Phy-105 HW #6 & 7 Solutions Kruse, April '11

1 Schwarzschild radius, $r_{sch} = \frac{2GM}{c^2}$

For:

(a) $M = 40 M_{\odot} = 40\,(2\times10^{30}\,kg)$

$$r_{sch} = \frac{2(6.672\times10^{-11})(2\times10^{30}\,kg)\,40}{(3\times10^{8})^2} \quad \text{(in SI units)}$$

$\Rightarrow r_{sch} = 1.2\times10^{5}\,m \quad (= 120\,km)$

(b) ~~For~~ the Sun, $r_{sch} = \frac{1}{40}(1.2\times10^{5}\,m)$

$\Rightarrow r_{sch} = 3.0\times10^{3}\,m \quad (3\,km)$

(c) the Earth; $M = 5.97\times10^{24}\,kg$

$$\Rightarrow r_{sch} = \frac{2(6.672\times10^{-11})(5.97\times10^{24})}{(3\times10^{8})^2}$$

$\Rightarrow r_{sch} = 8.8\times10^{-3}\,m \quad (0.88\,cm\,!)$

(d) a proton; $m = 1.67\times10^{-27}\,kg$

$$\Rightarrow r_{sch} = \frac{2(6.672\times10^{-11})(1.67\times10^{-27})}{(3\times10^{8})^2}$$

$\Rightarrow r_{sch} = 2.5\times10^{-54}\,m$

(this is smaller, by several orders of magnitude, than the so-called "Planck length", something we'll mention again later, and which gives us limitations on the physics we can possibly know)

2 (a) We derived in class (though you should go through the steps here), the gravitational redshift

$$z = \frac{\Delta\lambda}{\lambda_0}, \quad \text{where } \lambda = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} \lambda_0$$

For small corrections to λ_0 we get:
$$z \approx \frac{GM}{rc^2}$$

For $M = 2M_\odot = 4\times10^{30}$ kg, and, $r = R_{WD} = 3\times10^{8}$ m, we get:

$$z = \frac{(6.67\times10^{-11})(4\times10^{30})}{(3\times10^{8})(3\times10^{8})^2} \quad \text{(in SI units)}$$

$$\Rightarrow \underline{\underline{z \approx 1\times10^{-5}}}$$

(b) time dilation: $\Delta t = \Delta t_\infty - \Delta t_0 = \Delta t_\infty - \left(1 - \frac{2GM}{rc^2}\right)^{1/2}\Delta t_\infty$

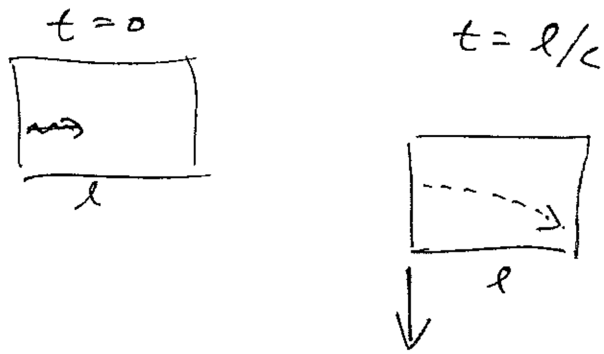
$$\left[d\tau = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} dt\right]$$

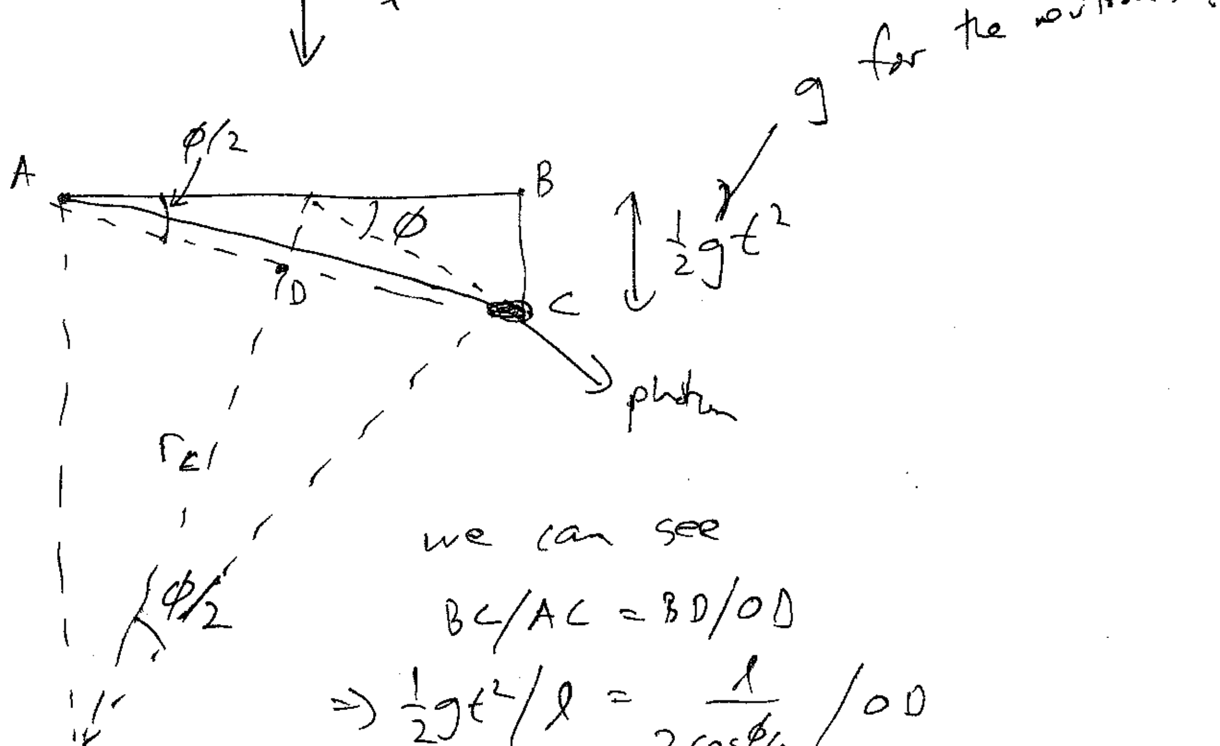
proper time $= dt_0$ — time measured by us $= dt_\infty$

$$\approx \left(\frac{GM}{rc^2}\right)\Delta t_\infty$$
$$= (1\times10^{-5})\,\Delta t_\infty .$$

That is, the time dilation factor is about 10^{-5}, meaning that the time as measured by us is slower by about 1 part in 100 000, ~~[illegible]~~ than the time as measured on the surface of the WD.

2(c) Consider the equivalence principle for a horizontally travelling γ — that is, the path of the photon in a locally ~~[illegible]~~ reference frame compared to that as seen in a frame with no gravity.





we can see

$$BC/AC = BD/OD$$

$$\Rightarrow \tfrac{1}{2}gt^2/l = \frac{l}{2\cos\phi/2} \Big/ OD$$

ϕ small $\Rightarrow$ $\cos\phi/2 \approx 1$, and $OD \approx r_c$

then, since $t = l/c$,

$$r_c = \frac{c^2}{g}, \quad \text{where } g = \frac{GM_{NS}}{R_{NS}^2}.$$

3 & 4 see class notes.

5 (a) The radial distance measured simultaneously ($dt=0$) between 2 points on the same radial line ($d\theta = d\phi = 0$) is the proper distance,

$$dL = \sqrt{-(ds)^2} = \frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}} = \frac{dr}{\sqrt{1-\frac{r_{sch}}{r}}}$$

$$\rightarrow \Delta L = \int_{r_{sh}}^{r} \frac{dr}{\sqrt{1-\frac{r_{sch}}{r}}}$$

This isn't an easy integral to evaluate, but ~~it~~ you can find it using mathematica for example, to get the given answer.

(b) As $r \to \infty$ $\quad \left(1-\frac{r_{sch}}{r}\right)^{1/2} \to 1 - \frac{1}{2}\frac{r_{sch}}{r}$

$$\Rightarrow \Delta L \cong r\left(1-\frac{1}{2}\frac{r_{sch}}{r}\right) + \frac{r_{sch}}{2}\ln\left(\frac{2-\frac{1}{2}\frac{r_{sch}}{r}}{\frac{1}{2}\frac{r_{sch}}{r}}\right) \quad \text{for large } r$$

$$= r - \frac{1}{2}r_{sch} + \frac{r_{sch}}{2}\ln\left(\frac{2r-\frac{1}{2}r_{sch}}{\frac{1}{2}r_{sch}}\right)$$

$$= r + \frac{r_{sch}}{2}\ln\left(2r-\frac{1}{2}r_{sch}\right) \underbrace{- \frac{1}{2}r_{sch} - \ln\left(\frac{1}{2}r_{sch}\right)}_{\text{constant}}$$

As $r \to \infty$ the $\ln 2r$ term increases much more slowly than r $\left(\lim_{r\to\infty}(r + \ln r) = r\right)$, so that

$\Delta L \to r$ as $r \to \infty$.

(16) & (17) do similarly as in class notes using conservation of angular momentum ($I_i \omega_i = I_f \omega_f$, $I = \frac{2}{5}MR^2$ for a sphere) and magnetic flux ($B_i R_i^2 = B_f R_f^2$).

(18) Using WD as starting point gives very different ~~res~~ result — but a more correct result.

As a star collapses to a WD, significant mass from the outer layers is lost, and the core becomes separated from those outer layers, so that angular momentum is <u>NOT</u> conserved.

9 (a) $$\left[\frac{1}{R}\left(\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right]R^2 = -kc^2$$ (where ρ and R are functions of t)

Since $R^3\rho = \text{const.} \Rightarrow R^3\rho = R_0^3\rho_0 = \rho_0 \Rightarrow \rho = \frac{\rho_0}{R^3}$

$$\Rightarrow \left(\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho_0 \frac{1}{R} = -kc^2$$

(b) Flat universe, $k=0$

$$\Rightarrow \left(\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G\rho_0 \frac{1}{R}$$

$$\Rightarrow R^{1/2}dR = \left(\frac{8}{3}\pi G\rho_0\right)^{1/2} dt$$

$$\Rightarrow \int_0^R R'^{1/2}dR' = \left(\frac{8}{3}\pi G\rho_0\right)^{1/2}\int_0^t dt'$$ (taking $R=0$ at $t=0$)

$$\Rightarrow \frac{2R^{3/2}}{3} = \left(\frac{8}{3}\pi G\rho_0\right)^{1/2} t$$

Using $H_0^2 = \frac{8}{3}\pi G\rho_0 = \frac{1}{t_H^2} \Rightarrow \frac{2R^{3/2}}{3} = \frac{t}{t_H}$

$$\Rightarrow \boxed{R(t) = \left(\frac{3}{2}\cdot\frac{t}{t_H}\right)^{2/3}}$$

Closed universe, $k>0$ - show given solution works

$$\frac{dR}{dt} = \frac{dR}{dx}\cdot\frac{dx}{dt} = \left(\frac{1}{2}\cdot\cancel{\frac{\Omega_0}{\Omega_0-1}}\sin x\right)\cdot\frac{2H_0(\Omega_0-1)^{3/2}}{\cancel{\Omega_0}}\cdot\frac{1}{(1-\cos x)}$$ $\left(\frac{dt}{dx} = \frac{\Omega_0}{2H_0(\Omega_0-1)^{3/2}}(1-\cos x)\right)$

$$= H_0(\Omega_0-1)^{1/2}\cdot\frac{\sin x}{1-\cos x}$$

Substituting this, and, $R(t) = \frac{1}{2}\frac{\Omega_0}{(\Omega_0-1)}(1-\cos x)$ into differential equation for $R(t)$:

$$H_0^2(\Omega_0-1)\frac{\sin^2 x}{(1-\cos x)^2} - \underbrace{\frac{8}{3}\pi G\rho_0}_{H_0^2}\frac{2(\Omega_0-1)}{\Omega_0}\cdot\frac{1}{(1-\cos x)} = -kc^2$$

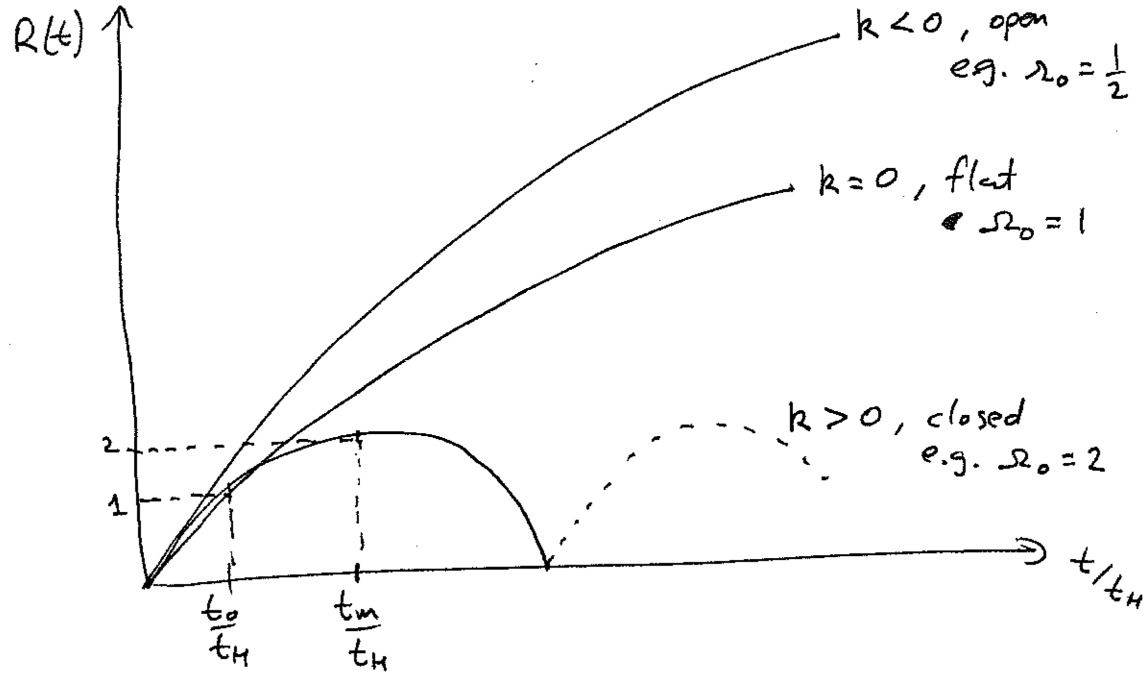
$$\Rightarrow H_0^2(\Omega_0-1)\underbrace{\left[\frac{\sin^2 x}{(1-\cos x)^2} - \frac{2}{(1-\cos x)}\right]} = -kc^2$$

$$\frac{\sin^2 x - 2 + 2\cos x}{(1-\cos x)^2} = \frac{\sin^2 x - 2\sin^2 x - 2\cos^2 x + 2\cos x}{(1-\cos x)^2} = \frac{-1-\cos^2 x + 2\cos x}{(1-\cos x)^2}$$
$$= -\frac{(1-\cos x)^2}{(1-\cos x)^2}$$
$$= -1$$

$$\Rightarrow H_0^2(\Omega_0-1)(-1) = -kc^2$$

From our definition of Ω_0 we saw in class that $H_0^2(\Omega_0-1) = kc^2$ thus verifying the soln. The open solution ($k>0$) is done similarly !!

(c)



Aside

To get an idea of what value of $\frac{t}{t_H}$ corresponds to a maximum in the closed universe case, (labelled as $\frac{t_m}{t_H}$, lets choose a closed universe with $\Omega_0 = 2$ (i.e. one with twice the critical density).

In this case we get: $R(t) = 1 - \cos x$, $t = \frac{1}{H_0}(x - \sin x)$

$$\Rightarrow \frac{t}{t_H} = x - \sin x$$

At $\frac{t_m}{t_H}$, $\frac{dR}{dt} = 0 \Rightarrow \frac{dR}{dx} \cdot \frac{dx}{dt} = 0$

$$\Rightarrow (\sin x) \frac{H_0}{1 - \cos x} = 0$$

$$\Rightarrow \sin x = 0$$

$$\Rightarrow x = \pi$$

(or ϕ, but this is not a valid solution as $(1 - \cos x)^{-1}$ blows up)

That is, $\boxed{\frac{t_m}{t_H} = \pi.}$

Also, $R(t) = 1 - \cos(\pi) = 2$. i.e. universe will be double the size it is now, if it is closed with $\Omega_0 = 2$.

This can be compared to $\frac{t_0}{t_H} \approx 0.5$ for age of universe in this model (see sketch)

(d) Using $R = \frac{1}{1+z}$ (both functions of time) we get for a:

Flat universe

$$\frac{1}{1+z} = \left(\frac{3}{2}\cdot\frac{t}{t_H}\right)^{2/3} \Rightarrow \frac{t}{t_H} = \frac{2}{3}\cdot\frac{1}{(1+z)^{3/2}}$$

Closed universe

$$\frac{1}{1+z} = \frac{1}{2}\cdot\frac{\Omega_0}{\Omega_0-1}(1-\cos x)$$

$$\Rightarrow 1-\cos x = \frac{2(\Omega_0-1)}{\Omega_0(1+z)} \Rightarrow \cos x = 1 - \frac{2(\Omega_0-1)}{\Omega_0(1+z)}$$

$$\Rightarrow \cos x = \frac{\Omega_0+\Omega_0 z - 2\Omega_0 + 2}{\Omega_0(1+z)}$$

$$\Rightarrow \cos x = \frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0(1+z)}$$

$$\Rightarrow x = \cos^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0(1+z)}\right)$$

Also need $\sin x$:

$$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1 - \frac{(\Omega_0 z - \Omega_0+2)^2}{(\Omega_0(1+z))^2}}$$

$$= \frac{\sqrt{\cancel{\Omega_0^2} + 2\Omega_0^2 z + \cancel{\Omega_0^2 z^2} - (\cancel{\Omega_0^2 z^2} + 2(2-\Omega_0)(\Omega_0 z) + \overset{4+4\Omega_0+\Omega_0^2}{\cancel{(2-\Omega_0)^2}})}}{\Omega_0(1+z)}$$

$$= \frac{\sqrt{4\Omega_0^2 z - 4\Omega_0 z - 4 + 4\Omega_0}}{\Omega_0(1+z)}$$

$$= \frac{2\sqrt{\Omega_0^2 z - \Omega_0 z - 1 + \Omega_0}}{\Omega_0(1+z)}$$

$$= \frac{2\sqrt{(\Omega_0-1)(\Omega_0 z+1)}}{\Omega_0(1+z)}$$

Therefore:

$$\frac{t}{t_H} = \frac{1}{2}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}\left[\cos^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0(1+z)}\right) - \frac{2\sqrt{(\Omega_0-1)(\Omega_0 z+1)}}{\Omega_0(1+z)}\right]$$

Open Universe

Do similarly, first from $\cosh x = \ldots$, then finding $\sinh x$

(e) Age of universe

set $z=0$ (i.e. "now") in $\frac{t(z)}{t_H}$.

$\Rightarrow$ for flat universe, $\boxed{\frac{t_0}{t_H} = \frac{2}{3} \Rightarrow t_0 = 9.1\times 10^9 \text{ yrs}}$

for closed universe, with $\Omega_0 = 2$

$$\frac{t_0}{t_H} = \cos^{-1}(0) - 1 \Rightarrow \boxed{\frac{t_0}{t_H} = \frac{\pi}{2} - 1 \Rightarrow t_0 \approx 7.8\times 10^9 \text{ yrs}}$$

for closed universe, with $\Omega_0 = \frac{1}{2}$

$$\frac{t_0}{t_H} = \frac{\frac{1}{2}}{2\left(\frac{1}{2}\right)^{3/2}}\cdot\left(-\cosh^{-1}(3) + \frac{2\sqrt{\frac{1}{2}}}{\frac{1}{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(-1.76 + \frac{4}{\sqrt{2}}\right)$$

$$\Rightarrow \boxed{\frac{t_0}{t_H} = 0.755 \Rightarrow t_0 \approx 10.3\times 10^9 \text{ yrs}}$$

Problem 10
(a) Maximum occurs when $dR/dt = 0$. You should find that this occurs when $x = \pi$, giving:

$$R_{max} = \frac{\Omega_0}{\Omega_0 - 1}$$

(b) If $\rho_0 = 2\rho_c$, then $\Omega_0 = 2$, giving $R_{max} = 2$.
That is, the universe would expand to twice its current "size" before beginning to contract. Since R_{max} occurs at $x = \pi$, this corresponds to a time of $t = \pi t_H$ or $t = \pi/H_0$, giving the time between the "Big Bang" and the "Big Crunch" of $t_{BC} = 2\pi t_H$. Note that in this case of $\Omega_0 = 2$, the present time is $t_0 = (\pi/2 - 1)t_H$, so that the Uinverse would currently be about 10% through its evolution to the Big Crunch.

Problem 11
The time at which the energies of particles in the Universe were $\sim 10^3 GeV$, is given by,

$$T(t) = (1.52 \times 10^{10} K s^{1/2}) t^{-1/2},$$

where, $T(t)$ is given by,

$$E = kT \quad \Rightarrow \quad T = \frac{10^{12} eV}{8.63 \times 10^{-5} eV/K} = 1.2 \times 10^{16} K$$

giving:

$$\sqrt{t} = \frac{1.52 \times 10^{10} K s^{1/2}}{1.2 \times 10^{16} K} \quad \Rightarrow \quad t = 1.6 \times 10^{-12} s$$

Since our knowledge of particle physics becomes somewhat uncertain at energies higher than about $10^3 GeV$ (for example the electroweak symmetry is broken (giving rise to the masses of the fundamental particles) around this energy, the mechanism for which we are not yet sure of – the Higgs mechanism being one possibility – and which is one of the things we're currently trying to understand at the highest energy particle accelerators (at Fermilab and CERN) which produce collision energies of around $10^3 GeV$).....anyway, it means the certainty in our knowledge of the Universe is currently limited to times greater than about $10^{-12} s$.

Problem 12
Should find: $t \approx 230s$, $R(t) \approx 3 \times 10^{-9}$, and, $z \approx 4 \times 10^8$