

Midterm 1
PHY 760 - Fall 2013
October 7, 2013

This closed book exam consists of four Short Answer questions (10 pts.) and three Problems (20 pts.). Please sign the following pledge BEFORE you begin the exam:

I certify that I have abided by the rules and the spirit of the Duke Community Standard.

Sign: Key

Short Answer 1: [10 pts.]

- Give the conditions a matrix U must satisfy in order to be unitary and state what this implies for its eigenvalues.
- Give the condition a matrix M must satisfy in order to be invertible and state what this implies for its eigenvalues.
- Explain why an $n \times n$ real antisymmetric matrix with n odd must have a vanishing determinant.

a) $U^T = U^{-1} \Rightarrow |\lambda_i| = 1$ where λ_i is an eigenvalue.

b) $\det M \neq 0 \Rightarrow \lambda_i \neq 0$ all λ_i

c) $A^T = -A \quad \det A^T = \det (-A) = (-)^n \det A = \det A$

\Rightarrow odd $n \quad \underline{\det A = 0}$

Short Answer 2: [10 pts.]

For the function

$$f(t) = \begin{cases} 1 & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$$

compute its Fourier transform and use the result to evaluate the integral

$$\int_{-\infty}^{\infty} dw \frac{\sin^2 \omega}{\omega^2}.$$

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} \frac{dw}{\sqrt{2\pi}} e^{i\omega t} f(t) \\ &= \int_{-1}^1 \frac{dw}{\sqrt{2\pi}} e^{i\omega t} = \frac{1}{\sqrt{2\pi}} \frac{e^{i\omega} - e^{-i\omega}}{i\omega} = \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega}\end{aligned}$$

$$\int_{-\infty}^{\infty} dw |\hat{f}(\omega)|^2 = \int_{-\infty}^{\infty} dt |f(t)|^2$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} dw \frac{\sin^2 \omega}{\omega^2} = \int_{-1}^1 dt = 2$$

$$\boxed{\int_{-\infty}^{\infty} dw \frac{\sin^2 \omega}{\omega^2} = \pi}$$

Short Answer 3: [10 pts.]

For what values of a and b (where a and b are real) will the following infinite series/products converge?

$$i) \sum_{n=1}^{\infty} \left(1 + \frac{a}{n}\right)^{n^2}$$

$$ii) \prod_{n=1}^{\infty} \frac{1+a/n}{1+b/n}$$

i) root test

$$\lim_{n \rightarrow \infty} c_n^{1/n} = \lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n = e^a < 1$$

converges if $\boxed{a < 0}$

$$\begin{aligned} ii) \ln \prod_{n=1}^N \frac{1+a/n}{1+b/n} &= \sum_{n=1}^N \ln \left(\frac{1+a/n}{1+b/n} \right) \\ &= \sum_{n=1}^N \frac{1}{n} (a-b) + O\left(\frac{1}{n^2}\right) \\ &= \ln N(a-b) + \text{finite} \end{aligned}$$

$$\Rightarrow \prod_{n=1}^N \frac{1+a/n}{1+b/n} \approx N^{a-b} \times \text{finite}$$

$$\Rightarrow \prod_{n=1}^{\infty} \frac{1+a/n}{1+b/n} = \begin{cases} \infty & a > b \\ 1 & a = b \\ 0 & a < b \end{cases}$$

Converges when $\boxed{a \leq b}$

This assumes $b \neq$ negative integer

the product clearly has poles at

$$\therefore b = -\text{integer}$$

Short Answer 4: [10 pts.]

Derive the convolution theorem for the Laplace transform:

$$\mathcal{L}^{-1}[\bar{f}(s)\bar{g}(s)] = \int_0^t du f(u)g(t-u),$$

where $\mathcal{L}^{-1}[\dots]$ denotes the inverse Laplace transform.

$$\begin{aligned}& \mathcal{L}\left[\int_0^t du f(u) g(t-u) \right] \\&= \int_0^\infty dt e^{-st} \int_0^t du f(u) g(t-u) \\&= \int_0^\infty du \int_u^\infty dt e^{-st} f(u) g(t-u) \\&= \int_0^\infty du \int_0^\infty dt' e^{-s(t'+u)} f(u) g(t') \\&= \int_0^\infty du e^{-su} f(u) \int_0^\infty dt' e^{-st'} g(t') \\&= \bar{f}(s) \bar{g}(s)\end{aligned}$$

Problem 1: [20 pts.]

For the matrix M given by

$$M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

with a and b complex numbers.

- a) Give a closed form expression for $\cos(M)$.
- b) State the conditions on a and b for M to have an orthonormal basis of eigenvectors.
- c) Impose the condition found in part b) and verify that M can be written as

$$M = \sum_{i=1}^2 \lambda_i |v_i\rangle\langle v_i|,$$

where λ_i and $|v_i\rangle$ are the eigenvalues and normalized eigenvectors of M , respectively.

a) $M^2 = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = ab \mathbb{1}$

$$\cos M = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n M^{2n}}{2^n n!}}{2} = \boxed{\cos \sqrt{ab} \mathbb{1}}$$

b) $[M, M^\dagger] = 0 \Rightarrow \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a^* & b^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = (|a|^2 - |b|^2) \mathbb{1}$

$$\Rightarrow |a| = |b| \quad \text{or} \quad \boxed{b = a e^{i\theta}}$$

c) $M = \begin{pmatrix} 0 & a \\ a e^{i\theta} & 0 \end{pmatrix}, \quad \det M - \lambda \mathbb{1} = \lambda^2 - a^2 e^{i2\theta} = 0$
 $\Rightarrow \lambda = \pm a e^{i\theta/2}$

eigenvectors = $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm e^{i\theta/2} \end{pmatrix} \quad \begin{pmatrix} 0 & a \\ a e^{i\theta} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \pm e^{i\theta/2} \end{pmatrix} = \frac{\pm a e^{i\theta/2}}{a} \begin{pmatrix} 1 \\ \pm e^{i\theta/2} \end{pmatrix}$
 $= \pm a e^{i\theta/2} \begin{pmatrix} 1 \\ \pm e^{i\theta/2} \end{pmatrix}$

$$\sum_i \lambda_i |v_i\rangle\langle v_i| = \frac{a e^{i\theta/2}}{2} \begin{pmatrix} 1 & e^{-i\theta/2} \\ e^{i\theta/2} & 1 \end{pmatrix} + \frac{a e^{i\theta/2}}{2} \begin{pmatrix} 1 & -e^{i\theta/2} \\ -e^{i\theta/2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a \\ a e^{i\theta} & 0 \end{pmatrix}$$

$$= M$$

Problem 2: [20 pts.]

Starting with the product formula for the Gamma function:

$$\Gamma[z] = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

prove the properties

- i) $\Gamma[1] = 1$
- ii) $z \Gamma[z] = \Gamma[z+1]$.

For i) it may be helpful to use the identity $\sum_{n=1}^{\infty} \frac{1}{n} = \log N + \gamma + O(1/N)$. Here γ is the Euler-Mascheroni constant.

$$i) \quad \Gamma(1) = e^{-\gamma} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-1} e^{1/n}$$

$$\prod_{n=1}^N \left(1 + \frac{1}{n}\right)^{-1} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{N}{N+1} = (N+1)^{-1} \sim N^{-1} \text{ for large } N$$

$$\Gamma(1) = \lim_{N \rightarrow \infty} e^{-\gamma} N e^{\sum_{n=1}^N 1/n} = \lim_{N \rightarrow \infty} e^{-\gamma - \ln N + \sum_{n=1}^N \frac{1}{n} + O(1/N)} = 1$$

$$ii) \quad \Gamma(z+1) = \underbrace{e^{-\gamma(z+1)}}_{z+1} \prod_{n=1}^{\infty} \left(1 + \frac{z+1}{n}\right)^{-1} e^{(z+1)/n}$$

$$= \underbrace{e^{-\gamma} \prod_{n=1}^{\infty} \left(\frac{z+1}{n}\right)^{-1} e^{z/n}}_{\Gamma(1)} \underbrace{\frac{e^{-\gamma z}}{z+1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}}$$

$$= e^{-\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

$$= z \Gamma(z)$$

Problem 3: [20 pts.]

a) Solve the 1-D diffusion equation for the density of diffusing particles, $\rho(x, t)$, given by

$$\left(\frac{\partial}{\partial t} - D \frac{\partial^2}{\partial x^2} \right) \rho(x, t) = 0,$$

by performing a Laplace transform with respect to t and a Fourier transform with respect to x , then performing the required inverse transforms. Express your solution as an integral over an arbitrary initial condition, $\rho(x, 0)$.

b) Find $\rho(x, t)$ for the initial condition

$$\rho(x, 0) = N \sqrt{\frac{1}{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

c) Show that the total number of particles

$$N = \int_{-\infty}^{\infty} dx \rho(x, t),$$

is conserved: $dN/dt = 0$.

$$a) \bar{\rho}(k, s) = \int_{-\infty}^{\infty} dk e^{ikx} \int_0^{\infty} dt e^{-st} \rho(x, t)$$

$$s \bar{\rho}(k, s) - \bar{\rho}(k, 0) + k^2 D \bar{\rho}(k, 0) = 0$$

$$\Rightarrow \boxed{\bar{\rho}(k, s) = \frac{\bar{\rho}(k, 0)}{s + k^2 D}}$$

$$\mathcal{L}^{-1}[\bar{\rho}(k, s)] = e^{-k^2 D t} \bar{\rho}(k, 0)$$

$$\rho(x, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} e^{-k^2 D t} \bar{\rho}(k, 0)$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dy e^{-ik(x-y)} e^{-k^2 D t} \bar{\rho}(y, 0)$$

$$\boxed{\rho(x, t) = \int_{-\infty}^{\infty} dy \sqrt{\frac{1}{4\pi D t}} e^{-\frac{(x-y)^2}{4Dt}} \bar{\rho}(y, 0)}$$

(next page)

$$\begin{aligned}
 c) & N \int_{-\infty}^{\infty} dy \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(x-y)^2}{4Dt}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \\
 & = N \frac{e^{-\frac{x^2}{4\sigma^2}}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dy \frac{1}{\sqrt{4\pi D t}} \exp \left[-y^2 \left(\frac{1}{2\sigma^2} + \frac{1}{4Dt} \right) + \frac{xy}{2Dt} \right] \\
 & = \frac{N}{\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{4\pi D t}} \sqrt{\frac{8\pi Dt\sigma^2}{2\sigma^2 + 4Dt}} \exp \left[-\frac{x^2}{4\sigma^2} - \frac{x^2}{2Dt(\sigma^2 + 4Dt)} \right]
 \end{aligned}$$

$$P(x,t) = \frac{N}{\sqrt{2\pi(\sigma^2 + 4Dt)}} \exp \left[-\frac{x^2}{2\sigma^2 + 4Dt} \right]$$

$$d) \hat{P}(k,t) = \int_{-\infty}^{\infty} dx e^{ikx} P(x,t)$$

$$\frac{\partial}{\partial t} \hat{P}(k,t) + ik^2 \hat{P}(k,0) = 0$$

$$\Rightarrow \frac{\partial}{\partial t} \hat{P}(0,t) = \boxed{\frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx P(x,t) = 0}$$

can also see this from solution in c)

which N times a normalized Gaussian
with variance $\sigma^2(t) = \sigma^2 + 4Dt$