Quantum versus classical probability

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Reconstructing quantum theory: In search of a physical principle

Motivation from quantum foundations

Special relativity

Mathematical apparatus
- Lorentz transformations

Physical principle
- space and time not absolute but observer-dependent
- causality (light cone) preserved in all frames

Quantum mechanics

- States
- Unitary evolution
- Measurement

?
The issue is pertinent in the quest for a theory of quantum gravity

Relevance for quantum gravity

Mathematical apparatus
- General relativity
  - Pseudo-Riemannian manifold
  - Einstein equation
- Quantum mechanics
  - States
  - Unitary evolution
  - Measurement

Physical principle
- General relativity
  - Local flatness
  - Equivalence principle
  - ?
- Quantum mechanics
  - ?
There have been many reconstructive approaches – none of them entirely satisfactory

Examples for reconstruction

<table>
<thead>
<tr>
<th>Name</th>
<th>Protagonists</th>
<th>Principal idea</th>
<th>Shortcomings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantum logic</td>
<td>• Birkhoff, v. Neumann&lt;br&gt;• Jauch et al&lt;br&gt;„Geneva School“</td>
<td>• Generalise classical logic: lattice theory&lt;br&gt;• Propositions → subspaces of Hilbert sp.</td>
<td>• Skew field remains unspecified (R, C, H)&lt;br&gt;• Proof only for d≥3</td>
</tr>
<tr>
<td>Algebraic</td>
<td>• Jordan, v. Neumann, Wigner&lt;br&gt;• Gelfand, Neumark, Segal&lt;br&gt;• Haag, Kastler</td>
<td>• Algebra of operators (C* algebra)&lt;br&gt;• Hermitian elements → observables</td>
<td>• Physical understanding?</td>
</tr>
<tr>
<td>Operational, convex cone</td>
<td>• Mackey&lt;br&gt;• Ludwig et al&lt;br&gt;„Marburg School“&lt;br&gt;• Varadarajan, Gudder</td>
<td>• Focus on primitive laboratory operations&lt;br&gt;• Convex geometry of state space</td>
<td>• Convexity does not suffice&lt;br&gt;• Additional mathematical assumptions needed</td>
</tr>
<tr>
<td>Test spaces</td>
<td>• Foulis, Randall et al&lt;br&gt;„Amherst School“</td>
<td>• Generalise probability theory</td>
<td>• How to single out quantum theory from the multitude of conceivable generalisations of classical probability?</td>
</tr>
</tbody>
</table>
Interest in quantum foundations has been revived by the advent of quantum computation

Motivation from quantum computing

Today's talk

classical probability theory
sum rule, Bayes rule

quantum probability
interference, entanglement, Bell, Kochen-Specker

other non-classical probability theories

classical information
Shannon

quantum information
Holevo, Schumacher

generalised information

Physics / computer science / logic

classical computing
Turing, network model

quantum computing
no-cloning, teleportation

other non-classical forms of information processing
Quantum theory and classical probability are often seen as two very different theories...

Major differences

- amplitudes, interference
- non-commutativity
- uncertainty relations
- entanglement
- violation of Bell inequalities
- Kochen-Specker theorem
- ...

classical probability

quantum theory
...yet they share a substantial common core
Both classical and quantum theory deal with propositions and logical relations

Logical structure

<table>
<thead>
<tr>
<th>Classical concept</th>
<th>Quantum analog</th>
<th>Generic name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample space</td>
<td>Hilbert space</td>
<td>Proposition system</td>
</tr>
<tr>
<td>Subset</td>
<td>Subspace</td>
<td>Hypothesis, proposition</td>
</tr>
<tr>
<td>Element</td>
<td>1-dim. subspace (ray)</td>
<td>Most accurate hypothesis</td>
</tr>
<tr>
<td>Empty set</td>
<td>Zero</td>
<td>Absurd hypothesis</td>
</tr>
<tr>
<td>Disjointedness</td>
<td>Orthogonality</td>
<td>Contradiction, exclusion</td>
</tr>
<tr>
<td>Set inclusion</td>
<td>Embedding</td>
<td>Implication, refinement</td>
</tr>
<tr>
<td>Set complement</td>
<td>Orthogonal complement</td>
<td>Complement</td>
</tr>
<tr>
<td>Cardinality</td>
<td>Dimension</td>
<td>Granularity</td>
</tr>
</tbody>
</table>

- orthomodular poset / orthoalgebra / test space
- **weaker than classical:** $\land$, $\lor$ defined iff jointly decidable
Probabilities satisfy sum and Bayes rules – central to ensure consistency

Reasoning in the face of uncertainty

Bayesian probability
- embodies agent’s state of knowledge
- degree of belief rather than limit of relative frequency
- can be legitimately assigned not just to ensembles but also to individual systems

Consistency
Different ways of using the same information must lead to the same conclusions, irrespective of the particular path chosen

Quantitative rules
- Sum rule
- Bayes rule

quantum:
only if probabilities pertain to propositions that are jointly decidable

Cox 1946, Jaynes 2003
Agents can intervene by selection or interrogation

Agent intervention

Generic setup

\begin{itemize}
\item input register
\item \texttt{000 11..}
\item algorithm
\item \texttt{1111 10..}
\item machine
\item \texttt{1-p}
\item \texttt{100 01..}
\item output register
\item \texttt{p}
\end{itemize}

Two examples

\begin{itemize}
\item \textbf{Selection}
\begin{itemize}
\item sequence of reproducible measurements
\item register = record of results
\item selection rule, possibly stochastic
\item keep or discard the system
\end{itemize}
\item \textbf{Interrogation}
\begin{itemize}
\item finite or infinite sequence of reproducible measurements, possibly stochastic
\item register = record of results
\item measurement setups may be controlled by intermediate results (feed forward, as in 1-way q. comp.)
\item no selection, no waste
\end{itemize}
\end{itemize}
**Arbitrary stochastic selection rules are allowed – selection can prepare arbitrary states**

**Most general selection**

\[
S(\sum k_i) = \text{dim } M(\{k_i\}) + \sum S(k_i)
\]

- **classical**: 0, \(k_i^2\) [✓]
- **quantum**: \(\sum_{i \neq j} k_i k_j\), \(k_i^2\) [✓]
Arbitrary interrogation algorithms are allowed

Most general interrogation

\[
\text{prob}(x|\rho) = \sum_{l,r} \text{prob}(x|r,l,\sigma) \cdot \text{prob}(r|l,\sigma) \cdot \text{prob}(l|l,\sigma)
\]

Convexity
- arbitrary mixtures constitute valid posteriors
- states form a convex set
Which additional assumptions are needed to arrive at the classical and quantum case, respectively?

Key question

Not ℏ: We consider probability theory, not dynamics!
Only in the quantum case it is possible to prepare arbitrary states by mere interrogation

Response to interrogation

<table>
<thead>
<tr>
<th>Classical invariability</th>
<th>Quantum malleability</th>
</tr>
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<tbody>
<tr>
<td>Interrogation has no effect:</td>
<td>Interrogation can steer a system from any pure state to any other pure state:</td>
</tr>
<tr>
<td>[ \text{prob}(x</td>
<td>I,\sigma) = \text{prob}(x</td>
</tr>
<tr>
<td>based on joint decidability</td>
<td>v. Neumann 1932</td>
</tr>
</tbody>
</table>

- Preceding interrogation can make any measurement emulate any other measurement of the same resolution:
  \[ \forall \{x_i\},\{y_i\} \in M(\{k_i\}) \exists I: \right \|
  \text{prob}(y_i|I,\sigma) = \text{prob}(x_i|\sigma) \quad \forall i,\sigma \]

based on quantum Zeno effect
Responsiveness to interrogation adds another layer to agent-dependency

Agent-dependency

**Classical**

Orthodox: \[ \text{data} \rightarrow \text{conclusion} \]

Bayes: \[ \text{prior expectation} + \text{data} \rightarrow \text{conclusion} \]

**Quantum**

Orthodox: \[ \text{questions} \rightarrow \text{elicit} \] may inform \[ \text{prior expectation} + \text{answers (data)} \rightarrow \text{conclusion} \]
In quantum theory conclusions depend on questions asked

Wheeler’s game of 20 questions (1/2)

About the game of twenty questions. You recall how it goes—one of the after-dinner party sent out of the living room, the others agreeing on a word, the one fated to be questioner returning and starting his questions. "Is it a living object?" "No." "Is it here on earth?" "Yes." So the questions go from respondent to respondent around the room until at length the word emerges: victory if in twenty tries or less; otherwise, defeat.

Then comes the moment when we are fourth to be sent from the room. We are locked out unbelievably long. On finally being readmitted, we find a smile on everyone’s face, sign of a joke or a plot. We innocently start our questions. At first the answers come quickly. Then each question begins to take longer in the answering—strange, when the answer itself is only a simple "yes" or "no". At length, feeling hot on the trail, we ask, "Is the word ‘cloud’?" "Yes", comes the reply, and everyone bursts out laughing. When we were out of the room, they explain, they had agreed not to agree in advance on any word at all. Each one around the circle could respond "yes" or "no" as he pleased to whatever question we put to him. But however he replied he had to have a word in mind compatible with his own reply—and with all the replies that went before. No wonder some of those decisions between "yes" and "no" proved so hard!

Wheeler 1983 ("Law without law")
In quantum theory conclusions depend on questions asked

Wheeler’s game of 20 questions (2/2)

1. The word does not already exist "out there" – rather, information about the word is brought into being through the questions raised.

2. The answers given are internally consistent; convergence to some word is possible.

3. The questioner has influence on the outcome: a different sequence of questions will lead to a different word.

...like in quantum theory:

- There is no preexisting reality that is merely revealed, rather than influenced, by the act of measurement.
- The image of reality that emerges through acts of measurement reflects as much the history of intervention as it reflects the external world.

Wheeler 1983 ("Law without law")
This new form of agent-dependency leads to a radically different world view

World view

<table>
<thead>
<tr>
<th>Classical</th>
<th>Quantum</th>
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<tbody>
<tr>
<td>observe without disturbance</td>
<td>image of reality</td>
</tr>
<tr>
<td>reality with independent existence</td>
<td>(partial) record of answers: 0111001 1101100</td>
</tr>
<tr>
<td>manipulate in experiment: grab, dissect into parts, select, transform</td>
<td>ongoing interrogation</td>
</tr>
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Duke University, August 2009

J. Rau | Quantum vs classical probability
Can agent-dependency serve as a foundational principle for quantum theory?

Conjecture

Special relativity

Mathematical apparatus
- Lorentz transformations

Physical principle
- space and time not absolute but observer-dependent
- causality (light cone) preserved in all frames

Quantum mechanics

- States
- Unitary evolution
- Measurement

- image of reality not absolute but agent-dependent
- consistency of reasoning preserved for all agents
Only in the quantum case it is possible to prepare arbitrary states by mere interrogation

Response to interrogation

Hidden, deterministic, static interrogation

Classical invariability
- Interrogation has no effect:
  \[\text{prob}(x|I,\sigma) = \text{prob}(x|\sigma) \quad \forall \ I, x, \sigma\]
  
  based on joint decidability

Quantum malleability
- Interrogation can steer a system from any pure state to any other pure state:
  \[\forall e,f \exists I: \text{prob}(x|I,e) = \text{prob}(x|f) \quad \forall x\]
  
  v. Neumann 1932

- Preceding interrogation can make any measurement emulate any other measurement of the same resolution:
  \[\forall \{x_i\},\{y_i\}\in M(\{k_i\}) \exists I:\]
  \[\text{prob}(y_i|I,\sigma) = \text{prob}(x_i|\sigma) \quad \forall i,\sigma\]
  
  based on quantum Zeno effect
Maximum agent-dependency rests on the Zeno effect, which in turn presupposes smoothness

Smoothness

For finite granularity $d$:

1. Set of pure states $X(d)$ is a continuous, compact, simply connected manifold

2. Probabilities change in a continuous fashion:

$$\forall \varepsilon > 0 \exists \delta > 0: \text{prob}(x|e) > 1 - \varepsilon \ \forall \ e \in B(e_0; \delta)$$

$\Leftrightarrow$ Robustness under small preparation inaccuracies
Continuity in combination with common core leads to quantum theory in Hilbert space over R, C or H

Analysis of symmetry group

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Implication for group</th>
</tr>
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<tbody>
<tr>
<td>$X(d) \sim M{1,d-1}$</td>
<td>transitive on $X(d)$</td>
</tr>
<tr>
<td>$X(d)$ continuous</td>
<td>Lie group</td>
</tr>
<tr>
<td>$X(d)$ compact and simply connected (hull of convex set)</td>
<td>compact and simple up to Abelian factor</td>
</tr>
<tr>
<td>dimensions match</td>
<td>$\dim X(d) = \dim G(d) - \dim G(d-1) - \dim G(1)$</td>
</tr>
<tr>
<td>probabilities change in a continuous fashion</td>
<td>$\dim G(d) = \dim X(2)/2 \cdot d(d-1) + \dim G(1) \cdot d$</td>
</tr>
</tbody>
</table>

**Definition**
Automorphisms of a proposition system preserve
- logical relations $\bot, \subset, \setminus$
- granularity $d$

**Irreducible building block**
Group acts transitively on collections $M\{\{k_i\}\}$:

$M\{\{k_i\}\} \sim G(\Sigma k_i)/\otimes_i G(k_i)$

$G(d) \in \{SO(n \cdot d), U(n \cdot d), Sp(n \cdot d) \mid n \in \mathbb{N}\}$
Classical and quantum case derive from common core via additional assumptions on agent-dependency

Conclusion

\[ \text{prob}(x|I, \sigma) = \text{prob}(x|\sigma) \]

\[ \forall e, f \exists I: \text{prob}(x|I, e) = \text{prob}(x|f) \]

R, C or H?
Summary

- Quantum theory shares with classical probability theory a substantial core of common properties.

- Quantum theory and classical probability theory mark opposite extremes in their response to interrogation – quantum theory maximises agent-dependency.

- No reference to unitary evolution – primitive operations are measurements only.\(^1\)

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1. see also: T. Rudolph; measurement-based quantum computing
Further details: