r'}{4\pi \varepsilon_0 r'^3} \]

   \[ A' = 0 \]

   Find the 4-vector potential, $A^\alpha = (\Phi/c, A)$, of a moving dipole by placing the dipole at the origin of a frame $K'$ moving at velocity $v$ in the $\hat{x}$ direction with respect to frame $K$ and applying a Lorentz transformation.

3. [Jackson 4.10]

   Two concentric conducting spheres of inner and outer radii $a$ and $b$, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemi-spherical shell of dielectric (of dielectric constant $\varepsilon/\varepsilon_0$), as shown in the figure.

   a. Find the electric field everywhere between the spheres.
   
   b. Calculate the surface-charge distribution on the inner sphere.
   
   c. Calculate the polarization-charge density induced on the surface of the dielectric $r = a$ (i.e. $\sigma_b$).

   Hints: (1) Guess that the $E$ is radial. (2) The Gauss’s law is \( \oint_S E \cdot \hat{n} \, da = q^{\text{inside}}/\varepsilon_0 \) and also \( \oint_S D \cdot \hat{n} \, da = q^{\text{inside}}/\varepsilon_0 \).