# Wigner Functional Method in Quantum Field Theory

Berndt Müller

**YIPQS Molecule: "*Entropy Production*"**

YITP Kyoto - 13 August 2008

Based on: S. Mrówczyński & BM, PRD 50, 7542 (1994)

# An impossible dream?

Simulations of real-time dynamics of quantum fields far off equilibrium have used a variety of approximation schemes:

- Boltzmann equations for (quasi-)particle excitations:
  - Good in dilute systems, but loss of coherence and off-shell effects.
- Classical nonlinear (lattice) field equations:
  - Good for highly occupied modes, but loss of all quantum effects.
- Real-time Green function techniques:
  - Practical for homogeneous or low-dimensional systems only.

Is it possible to devise a scheme which encompasses the classical field and particle limit (useful for phenomenology), yet includes essential quantum effects, such as the uncertainty relation, phases, or correlations, yet can be treated by similar methods similar to those that have already been developed?

# Wigner function

Reminder - Usual definition of Wigner function for a point particle:

$$W(q,p;t) = \int du\, e^{-ipu} \langle q + \frac{1}{2}u|\, \hat{\rho}(t)\, |q - \frac{1}{2}u\rangle$$

Properties:

$$W(q,p) = W(q,p)^*$$

$$\int \frac{dqdp}{2\pi}\, W(q,p) = \mathrm{Tr}\,\hat{\rho} = 1$$

Equation of motion (for $H = p^2/m + V$) :

$$\begin{aligned} \frac{\partial W}{\partial t} &= -\frac{p}{m}\frac{\partial W}{\partial q} - \frac{i}{\hbar}\left[\frac{\partial V}{\partial q}\left(q + \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) - \frac{\partial V}{\partial q}\left(q - \frac{i\hbar}{2}\frac{\partial}{\partial p}\right)\right] W \\ &\xrightarrow{\hbar\to 0} -\frac{p}{m}\frac{\partial W}{\partial q} + \frac{\partial V}{\partial q}\frac{\partial W}{\partial p} \end{aligned}$$

# Wave packets

Consider a Gaussian wavepacket: $\Psi_{P,a}(q) = \sqrt{2\Delta/\pi}\, e^{iPq-\Delta(q-a)^2}$

The Wigner transform is: $W(q,p) = 2\exp\left(-2\Delta(q-a)^2 - \frac{1}{2\Delta}(p-P)^2\right)$

Pure quantum state:

$$\mathrm{Tr}\,\hat{\rho}^2 = \mathrm{Tr}\,\hat{\rho} = 1 \qquad \leftrightarrow \qquad \int \frac{dqdp}{2\pi} W(q,p)^2 = \int \frac{dqdp}{2\pi} W(q,p) = 1$$

$$\langle (q-a)^2 \rangle = \frac{1}{4\Delta}$$
$$\langle (p-P)^2 \rangle = \Delta$$

$\Delta \to \infty$: Good position

$\Delta \to 0$: Good momentum

$$\left(\langle (\Delta q)^2 \rangle \langle (\Delta P)^2 \rangle\right)^{1/2} = \frac{1}{2}$$

Minumum uncertainty wavepacket

# Wigner functional

Adapt quantum phase space formulation (Wigner function) to the field representation of quantum field theory in order make the classical field limit more transparent.

*Wigner functional* = Adaptation to QFT. Start with a scalar quantum field Φ.

Position space representation:

$$W[\Phi(x),\Pi(x);t] \quad = \quad \int \mathcal{D}\varphi(x)\,\exp\left[-i\int dx\;\Pi(x)\varphi(x)\right]\;\langle\Phi(x)+\frac{1}{2}\varphi(x)|\;\hat{\rho}(t)\;|\Phi(x)-\frac{1}{2}\varphi(x)\rangle$$

Momentum space representation:

$$W[\Phi(p),\Pi(p);t] = \int \mathcal{D}\varphi(p)\,\exp\left[-i\int_0^\infty dp\;\left(\Pi^*(p)\varphi(p)+\Pi(p)\varphi^*(p)\right)\right]\langle\Phi(p)+\frac{1}{2}\varphi(p)|\;\hat{\rho}(t)\;|\Phi(p)-\frac{1}{2}\varphi(p)\rangle$$

# Basics

Using $$\int \frac{\mathcal{D}\Pi}{2\pi} \exp\left[\pm i \int dx\, \Pi(x)\varphi(x)\right] = \delta(\varphi(x))$$ it is obvious that

$$W[\Phi(x),\Pi(x);t] \;=\; \int \mathcal{D}\varphi(x) \exp\left[-i\int dx\, \Pi(x)\varphi(x)\right] \langle \Phi(x)+\frac{1}{2}\varphi(x)|\, \hat{\rho}(t)\, |\Phi(x)-\frac{1}{2}\varphi(x)\rangle$$

implies $$Z \equiv \mathrm{Tr}\,\hat{\rho} = \int \mathcal{D}\Phi(x) \langle \Phi(x)|\, \hat{\rho}\, |\Phi(x)\rangle = \int \mathcal{D}\Phi(x) \frac{\mathcal{D}\Pi(x)}{2\pi}\, W[\Phi,\Pi;t]$$

Similarly, by inserting complete sets of states and partial integrations:

$$\langle \mathcal{O}(\hat{\Phi},\hat{\Pi})\rangle = \frac{1}{Z}\,\mathrm{Tr}\big[\hat{\rho}(t)\mathcal{O}(\hat{\Phi},\hat{\Pi})\big] = \frac{1}{Z}\int \mathcal{D}\Phi(x) \frac{\mathcal{D}\Pi(x)}{2\pi}\, \mathcal{O}(\Phi,\Pi)\, W[\Phi,\Pi;t]$$

Wigner functional expression corresponds to symmetrized quantum operators.

# Equation of motion - I

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)] \qquad \hat{H} = \frac{1}{2}\int dx\Big(\hat{\Pi}^2(x) + \big(\nabla\hat{\Phi}(x)\big)^2 + m^2\hat{\Phi}^2(x) - \hat{\mathcal{L}}_I(x)\Big)$$

$$\begin{aligned}
\frac{\partial}{\partial t}W[\Phi,\Pi;t] &= \int \mathcal{D}\varphi \exp\left[-\frac{i}{\hbar}\int dx\, \Pi(x)\varphi(x)\right] \langle\Phi + \frac{1}{2}\varphi|\, [\hat{H},\hat{\rho}]\, |\Phi - \frac{1}{2}\varphi\rangle \\
&\equiv G_\Pi + G_\nabla + G_m + G_I
\end{aligned}$$

$$\begin{aligned}
G_m &= \frac{m^2}{2}\int dx \int \mathcal{D}\varphi \exp\left[-\frac{i}{\hbar}\int dx\, \Pi(x)\varphi(x)\right] \\
&\times \Big(\big(\Phi(x) + \frac{1}{2}\varphi(x)\big)^2 - \big(\Phi(x) - \tfrac{1}{2}\varphi(x)\big)^2\Big) \left\langle\Phi + \tfrac{1}{2}\varphi|\, \hat{\rho}\, |\Phi - \tfrac{1}{2}\varphi\right\rangle
\end{aligned}$$

$$G_m = i\hbar m^2 \int dx\, \Phi(x)\, \frac{\delta}{\delta\Pi(x)} W[\Phi,\Pi;t]$$

# Equation of motion - II

Other terms are derived similarly, final result is:

$$\left[\frac{\partial}{\partial t} + \int dx \left(\Pi(x)\,\frac{\delta}{\delta\Phi(x)} - \left(m^2\Phi(x) - \nabla^2\Phi(x)\right)\frac{\delta}{\delta\Pi(x)} + \mathcal{K}_I(x)\right)\right] W[\Phi,\Pi;t] = 0$$

with $$\mathcal{K}_I(x) \equiv -\frac{i}{\hbar}\mathcal{L}_I\left(\Phi(x) + \frac{i\hbar}{2}\frac{\delta}{\delta\Pi(x)}\right) + \frac{i}{\hbar}\mathcal{L}_I\left(\Phi(x) - \frac{i\hbar}{2}\frac{\delta}{\delta\Pi(x)}\right)$$

Standard example: $\Phi^4$ theory:

$$\mathcal{L}_I(\Phi) = -\frac{\lambda}{4!}\Phi^4 \qquad \longrightarrow \qquad \mathcal{K}_I = \frac{\lambda}{6}\left(-\Phi^3(x)\frac{\delta}{\delta\Pi(x)} + \frac{\hbar^2}{4}\Phi(x)\frac{\delta^3}{\delta\Pi^3(x)}\right)$$

Thus, evolution of *W* is described by a classical transport equation with quantum corrections:

$$\frac{\partial}{\partial t}W[\Phi,\Pi;t] = -\int dx \left(\frac{\delta H}{\delta\Pi(x)}\,\frac{\delta}{\delta\Phi(x)} - \frac{\delta H}{\delta\Phi(x)}\,\frac{\delta}{\delta\Pi(x)} + O\left(\hbar^2\frac{\delta^3}{\delta\Pi^3}\right)\right)W[\Phi,\Pi;t]$$

# Free field at $T \neq 0$

Density matrix: $\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}_0}$ with $\hat{H}_0 = \int_0^\infty \frac{dp}{2\pi} \left( \hat{\Pi}^\dagger(p)\hat{\Pi}(p) + (p^2 + m^2)\hat{\Phi}^\dagger(p)\hat{\Phi}(p) \right)$

Momentum space Wigner functional ($\hbar = 1$):

$$\widetilde{W}_\beta[\Phi, \Pi] = C \exp\left[ -\frac{1}{2}\beta \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, \tilde{\Delta}_\beta(p) \left( \Pi^*(p)\Pi(p) + E^2(p)\Phi^*(p)\Phi(p) \right) \right]$$

with

$$C \equiv \exp\left[ \int_{-\infty}^{\infty} \frac{dp}{2\pi} \ln \mathrm{th} \frac{\beta E(p)}{2} \right] \qquad \tilde{\Delta}_\beta(p) = \frac{2}{\beta E(p)} \mathrm{th} \frac{\beta E(p)}{2}$$

$$\beta \to 0: \qquad \tilde{\Delta}_\beta(p) \to 1.$$ (classical limit)

Coordinate space form contains “nonlocal” Hamiltonian:

$$W_\beta[\Phi(x), \Pi(x)] = C' \exp\left[ -\frac{1}{2}\beta \int dx\, dx'\, \Delta(x - x') \left( \Pi(x)\Pi(x') + \nabla\Phi(x)\nabla\Phi(x') + m^2\Phi(x)\Phi(x') \right) \right]$$

# Fluctuations

Thermal quantum fluctuations are larger than classical fluctuations:

$$\langle \hat{\Phi}^\dagger(p)\hat{\Phi}(p)\rangle = \frac{1}{2\ E(p)\ \mathrm{th}\frac{\beta E(p)}{2}} \geq \frac{1}{2\ E(p)}$$

$$\langle \hat{\Pi}^\dagger(p)\hat{\Pi}(p)\rangle = \frac{E(p)}{2\ \mathrm{th}\frac{\beta E(p)}{2}} \geq \frac{E(p)}{2}$$

implying the uncertainty relation:

$$\sqrt{\langle \hat{\Phi}^\dagger(p)\hat{\Phi}(p)\rangle\ \langle \hat{\Pi}^\dagger(p)\hat{\Pi}(p)\rangle} = \frac{1}{2\ \mathrm{th}\frac{\beta E(p)}{2}} \geq \frac{1}{2}$$

# Generating functional

Perturbation theory is based on generating functional

$$\begin{aligned}\mathcal{Z}[j(x)] &\equiv \int \mathcal{D}\Phi \frac{\mathcal{D}\Pi}{2\pi}\, W[\Phi,\Pi] \exp\left(\beta \int dx\; \Phi(x) j(x)\right) \\ &= \int \mathcal{D}\Phi \frac{\mathcal{D}\Pi}{2\pi} \exp\left[-\beta \int dx\; dx'\; \left(\mathcal{H}(x,x') - \delta(x-x')\Phi(x) j(x')\right)\right]\end{aligned}$$

Two point function: $$\langle \hat{\Phi}(x)\hat{\Phi}(y)\rangle = \frac{1}{\beta^2}\, \frac{1}{\mathcal{Z}[j]} \frac{\delta^2 \mathcal{Z}[j]}{\delta j(y)\delta j(x)}\bigg|_{j(x)=0}$$

Free field: $$\mathcal{Z}_0[j(x)] = \mathcal{N} \exp\left[\frac{\beta}{2}\int dx\; dx'\; j(x)\mathcal{G}(x-x') j(x')\right]$$

with $$\mathcal{G}(x) = \int \frac{dp}{2\pi}\, \frac{e^{-ipx}}{\tilde{\Delta}_\beta(p)\, \left[p^2+m^2\right]} \xrightarrow{m\to 0} \frac{1}{4\pi|\mathbf{x}|}\, \frac{\mathrm{sh}(2\pi|\mathbf{x}|/\beta)}{\mathrm{ch}(2\pi|\mathbf{x}|/\beta)-1}$$

# Particles

Consider one-particle state: $\hat{\rho} = a^\dagger(p)|0\rangle\langle 0|a(p)$

Wigner functional:

$$W_1[\Phi(p),\Pi(p)] = C\left(2\frac{|\Pi(p)|^2 + E(p)^2\Phi(p)|^2}{E(p)} - 1\right)\exp\left(-\frac{|\Pi(p)|^2 + E(p)^2\Phi(p)|^2}{E(p)}\right)$$

Partially occupied state: $\hat{\rho} = (1-f(p))|0\rangle\langle 0| + f(p)a^\dagger(p)|0\rangle\langle 0|a(p)$

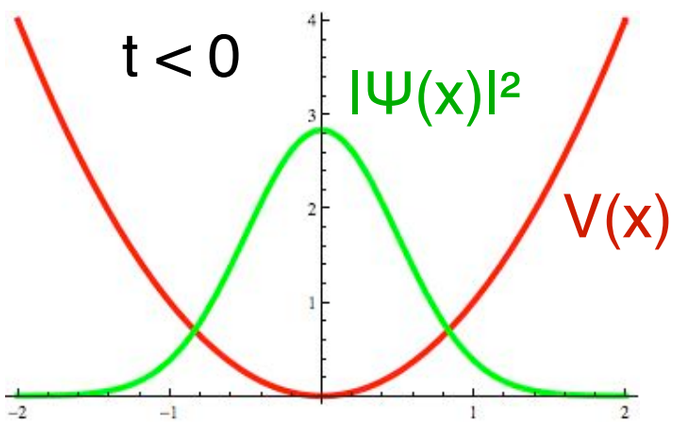
Wigner functional:

$$\begin{aligned} W_f[\Phi(p),\Pi(p)] &= C\left[1 + 2f(p)\left(\frac{|\Pi(p)|^2 + E(p)^2\Phi(p)|^2}{E(p)} - 1\right)\right]\exp\left(-\frac{|\Pi(p)|^2 + E(p)^2\Phi(p)|^2}{E(p)}\right) \\ &\approx C'\exp\left(-\frac{|\Pi(p)|^2 + E(p)^2\Phi(p)|^2}{[1+2f(p)]\,E(p)}\right) \end{aligned}$$

Agrees with thermal expression for $f(p) = (e^{-\beta E(p)}-1)^{-1}$.

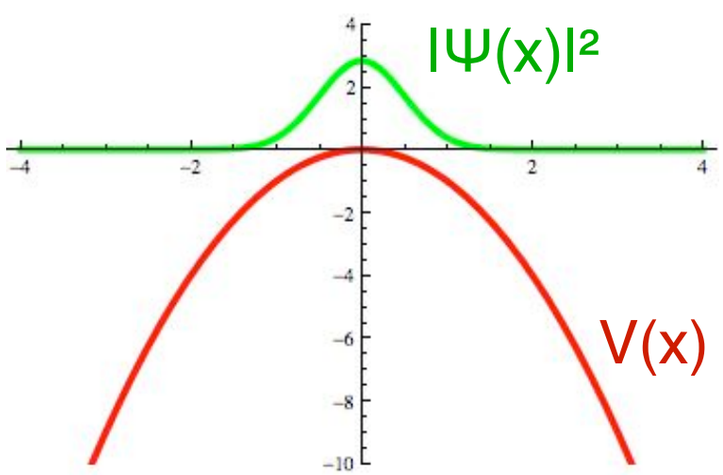
# Roll-over transition

Decay of an unstable “vacuum state” is a common problem e.g. in cosmology (end of inflation) or condensed matter physics (phase transitions). Paradigm case: “roll-over”.

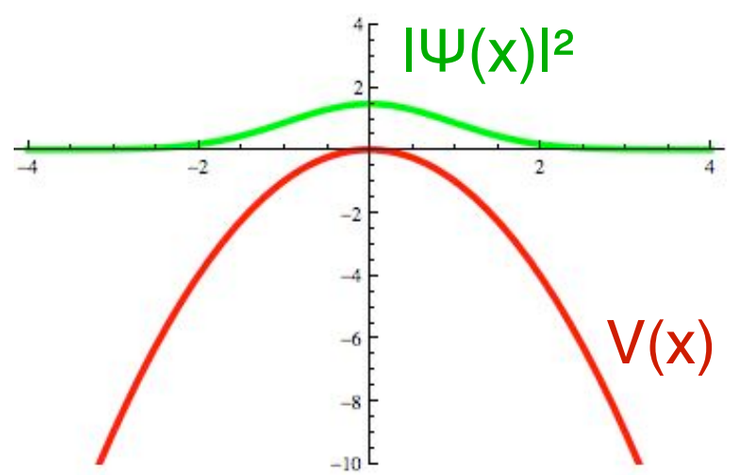



$$\hat{H}(t) = \frac{p^2}{2} + \frac{m(t)^2}{2} x^2$$

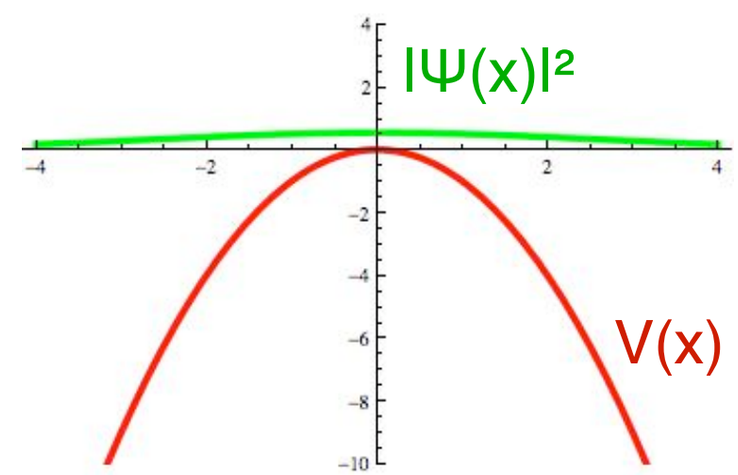
with $\quad m^2(t) = m^2\,\theta(-t) - \mu^2\,\theta(t)$




t = 0

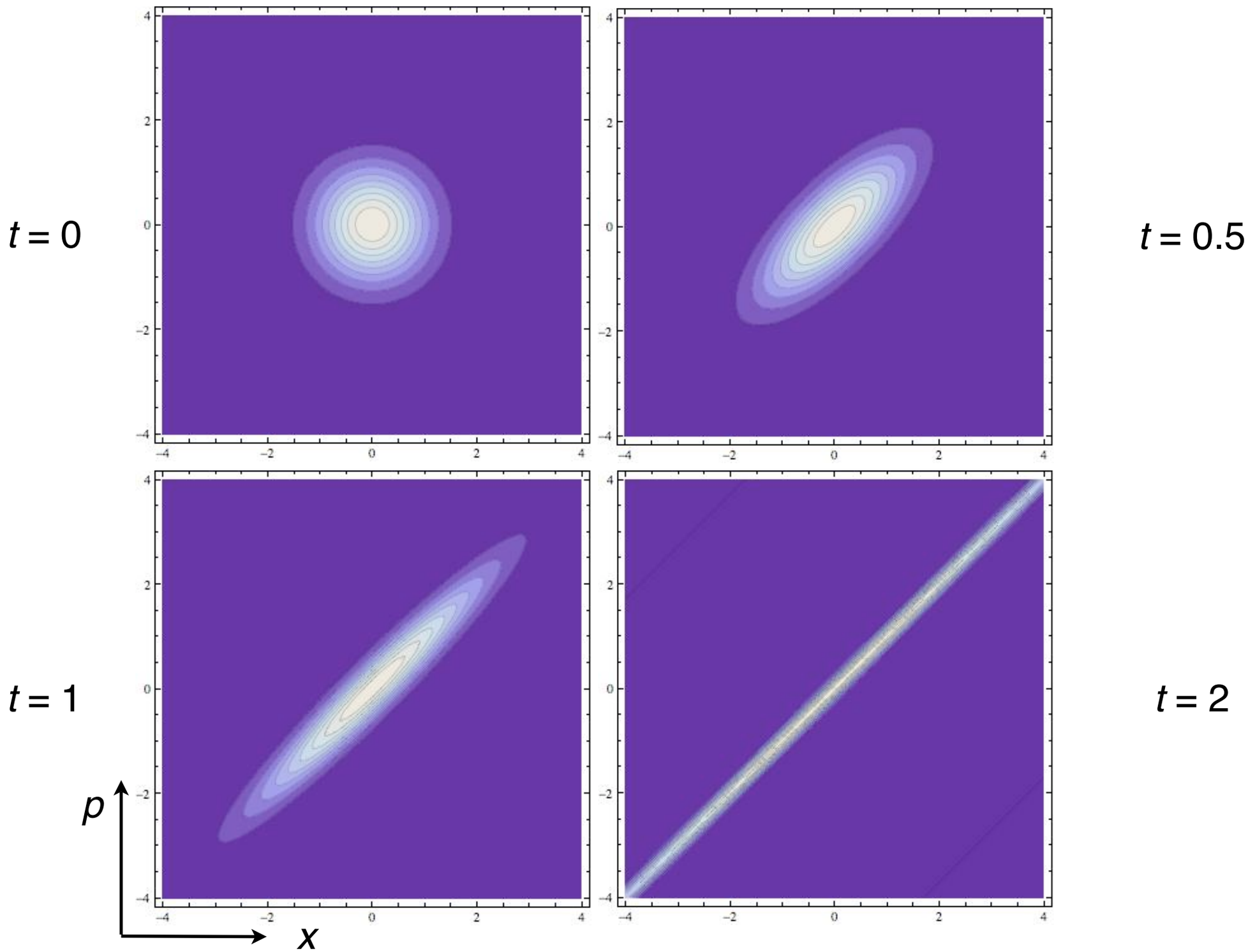



t = 1




t = 2

# Wigner function



# Roll-over in QFT

$$\hat{H}(t) = \int_0^\infty \frac{dp}{2\pi}\Big(\hat{\Pi}^\dagger(p)\hat{\Pi}(p) + (m^2(t) + p^2)\ \hat{\Phi}^\dagger(p)\hat{\Phi}(p)\Big)$$

$$m^2(t) = m^2\,\theta(-t) - \mu^2\,\theta(t)$$

Initial state (not necessarily vacuum):

$$\widetilde{W}[\Phi, \Pi, t < 0] = C\,\exp\left[-\frac{1}{2}\beta\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\Delta_\beta(p)\,\Big(\Pi^*(p)\Pi(p) + E^2(p)\Phi^*(p)\Phi(p)\Big)\right]$$

Hamiltonian for $t > 0$:

$$\hat{H}(t>0) \quad = \quad \int_0^\mu \frac{dp}{2\pi}\Big(\hat{\Pi}^\dagger(p)\hat{\Pi}(p) - \omega_-^2(p)\ \hat{\Phi}^\dagger(p)\hat{\Phi}(p)\Big) + \int_\mu^\infty \frac{dp}{2\pi}\Big(\hat{\Pi}^\dagger(p)\hat{\Pi}(p) + \omega_+^2(p)\ \hat{\Phi}^\dagger(p)\hat{\Phi}(p)\Big)$$

$$\omega_\pm(p) \equiv \sqrt{\pm(p^2 - \mu^2)}\ \text{is always real}$$

# Solution

Classical field trajectories:

$$W[\Phi(p),\Pi(p);t] = W[\Phi(p,-t),\Pi(p,-t);0]$$

$$\frac{d}{dt}\,\Pi(p,t) \pm \omega_\pm^2(p)\;\Phi(p,t) = 0$$

$$\frac{d}{dt}\,\Phi(p,t) = \Pi(p,t)$$

with

$$\Phi(p,0) = \Phi(p),$$

$$\Pi(p,0) = \Pi(p)$$

$$\widetilde{W}[\Phi,\Pi;t] = C\,\exp\left[-\frac{1}{2}\beta\int_{-\infty}^{\infty}\frac{dp}{2\pi}\;\tilde{\Delta}_\beta(p)\;\Big(\Pi^*(p,-t)\Pi(p,-t) + E^2(p)\Phi^*(p,-t)\Phi(p,-t)\Big)\right]$$

$$\begin{aligned}\Phi(p,t) &= \Phi(p)\;\mathrm{ch}(\omega_-(p)t) - \frac{1}{\omega_-(p)}\;\Pi(p)\;\mathrm{sh}(\omega_-(p)t)\\ \Pi(p,t) &= \Pi(p)\;\mathrm{ch}(\omega_-(p)t) - \omega_-(p)\;\Phi(p)\;\mathrm{sh}(\omega_-(p)t)\end{aligned}$$

$$0 < p < \mu$$

$$\begin{aligned}\Phi(p,t) &= \Phi(p)\;\cos(\omega_+(p)t) - \frac{1}{\omega_+(p)}\;\Pi(p)\;\sin(\omega_+(p)t)\\ \Pi(p,t) &= \Pi(p)\;\cos(\omega_+(p)t) + \omega_+(p)\;\Phi(p)\;\sin(\omega_+(p)t)\end{aligned}$$

$$p > \mu$$

# Field correlator

Distribution of field amplitudes:

$$\widetilde{W}[\Phi;t] \equiv \int \frac{\mathcal{D}\Pi}{2\pi}\widetilde{W}[\Phi,\Pi;t] = C'\exp\left[-\frac{\beta}{2}\int_0^\infty \frac{dp}{2\pi}\,\tilde{\Delta}_\beta(p)\,\chi(p,t)\,\Phi^*(p)\Phi(p)\right]$$

with

$$\chi(p,t) = \Theta(\mu-p)\,\frac{E^2(p)\omega_-^2(p)}{\omega_-^2(p)\,\mathrm{ch}^2(\omega_-(p)t)+E^2(p)\,\mathrm{sh}^2(\omega_-(p)t)} + \Theta(p-\mu)\,\frac{E^2(p)\omega_+^2(p)}{\omega_+^2(p)\,\cos^2(\omega_+(p)t)+E^2(p)\,\sin^2(\omega_+(p)t)}$$

Field fluctuations:

$$\langle\hat{\Phi}^\dagger(p)\hat{\Phi}(p)\rangle_t = \langle\hat{\Phi}^\dagger(p)\hat{\Phi}(p)\rangle_0\left(\mathrm{ch}^2(\omega_-(p)t)+\frac{E^2(p)}{\omega_-^2(p)}\mathrm{sh}^2(\omega_-(p)t)\right)$$

Field correlation function:

$$\mathcal{G}(x,t) = \int\frac{dp}{2\pi}\,\frac{e^{-ipx}}{\tilde{\Delta}_\beta(p)\,\chi(p,t)} \xrightarrow{t\to\infty} \frac{\beta m\mu}{64\,\mathrm{th}\,(\beta m/2)}\left(1+\frac{\mu^2}{m^2}\right)(\pi\mu t)^{-3/2}\exp\left(2\mu t-\frac{\mu\mathbf{x}^2}{4t}\right)$$

# Mean field approximation

Reduce nonlinear interactions by means of expectation values:

$$\frac{\lambda}{4!}\hat{\Phi}^4(x) \to \frac{\lambda}{4}\langle\hat{\Phi}^2(x)\rangle\,\hat{\Phi}^2(x) \qquad \Longleftrightarrow \qquad m_*^2 = m^2 + \frac{\lambda}{2}\langle\hat{\Phi}^2(x)\rangle$$

$$\langle\hat{\Phi}^2(x)\rangle = \frac{1}{Z}\int \mathcal{D}\Phi(x)\frac{\mathcal{D}\Pi(x)}{2\pi}\,\Phi^2(x)\,W[\Phi,\Pi;t]$$

Self-consistent equation for m*:

$$\begin{aligned} m_*^2 &= m^2 + \frac{\lambda}{2}\,\mathcal{G}(0) \\ &= m_{\rm ren}^2 + \frac{\lambda}{2}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\frac{\sqrt{p^2+m_*^2}}{\exp\left(\beta\sqrt{p^2+m_*^2}\right)-1}\,\frac{1}{p^2+m_*^2} \end{aligned}$$

# Semi-classical approx.

Transport equation:

$$\left(\hat{K}_{\rm c} + \hbar^2 \hat{K}_{\rm q}\right) W = 0 \qquad \text{with} \qquad W = W_{\rm c} + \hbar^2 W_{\rm q}$$

$$\hat{K}_{\rm c} W_{\rm c} = 0 \quad \Longrightarrow \quad \hat{K}_{\rm c} W_{\rm q} = -\hat{K}_{\rm q} W_{\rm c} + \mathcal{O}(\hbar^2)$$

Thus: $$W_{\rm q}(q,p;t) = \int dt' dq' dp' \; G_c(q,q',p,p';t,t') \; \hat{K}_{\rm q} W_{\rm c}(q',p';t')$$

Classical Green function satisfies:

$$\hat{K}_{\rm c} G_c(q,q',p,p';t,t') \equiv \frac{\partial G_c}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial G_c}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial G_c}{\partial p} = \delta(t-t') \; \delta(q-q') \; \delta(p-p')$$

Solution is simply:

$$G_c(q,q',p,p';t,t') = \theta(t-t') \; \delta\big(q - \tilde{q}(t) - q' + \tilde{q}(t')\big) \; \delta\big(p - \tilde{p}(t) - p' + \tilde{p}(t')\big)$$

# Non-abelian gauge field

Idea: Use generalized Gaussian wave packets to represent Wigner function in the link representation of lattice gauge fields [Gong, BM, Biró, Nucl. Phys. A568 (1994) 727].

Lattice (KS) Hamiltonian:

$$H = \frac{g^2}{a}\left(\sum_l \frac{1}{2}E_l^a E_l^a + \frac{4}{g^4}\mathrm{Re}\sum_p (1 - \mathrm{tr}\, U_p)\right)$$

can be scaled to dimensionless variables:

$$ag^2 H \to H, \qquad t/a \to t, \qquad g^2 E \to E, \qquad g^2\hbar \to \hbar$$

Wave packet *ansatz*:

$$\Phi[U_l] = \prod_l \phi_l(U_l) = \prod_l \frac{1}{\sqrt{N_l}} \exp\left(\frac{b_l}{2}\mathrm{tr}(U_l U_{l0}^{-1}) - \frac{1}{\hbar}\mathrm{tr}(E_{l0} U_l U_{l0}^{-1})\right)$$

Continuum limit:

$$\Phi\left[(A(x)\right] = \frac{1}{\sqrt{N}} \exp\left[\int d^3x \sum_a \left(-\frac{b(x)}{8}(A^a(x) - A_0^a(x))^2 + \frac{i}{\hbar}E_0(x)^a A^a(x)\right)\right]$$

# Time evolution

Semiclassical transport equation for Wigner functional is realized by variational principle for wave packets. $\hbar$ then enters in two distinct ways:

$$\delta \int_{t1}^{t2} \langle \Phi | (i\hbar\, \partial_t - H) | \Phi \rangle = 0 \qquad \text{and} \qquad [E_l^a, A_m^b] = -i\hbar \delta^{ab} \delta_{lm}$$

$$\Phi[U_l] = \prod_l \phi_l(U_l) = \prod_l \frac{1}{\sqrt{N_l}} \exp\left( \frac{b_l}{2} \mathrm{tr}(U_l U_{l0}^{-1}) - \frac{1}{\hbar} \mathrm{tr}(E_{l0} U_l U_{l0}^{-1}) \right)$$

Variation is performed with respect to parameters: $b_l$(t), $U_{l0}$(t), $E_{l0}$(t).

Write $b = v + \mathrm{i}w$ ; then express $H$ in terms of parameters:

$$H_{\mathrm{eff}} = \sum_l \left[ \frac{1}{2}(1 - \frac{f(v_l)}{2v_l}) E_{l0}^2 + \hbar^2 \frac{3f(v_l)}{16 v_l} |b_l|^2 \right] + 4 \sum_p (1 - f_p\, U_{p0})$$

where $f(v) = I_2(2v)/I_1(2v)$ and $f_p = \prod_{l \in p} f(v_l)$

Limits: $\hbar \to \infty \Rightarrow v_l \to 0$ i.e. wave packet covers all SU(2); $\hbar \to 0 \Rightarrow v_l \to \infty$ .

# Eqs. of motion

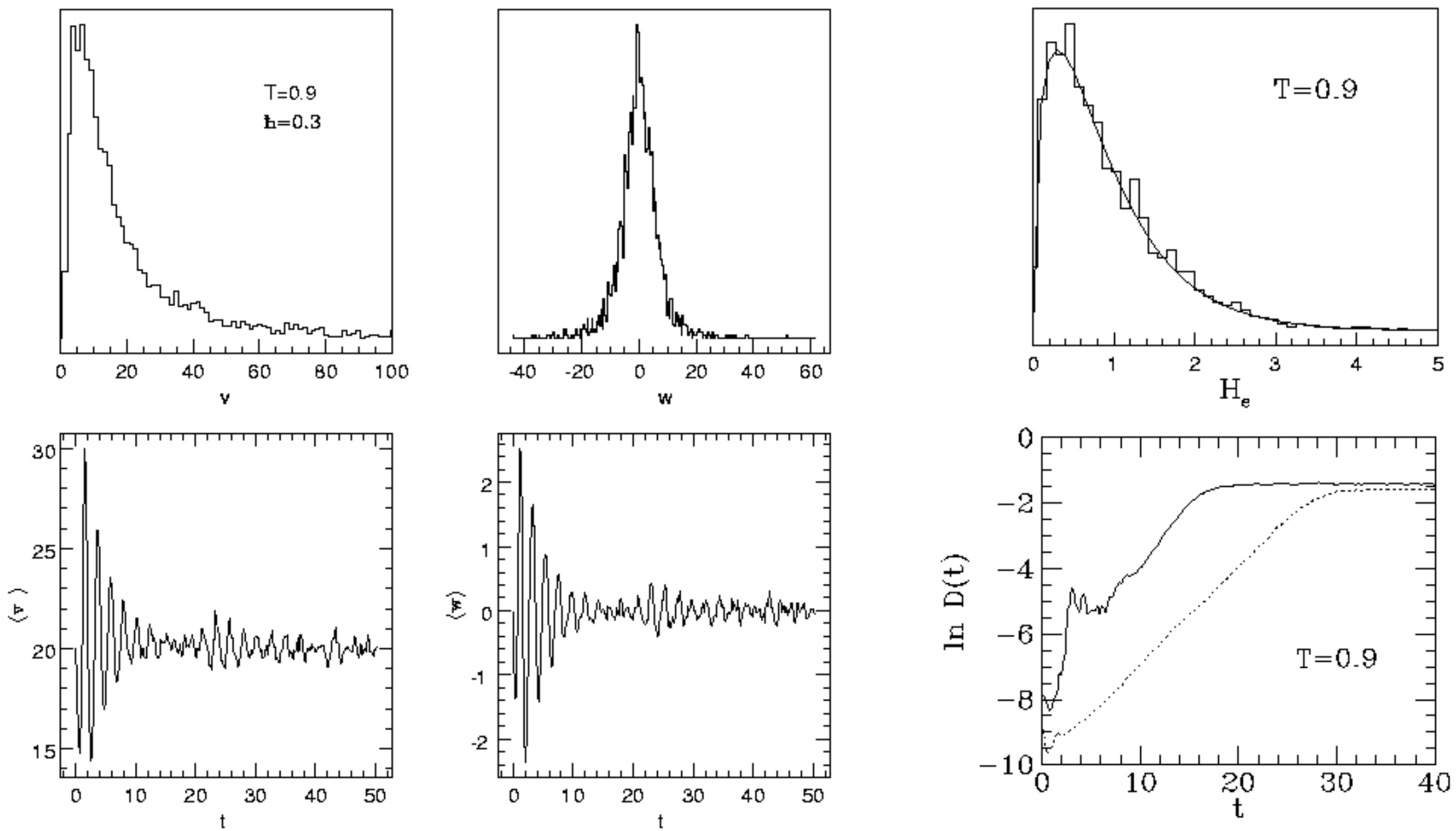
$$
\begin{aligned}
\frac{dE_l^a}{dt} &= \frac{i}{f(v_l)} \sum_{p(l)} (f_p \tau^a U_p) - \frac{3\hbar}{8} \frac{w_l}{v_l} E_l^a, \\
\frac{dU_l}{dt} &= \frac{i}{2} \left( \frac{1}{f(v_l)} - \frac{1}{2v_l} \right) E_l U_l, \\
\frac{dv_l}{dt} &= \frac{3\hbar}{8} \frac{f(v_l)}{f'(v_l)} \frac{w_l}{v_l}, \\
\frac{dw_l}{dt} &= \left( \frac{1}{f(v_l)} - \frac{1}{2v_l} \right) E_l^2 + \frac{E_l^2}{4v_l} + \frac{2}{f(v_l)} \sum_{p(l)} f_p U_p - \frac{3}{16} \hbar^2 \left( v_l + \frac{w_l^2}{v_l} \right), \\
&\quad - \frac{f(v_l)}{f'(v_l)} \left( \frac{E_l^2}{4v_l^2} + \frac{3}{16} \hbar^2 \left( 1 - \frac{w_l^2}{v_l^2} \right) \right),
\end{aligned}
$$

$\hbar \to 0$, $v_l \to \infty \Rightarrow$ classical limit: $\frac{dE_l^a}{dt} = i \sum_{p(l)} (\tau^a U_p), \quad \frac{dU_l}{dt} = i E_l U_l.$

Reminder: $\hbar \leftrightarrow g^2 \hbar = 4\pi\alpha_s.$

# Results - 1

Run simulation from randomly chosen initial conditions (but identical width $b_l$) until the system thermalizes. Measure $T$ from electric energy distribution and determine maximal Lyapunov exponent. Compare with classical limit. Fit gives: $\langle v \rangle \approx 7/\hbar$.



# Results - 2

$$\lambda(T,\hbar) = \lambda_c(T)(1+\delta\lambda)$$

δλ > 0 and grows with *ħ* !

| $\hbar$ | $T$ | $\lambda$ | $\delta\lambda$ | $\langle v \rangle$ | $E$ |
|---|---|---|---|---|---|
| 0.1 | 0.76 | 0.255 | 0.01 | 70 | 2.13 |
| 0.3 | 0.64 | 0.24 | 0.12 | 23 | 2.48 |
| 0.3 | 0.73 | 0.3 | 0.22 | 22 | 2.92 |
| 0.3 | 0.85 | 0.34 | 0.2 | 21 | 3.31 |
| 0.3 | 0.85 | 0.36 | 0.27 | 21 | 3.25 |
| 0.3 | 0.85 | 0.34 | 0.2 | 22 | 3.25 |
| 0.3 | 0.9 | 0.41 | 0.37 | 20 | 3.74 |
| 0.7 | 0.5 | 0.19 | 0.16 | 9.0 | 3.24 |
| 0.7 | 0.8 | 0.41 | 0.54 | 7.4 | 4.40 |

# Conclusions

Wigner functional method provides a practical method to simulate quantum corrections to real-time evolution of fields in quantum field theory. Various approximation methods are available:

- Gaussian wave packets (with dynamical width)
- Green function techniques
- Mean field approximation

WF method permits smooth interpolation between field eigenstates and particle excitations. This allows for the approximate treatment of quantum coherence and uncertainty relation effects.

WF method has been applied to SU(2) gauge field with “Gaussian” SU(2) wave packets and variational time evolution. Results of a limited exploration suggest that $\lambda_{max}$ grows with $\hbar = 4\pi\alpha_s$.

Generalization to fermions, using techniques of AMD or FMD, possible?