

## PHY-105: Equations of Stellar Structure

To summarise where we currently are, we have the following equations to describe stellar structure:

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}, \quad \frac{dM_r}{dr} = 4\pi r^2\rho$$

which are the equations of “hydrostatic equilibrium” and “conservation of mass” respectively. In addition:

$$\frac{dL_r}{dr} = 4\pi r^2\rho\epsilon$$

which describes the luminosity with some radius  $r$  given the energy generation  $\epsilon(r)$ . As we’ve seen a lot goes into  $\epsilon$  – we first have to know what nuclear interactions occur, and at what relative rates, and then what energy production rate results from these interactions. We’ve seen this is highly dependent on temperature, so we also need to know  $T(r)$ . From our knowledge of how energy is transported throughout the stellar material we can determine how this affects the temperature. We showed (or at least qualitatively discussed) that:

$$\frac{dT}{dr} = -\frac{3\kappa L_r\rho}{16\pi a c r^2 T^3}$$

for the temperature variation due to energy transported by **radiation**. Note that there will also be an equation for  $\frac{dT}{dr}$  for convection which is not relevant for most stellar interiors but can become important near a star’s surface. Here we won’t consider this however. Energy transport by conduction can be neglected in all cases.

For the temperature gradient ( $\frac{dT}{dr}$ ) equation, the factor  $\kappa$  is the “opacity” of the stellar material (which is also a function of  $r$ ) – see previous handout for more discussion of opacity.

These 4 equations have 7 unknowns (at a given  $r$ ):  $P$ ,  $M_r$ ,  $L_r$ ,  $T$ ,  $\rho$ ,  $\epsilon$ ,  $\kappa$ . So in general we require expressions for  $P$ ,  $\kappa$ , and  $\epsilon$  in terms of  $\rho$ ,  $T$ , and the compositions. These can be complicated, but for example if we assume an ideal gas the  $P = (N/V)kT$  and we also have for the radiation pressure,  $P_{rad} = \frac{1}{3}aT^4$ . We’ve also seen that  $\epsilon$  and  $\kappa$  can be calculated using known physics. So that we have in the end 7 equations for the 7 unknowns as functions of  $r$ .

I don’t want to get too much further into the solving of these equations for the sake of time. But we will refer back to these equations when we next discuss the final stages of stellar evolution.

For now, let me just make a couple of further comments. The solution of these equations requires “boundary conditions”. The *central boundary conditions* we use are  $M_r \rightarrow 0$  and  $L_r \rightarrow 0$  as  $r \rightarrow 0$ . The surface boundary conditions are trickier. If stars are truly isolated then  $\rho$ ,  $P \rightarrow 0$  at the surface. But no sharp edge really exists for stars. However, we note that for the Sun  $\rho \sim 10^{-4} kgm^{-3}$  at the visible surface which is much less than the average density  $\bar{\rho} \sim 1.5 \times 10^3 kgm^{-3}$ . So we can take as approximations:  $\rho = 0$ ,  $P = 0$ ,  $T = 0$  at  $r = r_s$ .

We then see that:

$\frac{dM_r}{dr}$  depends on  $\rho$ ;

$\frac{dP}{dr}$  depends on  $M_r$ ,  $\rho$ ;

$\frac{dT}{dr}$  depends on  $T$ ,  $\rho$ ,  $\kappa$ ,  $L_r$ ;

$\frac{dL_r}{dr}$  depends on  $\rho$ ,  $\epsilon$ ;

$\rho$ ,  $\kappa$  and  $\epsilon$  depends on  $P$ ,  $T$ , and composition (at  $r$ ).

Which, with the boundary conditions gives us a theorem in astrophysics that states that:

“The mass ( $M_r(r)$ ) and composition of a star uniquely determines its radius, luminosity, and internal structure, as well as its subsequent evolution.”

This is known as the Vogt-Russell theorem. The dependence of a star’s evolution on mass and composition is a consequence of the change in composition due to nuclear burning.

We need to integrate these differential equations (numerically in practice – analytical solutions impossible) to obtain solutions.

Now we’ll discuss some consequences of these equations.....