PHY-105: Nuclear Reaction Rates

We have shown that nuclear reactions can provide the necessary energy to power the Sun (and therefore typical stars in general) and that the core temperature of stars is sufficient for nuclear reactions to occur (though recall quantum tunneling is necessary to explain this). However, to understand the rate at which energy is being produced from nuclear reactions (required in order to get an equation to solve for the star's luminosity) we need an understanding of the probability that a particular nuclear reaction actually occurs. This will depend on the stellar conditions in which these reactions occur: density, temperataure, abundances of interacting particles.

The probability of a given nuclear reaction occuring can be broken into two terms:

$$Prob[\text{nuclear reaction occuring}]$$

$$=$$

$$Prob[\text{particles have energy in the range (E, E + dE)}]$$

$$\times$$

$$Prob[\text{particles with this energy will overcome the Coulomb Barrier and interact}]$$

The first term we know how to calculate from the Maxwell-Boltzmann distribution (go back and review this). At a given temperature T, this term will be exponentially falling with E (at $E > kT$) due to the $e^{-E/kT}$ term in the M-B equation.

The second term is much trickier to calculate and the details are beyond this course, but lets look at least qualitatively at what it involves so you can gain some intuition for the final result. This term is defined by the "cross-section" for the interaction, $\sigma(E)$ (you can think of this as simply the probability a reaction occurs at a given energy, E):

$$\sigma(E) = \frac{\text{\# of reactions per nucleus per unit time}}{\text{\# of incident nuclei per unit area per time}}$$

It is apparent the units of σ is m^2. However, σ's are more commonly expressed in units of "barns" (b) where $1 \; barn = 10^{-28} m^2$. You'll see typical cross-sections expressed in mb, nb, pb, etc..

Let R_{ij} be the reaction rate (number of reactions occuring per unit mass of available nuclei per unti time) between two particles, $i \neq j$. R_{ij} will be directly proportional to the probability of nuclear reactions occuring as discussed on the previous page.

Suppose two interacting nuclei (note I'm using "nuclei" and "particles" interchangeably) have mass numbers A_i and A_j (and therefore masses of approximately $A_i m_H$ and $A_j m_H$ – approximately because the neutron mass is slightly larger than the proton mass), and charges $Z_i e$ and $Z_j e$ (where $e = 1.6 \times 10^{-19} C$, the charge of a proton (and an electron but negative)). Also, let $X_{i(j)}$ be the fraction by mass of the stellar material of nucleus $i(j)$. A calculation of σ (beyond this course) leads to expression of the form:

$$\sigma(E) = \frac{f(E)}{E} e^{-b/\sqrt{E}}$$

where $f(E)$ is some (typically) slowly varying function of E (so that for most purposes we can take it to be a constant), b is a constant that depends on the masses and charges of the interacting nuclei, and the exponential term comes from the probability to penetrate the Coulomb Barrier (which as you can see increases with increasing E). This results in an expression for R_{ij} of the form:

$$R_{ij} = C\rho \left(\frac{X_i X_j}{A_i A_j}\right) \left(\frac{1}{A Z_i Z_j}\right) \tau^2 e^{-\tau}$$

where C is a constant that depends on the nuclei properties, ρ is the density of the stellar material and,

$$\tau = 4.25 \times 10^3 \left(\frac{Z_i^2 Z_j^2 A}{T}\right)^{1/3}, \quad A = \frac{A_i A_j}{A_i + A_j}.$$

A few notes:

- When T is small, τ is large and $e^\tau \Rightarrow$ very small reaction rates.

- As τ decreases (T increases) R increases rapidly until T high enough when R starts to drop again (M-B exponential term starts to dominate). R_{max} occurs at the "Gamow peak" at $E_0 \sim T^{2/3}$

- In practice interested in T's at which R_{ij} is rising.

It is often more useful to write R_{ij} as a simple power law (from the power law expansion of R_{ij} centered in a particular temperature range. We can write for a given temperature range:

$$R_{ij} \approx R_0 X_i X_j \rho^\alpha T^\beta$$

where R_0 is a constant, and usually $\alpha = 1$ (for a collision between 2 nuclei) and β can range anywhere from ~ 1 to > 40.

Now, if ϵ_0 is the energy released per reaction, and ϵ_{ij} is the energy released per unit mass of material per second, then:

$$\epsilon_{ij} = \epsilon_0 R_{ij} \quad \Rightarrow \quad \epsilon_{ij} = \epsilon_0' X_i X_j \rho^\alpha T^\beta \quad (J\,kg^{-1}s^{-1})$$

The total nuclear energy generation rate will be the sum over all all possible reactions: $\epsilon_{nuclear} = \sum_{i,j} \epsilon_{i,j}$.

To determine the luminosity, L, of a star, consider all energy generated by stellar material. The contribution to L from a mass element dm is $dL = \epsilon dm$, where in general $\epsilon = \epsilon_{nuclear} + \epsilon_{grav}$. For typical stars during most of the evolution the gravitation term is negligible (but could for example be negative if the star is expanding).

For a spherically symmetric mass distribution, the mass of a thin shell of thickness dr is:

$$dm = \rho dV = 4\pi r^2 \rho dr \quad \Rightarrow \quad \frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon$$

where, L_r is the "interior luminosity" due to all energy generated within the star interior to radius r.

To calculate $\epsilon_{nuclear}$ we need to know the sequence of nuclear reaction steps by which one element is converted to another and the energy each step releases.

For typical stars in their early burning stage (where most of the core is hydrogen) the most common reaction is $4\,_1^1H \rightarrow \,_2^4He$ but this must proceed through a sequence of 2-body reactions. One possible chain of reactions is the so-called PP-I chain.

PHY-105: Summary of Nuclear Reactions in Stars

The most important series of nuclear reactions occuring in stars convert hydrogen to helium. Their are two basic reaction chains: the **proton-proton (PP)** chain and the **carbon-nitrogen-oxygen (CNO)** cycle.

The PP chain divides into three main branches called PP I, PP II, and PP III.

PP I reaction chain:

$$
\begin{array}{lrcl}
(1) & {}^1_1\text{H} + {}^1_1H & \rightarrow & {}^2_1\text{H} + e^+ + \nu_e \\
(2) & {}^2_1\text{H} + {}^1_1\text{H} & \rightarrow & {}^3_2\text{He} + \gamma \\
(3) & {}^3_2\text{He} + {}^3_2\text{He} & \rightarrow & {}^4_2\text{He} + 2\,{}^1_1\text{H}
\end{array}
$$
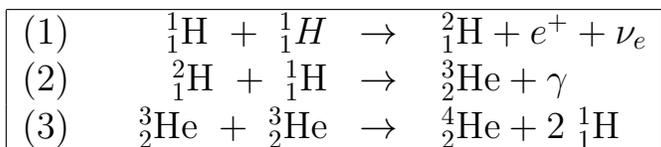
Note that the result of this chain is: $4\,{}^1_1\text{H} \rightarrow {}^4_2\text{He} + 2e^+ + 2\nu_e + 2\gamma$

PP II reaction chain: this starts with reactions (1) and (2), then:

$$
\begin{array}{lrcl}
(3') & {}^3_2\text{He} + {}^4_2\text{He} & \rightarrow & {}^7_4\text{Be} + \gamma \\
(4') & {}^7_4\text{Be} + e^- & \rightarrow & {}^7_3\text{Li} + \nu_e \\
(5') & {}^7_3\text{Li} + {}^1_1\text{H} & \rightarrow & 2\,{}^4_2\text{He}
\end{array}
$$
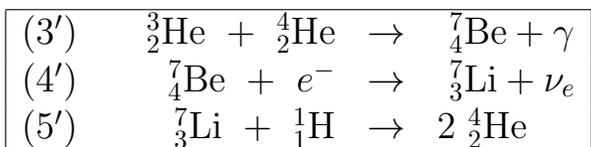
PP III reaction chain: this starts with reactions (1), (2) and (3'), then:

$$
\begin{array}{lrcl}
(4'') & {}^7_4\text{Be} + {}^1_1\text{H} & \rightarrow & {}^8_5\text{B} + \gamma \\
(5'') & {}^8_5\text{B} + e^- & \rightarrow & {}^8_4\text{Be} + e^+ + \nu_e \\
(6'') & {}^8_4\text{Be} & \rightarrow & 2\,{}^4_2\text{He}
\end{array}
$$
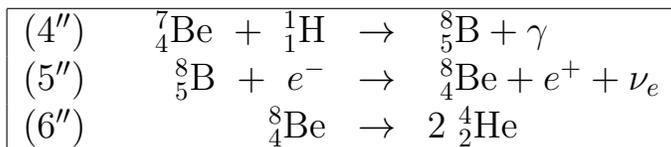
In the core of the Sun the following ratios of the PP chains occur:

(1) and (2)

69%   31%

(3)   (3')
[PP I]

99.7%   0.3%

(4') and (5')   (4''), (5'') and (6'')
[PP II]   [PP III]
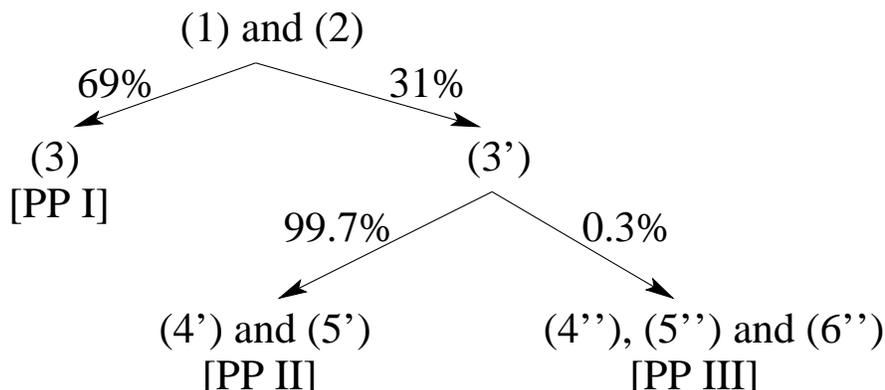
CNO cycle (proposed by Hans Bethe in 1938):

$$
\begin{array}{llrcl}
(1) & & {}^{12}_{6}\text{C} + {}^{1}_{1}\text{H} & \rightarrow & {}^{13}_{7}\text{N} + \gamma \\
(2) & & {}^{13}_{7}\text{N} & \rightarrow & {}^{13}_{6}\text{C} + e^{+} + \nu_{e} \\
(3) & & {}^{13}_{6}\text{C} + {}^{1}_{1}\text{H} & \rightarrow & {}^{14}_{7}\text{N} + \gamma \\
(4) & & {}^{14}_{7}\text{N} + {}^{1}_{1}\text{H} & \rightarrow & {}^{15}_{8}\text{O} + \gamma \\
(5) & & {}^{15}_{8}\text{O} & \rightarrow & {}^{15}_{7}\text{N} + e^{+} + \nu_{e} \\
(6) & & {}^{15}_{7}\text{N} + {}^{1}_{1}\text{H} & \rightarrow & {}^{12}_{6}\text{C} + {}^{4}_{2}\text{He}
\end{array}
$$
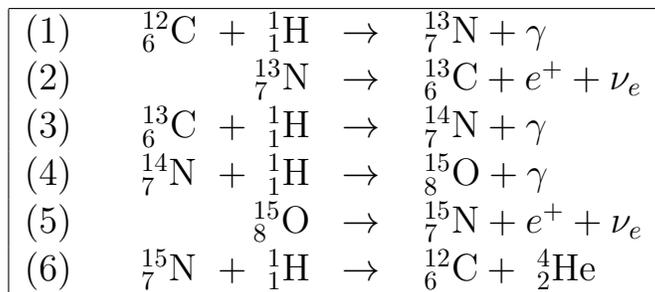
Note that essentially what is happening in the CNO cycle is 4 protons are being added in succession to a carbon nucleus. In two cases (reactions 2 and 5) the addition of the proton is immediately followed by a β-decay, and at the end of the cycle a carbon nucleus remains with a helium nucleus.

See class notes for more details on these reactions.