Midterm 2
PHY 760 - Fall 2013
October 7, 2013

This closed book exam consists of four Short Answer questions (10 pts.) and three Problems (20 pts.). Please sign the following pledge BEFORE you begin the exam:

I certify that I have abided by the rules and the spirit of the Duke Community Standard.

Sign: _____

Short Answer 1: [10 pts.]

Evaluate the integral

$$\int_0^\infty dt\, \frac{\cos at}{1+t^2}\,.$$

Short Answer 2: [10 pts.]

Find the solution to the 2-D Laplace's equation,

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y) = 0\,,$$

in the region $-\infty < x < \infty$, $y > 0$ subject to the boundary conditions

$$\phi(x,0) = \begin{cases} 0 & x > 0 \\ \phi_0 & x < 0 \end{cases}$$

Short Answer 3: [10 pts.]

Find the general solution to the following equations:

$$a) \qquad x\frac{\partial}{\partial x}u(x,y) + 3y\frac{\partial}{\partial y}u(x,y) = 0$$

with the boundary condition $u(x,1) = \cos(x)$.

$$b) \qquad R\frac{dQ(t)}{dt} + CQ(t) = V_0 e^{-t/\tau}$$

with the initial condition $Q(t) = 0$.

Short Answer 4: [10 pts.]

Find the Green's function satisfying

$$\left(\frac{\partial^2}{\partial x^2} + \omega^2\right) G(x, z) = \delta(x - z)$$

subject to the boundary conditions $G(0, z) = G(L, z) = 0$.

Problem 1: [20 pts.]

Suppose one is solving a differential equation of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

a) Given a solution $y_1(x)$ construct a second solution $y_2(x)$. You result should be expressed as an integral over $y_1(x)$.

b) Show that the Wronskian, $W = y_1(x)y_2'(x) - y_2(x)y_1'(x)$ is equal to

$$W = C \exp\left(-\int^x du\, p(u)\right).$$

where C is an arbitrary constant.

Problem 2: [20 pts.]

Consider the following differential equation:

$$(1 - x^2)y''(x) - 2xy'(x) + l(l + 1)y(x) = 0\,.$$

Assume l is a real number.

a) Find all singular points of the differential equation and determine whether they are regular or irregular singularities.

b) Write down the indicial equation for a series solution and give the roots of the indicial equation.

c) Find the recurrence relation for the coefficients of the series solution for both roots of the indicial equation. Does the Frobenius method yield two independent solutions for this differential equation?

Problem 3: [20 pts.]

Find the inverse Laplace transform of the function

$$\bar{f}(s) = \frac{1}{2} \log\left(\frac{s+1}{s-1}\right) .$$

Useful Equations

Laplace Transform

$$\begin{aligned}
\mathcal{L}[f(t)] &= \int_0^\infty dt e^{-st} f(t) \equiv \bar{f}(s) \\
\mathcal{L}[c] &= c/s \\
\mathcal{L}[t^a] &= \Gamma[a+1]/s^{a+1} \\
\mathcal{L}[e^{at}] &= 1/(s-a) \\
\mathcal{L}[t^b e^{at}] &= \Gamma[b+1]/(s-a)^{b+1} \\
\mathcal{L}[\delta(t-t_0)] &= e^{-st_0} \\
\mathcal{L}[\theta(t-t_0)] &= e^{-st_0}/s
\end{aligned}$$

$$\mathcal{L}\left[\frac{\partial^n f(t)}{\partial t^n}\right] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2}\frac{\partial f(0)}{\partial t} - ... - \frac{\partial^{n-1} f(0)}{\partial t^{n-1}}$$

Inverse Laplace Transform

$$\mathcal{L}^{-1}[\bar{f}(s)] = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{st}\bar{f}(s)$$

Eigenfunction methods

For $\mathcal{L}(x)$ a linear differential operator on the interval $a < x < b$:

$$\begin{aligned}
\mathcal{L}\phi_n(x) &= \lambda_n \phi_n(x) \\
\int_a^b dx \phi_m^*(x)\phi_n(x) &= \delta_{nm} \\
\sum_n \phi_n(x)\phi_n^*(z) &= \delta(x-z) \\
G(x,z) &= \sum_n \frac{\phi_n(x)\phi_n^*(z)}{\lambda_n}
\end{aligned}$$

Green's Function

$$\begin{aligned}
\mathcal{L}G(x,z) &= \delta(x-z) \\
\mathcal{L}\psi(x) &= f(x) \\
\psi(x) &= \int_a^b dz\, G(x,z) f(z)
\end{aligned}$$

For 2nd order linear ODE:

$$\begin{aligned}
G(x,z) &= \begin{cases} y_1(x)y_2(z)/W(y_1,y_2) & a < x < z \\ y_2(x)y_1(z)/W(y_1,y_2) & z < x < b \end{cases} \\
W(y_1,y_2) &= y_1 y_2' - y_2 y_1'
\end{aligned}$$