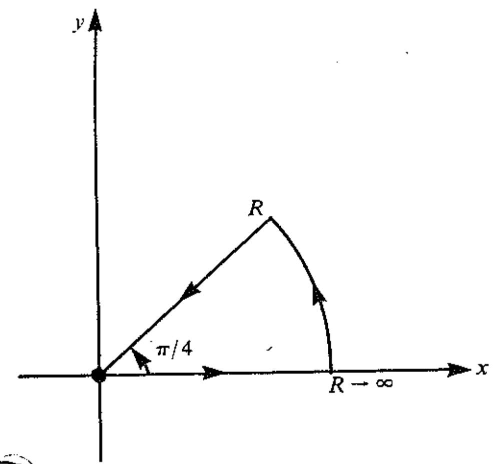

FIG. 7.16

7.2.24 Show that

$$\int_0^\infty \frac{1}{1+x^n}\,dx = \frac{\pi/n}{\sin(\pi/n)}.$$

Hint. Try the contour shown in Fig. 7.17.

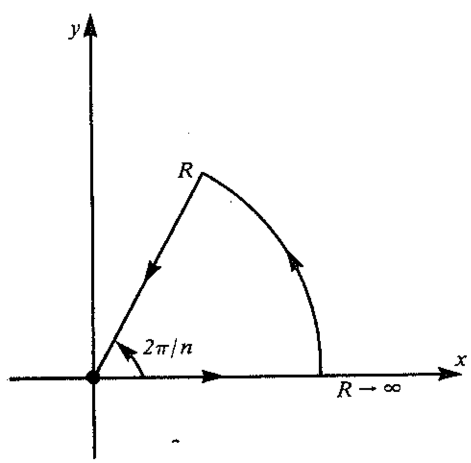

FIG. 7.17

7.2.25 (a) Show that

$$f(z) = z^4 - 2\cos 2\theta\, z^2 + 1$$

and has zeros at $e^{i\theta}$, $e^{-i\theta}$, $-e^{i\theta}$, and $-e^{-i\theta}$.

(b) Show that

$$\int_{-\infty}^\infty \frac{dx}{x^4 - 2\cos 2\theta\, x^2 + 1} = \frac{\pi}{2\sin\theta} = \frac{\pi}{2^{1/2}(1-\cos 2\theta)^{1/2}}.$$

Exercise 7.2.24 ($n = 4$) is a special case of this result.

7.2.26 Show that

$$\int_{-\infty}^\infty \frac{x^2\,dx}{x^4 - 2\cos 2\theta\, x^2 + 1} = \frac{\pi}{2\sin\theta} = \frac{\pi}{2^{1/2}(1-\cos 2\theta)^{1/2}}.$$

Exercise 7.2.21 is a special case of this result.

7.2.27 Apply the techniques of Example 7.2.4 to the evaluation of the improper integral

$$I = \int_{-\infty}^\infty \frac{dx}{x^2 - \sigma^2}.$$

(a) Let $\sigma \to \sigma + i\gamma$.
(b) Let $\sigma \to \sigma - i\gamma$.
(c) Take the Cauchy principal value.

7.2.28 The integral in Exercise 7.2.17 may be transformed into

$$\int_0^\infty e^{-y}\frac{y^2}{1+e^{-2y}}\,dy = \frac{\pi^3}{16}.$$

Evaluate this integral by the Gauss–Laguerre quadrature, Appendix A2, and compare your result with $\pi^3/16$.

ANS. Integral = 1.93775 (10 points).

7.3 DISPERSION RELATIONS

The concept of dispersion relations entered physics with the work of Kronig and Kramers in optics. The name dispersion comes from optical dispersion, a result of the dependence of the index of refraction on wavelength or angular frequency. The index of refraction n may have a real part determined by the phase velocity and a (negative) imaginary part determined by the absorption—see Eq. 7.79. Kronig and Kramers showed that the real part of $(n^2 - 1)$ could be expressed as an integral of the imaginary part. Generalizing this, we shall apply the label dispersion relations to any pair of equations giving the real part of a function as an integral of its imaginary part and the imaginary part as an integral of its real part—Eqs. 7.71*a* and 7.71*b* that follow. The existence of such integral relations might be suspected as an integral analog of the Cauchy–Riemann differential relations, Section 6.2.

The applications in modern physics are widespread. For instance, the real part of the function might describe the forward scattering of a gamma ray in a nuclear Coulomb field (a dispersive process). Then the imaginary part would describe the electron–positron pair production in that same Coulomb field (the absorptive process). As will be seen later, the dispersion relations may be taken as a consequence of causality and therefore are independent of the details of the particular interaction.

We consider a complex function $f(z)$ that is analytic in the upper half-plane and on the real axis. We also require that

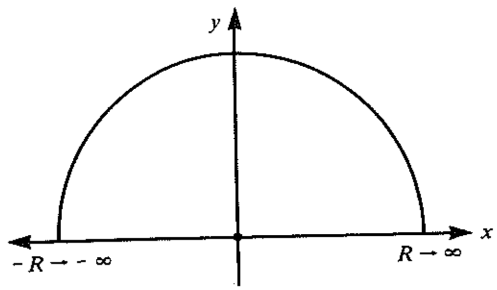

FIG. 7.18

$$\lim_{|z|\to\infty} |f(z)| = 0, \quad 0 \le \arg z \le \pi, \tag{7.66}$$

in order that the integral over an infinite semicircle will vanish. The point of these conditions is that we may express $f(z)$ by the Cauchy integral formula, Eq. 6.43,

$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z - z_0}\,dz. \tag{7.67}$$

The integral over the upper semicircle[1] vanishes and we have

$$f(z_0) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)}{x - z_0}\,dx. \tag{7.68}$$

The integral over the contour shown in Fig. 7.18 has become an integral along the x-axis.

Equation 7.68 assumes that z_0 is in the upper half-plane—interior to the closed contour. If z_0 were in the lower half-plane, the integral would yield zero by the Cauchy integral theorem, Section 6.3. Now, either letting z_0 approach the real axis from above ($z_0 \to x_0$), or placing it on the real axis and taking an average of Eq. 7.68 and zero, we find that Eq. 7.68 becomes

$$f(x_0) = \frac{1}{\pi i} P\int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\,dx, \tag{7.69}$$

where P indicates the Cauchy principal value.

Splitting Eq. 7.69 into real and imaginary parts[2] yields

$$\begin{aligned} f(x_0) &= u(x_0) + iv(x_0) \\ &= \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{v(x)}{x - x_0}\,dx - \frac{i}{\pi} P\int_{-\infty}^{\infty} \frac{u(x)}{x - x_0}\,dx. \end{aligned} \tag{7.70}$$

Finally, equating real part to real part and imaginary part to imaginary part, we obtain

[1] The use of a semicircle to close the path of integration is convenient, not mandatory. Other paths are possible.

[2] The second argument, $y = 0$, is dropped. $u(x_0, 0) \to u(x_0)$.

$$u(x_0) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{v(x)}{x - x_0}\,dx \tag{7.71a}$$

$$v(x_0) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{u(x)}{x - x_0}\,dx. \tag{7.71b}$$

These are the dispersion relations. The real part of our complex function is expressed as an integral over the imaginary part. The imaginary part is expressed as an integral over the real part. The real and imaginary parts are *Hilbert transforms* of each other. Note that these relations are meaningful only when $f(x)$ is a *complex* function of the real variable x. Compare Exercise 7.3.1.

From a physical point of view $u(x)$ and/or $v(x)$ represent some physical measurements. Then $f(z) = u(z) + iv(z)$ is an analytic continuation over the upper half-plane, with the value on the real axis serving as a boundary condition.

Symmetry Relations

On occasion $f(x)$ will satisfy a symmetry relation and the integral from $-\infty$ to $+\infty$ may be replaced by an integral over positive values only. This is of considerable physical importance because the variable x might represent a frequency and only zero and positive frequencies are available for physical measurements. Suppose[3]

$$f(-x) = f^*(x). \tag{7.72}$$

Then

$$u(-x) + iv(-x) = u(x) - iv(x). \tag{7.73}$$

The real part of $f(x)$ is even and the imaginary part is odd.[4] In quantum mechanical scattering problems these relations (Eq. 7.73) are called *crossing conditions*. To exploit these crossing conditions, we rewrite Eq. 7.71a as

$$u(x_0) = \frac{1}{\pi} P\int_{-\infty}^{0} \frac{v(x)}{x - x_0}\,dx + \frac{1}{\pi} P\int_{0}^{\infty} \frac{v(x)}{x - x_0}\,dx. \tag{7.74}$$

Letting $x \to -x$ in the first integral on the right-hand side of Eq. 7.74 and substituting $v(-x) = -v(x)$ from Eq. 7.73, we obtain

$$\begin{aligned} u(x_0) &= \frac{1}{\pi} P\int_{0}^{\infty} v(x)\left\{\frac{1}{x + x_0} + \frac{1}{x - x_0}\right\} dx \\ &= \frac{2}{\pi} P\int_{0}^{\infty} \frac{xv(x)}{x^2 - x_0^2}\,dx. \end{aligned} \tag{7.75}$$

Similarly,

[3] This is not just a happy coincidence. It ensures that the Fourier transform of $f(x)$ will be real. In turn, Eq. 7.72 is a consequence of obtaining $f(x)$ as the Fourier transform of a real function.

[4] $u(x, 0) = u(-x, 0)$, $v(x, 0) = -v(-x, 0)$. Compare these symmetry conditions with those that follow from the Schwarz reflection principle, Section 6.5.

$$v(x_0) = -\frac{2}{\pi} P \int_0^\infty \frac{x_0 u(x)}{x^2 - x_0^2} dx. \tag{7.76}$$

The original Kronig–Kramers optical dispersion relations were in this form. The asymptotic behavior ($x_0 \to \infty$) of Eqs. 7.75 and 7.76 lead to quantum mechanical *sum rules*, Exercise 7.3.4.

Optical Dispersion

The function $\exp[i(kx - \omega t)]$ describes a wave moving along the x-axis in the positive direction with velocity $v = \omega/k$. ω is the angular frequency, k the wave number or propagation vector, and $n = ck/\omega$ the index of refraction. From Maxwell's equations, electric permittivity ε, and Ohm's law with conductivity σ the propagation vector k for a dielectric becomes[5]

$$k^2 = \varepsilon \frac{\omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\omega\varepsilon}\right) \tag{7.77}$$

(with μ, the magnetic permeability taken to be unity). The presence of the conductivity (which means absorption) gives rise to an imaginary part. The propagation vector k (and therefore the index of refraction n) have become complex.

Conversely, the (positive) imaginary part implies absorption. For poor conductivity ($4\pi\sigma/\omega\varepsilon \ll 1$) a binomial expansion yields

$$k = \sqrt{\varepsilon}\frac{\omega}{c} + i\frac{2\pi\sigma}{c\sqrt{\varepsilon}}$$

and

$$e^{i(kx-\omega t)} = e^{i\omega(\sqrt{\varepsilon}/c - t)} e^{-2\pi\sigma x/c\sqrt{\varepsilon}},$$

an attenuated wave.

Returning to the general expression for k^2, we find that Eq. 7.77 the index of refraction becomes

$$n^2 = \frac{c^2k^2}{\omega^2} = \varepsilon + i\frac{4\pi\sigma}{\omega}. \tag{7.78}$$

We take n^2 to be a function of the *complex* variable ω (with ε and σ depending on ω). However, n^2 does not vanish as $\omega \to \infty$ but instead approaches unity. So to satisfy the condition, Eq. 7.66, one works with $f(\omega) = n^2(\omega) - 1$. The original Kronig–Kramers optical dispersion relations were in the form of

$$\begin{aligned} \mathscr{R}[n^2(\omega_0) - 1] &= \frac{2}{\pi} P \int_0^\infty \frac{\omega \mathscr{I}[n^2(\omega) - 1]}{\omega^2 - \omega_0^2} d\omega, \\ \mathscr{I}[n^2(\omega_0) - 1] &= -\frac{2}{\pi} P \int_0^\infty \frac{\omega_0 \mathscr{R}[n^2(\omega) - 1]}{\omega^2 - \omega_0^2} d\omega. \end{aligned} \tag{7.79}$$

[5] See J. D. Jackson, *Classical Electrodynamics*, 2nd ed., Section 7.7, New York: Wiley (1975). Equation 7.77 follows Jackson in the use of Gaussian units.

Knowledge of the absorption coefficient at all frequencies specifies the real part of the index of refraction and vice versa.

The Parseval Relation

When the functions $u(x)$ and $v(x)$ are Hilbert transforms of each other and each is square integrable,[6] the two functions are related by

$$\int_{-\infty}^\infty |u(x)|^2 dx = \int_{-\infty}^\infty |v(x)|^2 dx. \tag{7.80}$$

This is the Parseval relation.

To derive Eq. 7.80, we start with

$$\int_{-\infty}^\infty |u(x)|^2 dx = \int_{-\infty}^\infty \frac{1}{\pi}\int_{-\infty}^\infty \frac{v(s)\,ds}{s - x}\frac{1}{\pi}\int_{-\infty}^\infty \frac{v(t)\,dt}{t - x} dx,$$

using Eq. 7.71*a* twice.

Integrating first with respect to x, we have

$$\int_{-\infty}^\infty |u(x)|^2 dx = \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{\pi^2}\int_{-\infty}^\infty \frac{dx}{(s - x)(t - x)} v(s)\,ds\,v(t)\,dt. \tag{7.81}$$

From Exercise 7.3.8 the x integration yields a delta function:

$$\frac{1}{\pi^2}\int_{-\infty}^\infty \frac{dx}{(s - x)(t - x)} = \delta(s - t).$$

We have

$$\int_{-\infty}^\infty |u(x)|^2 dx = \int_{-\infty}^\infty \int_{-\infty}^\infty v(s)\,\delta(s - t)\,ds\,v(t)\,dt. \tag{7.82}$$

Then the s integration is carried out by inspection, using the defining property of the delta function.

$$\int_{-\infty}^\infty v(s)\,\delta(s - t)\,ds = v(t). \tag{7.83}$$

Substituting Eq. 7.83 into Eq. 7.82, we have Eq. 7.80, the Parseval relation. Again, in terms of optics, the presence of refraction over some frequency range ($n \neq 1$) implies the existence of absorption and vice versa.

Causality

The real significance of the dispersion relations in physics is that they are a direct consequence of assuming that the particular physical system obeys causality. Causality is awkward to define precisely but the general meaning is that the effect cannot precede the cause. A scattered wave cannot be emitted by the scattering center before the incident wave has arrived. For linear systems the most general relation between an input function G (the cause) and an output function H (the effect) may be written as

[6] This means that $\int_{-\infty}^\infty |u(x)|^2\,dx$ and $\int_{-\infty}^\infty |v(x)|^2\,dx$ are finite.

$$H(t) = \int_{-\infty}^{\infty} F(t-t')G(t')\,dt'. \tag{7.84}$$

Causality is imposed by requiring that

$$F(t-t') = 0 \qquad \text{for } t - t' < 0.$$

Equation 7.84 gives the time dependence. The frequency dependence is obtained by taking Fourier transforms. By the Fourier convolution theorem, Section 15.5,

$$h(\omega) = f(\omega)g(\omega),$$

where $f(\omega)$ is the Fourier transform of $F(t)$, and so on. Conversely, $F(t)$ is the Fourier transform of $f(\omega)$.

The connection with the dispersion relations is provided by the Titchmarsh theorem.[7] This states that if $f(\omega)$ is square integrable over the real ω-axis, then any one of the following three statements implies the other two.

1. The Fourier transform of $f(\omega)$ is zero for $t < 0$: Eq. 7.81.
2. Replacing ω by z, the function $f(z)$ is analytic in the complex z plane for $y > 0$ and approaches $f(x)$ almost everywhere as $y \to 0$. Further,

$$\int_{-\infty}^{\infty} |f(x+iy)|^2\,dx < K \qquad \text{for } y > 0,$$

   that is, the integral is bounded.
3. The real and imaginary parts of $f(z)$ are Hilbert transforms of each other: Eqs. 7.71*a* and 7.71*b*.

The assumption that the relationship between the input and the output of our linear system is causal (Eq. 7.81) means that the first statement is satisfied. If $f(\omega)$ is square integrable, then the Titchmarsh theorem has the third statement as a consequence and we have dispersion relations.

EXERCISES

7.3.1 The function $f(z)$ satisfies the conditions for the dispersion relations. In addition, $f(z) = f^*(z^*)$, the Schwarz reflection principle, Section 6.5. Show that $f(z)$ is identically zero.

7.3.2 For $f(z)$ such that we may replace the closed contour of the Cauchy integral formula by an integral over the real axis we have

[7] Refer to E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 2nd ed., New York: Oxford University Press 1937. For a more informal discussion of the Titchmarsh theorem and further details on causality see J. Hilgevoord, *Dispersion Relations and Causal Description*. Amsterdam: North-Holland Publishing Co. (1962).

$$f(x_0) = \frac{1}{2\pi i}\left\{\int_{-\infty}^{x_0-\delta} \frac{f(x)}{x-x_0}dx + \int_{x_0+\delta}^{\infty} \frac{f(x)}{x-x_0}dx\right\} + \frac{1}{2\pi i}\int_{C_{x_0}} \frac{f(x)}{x-x_0}dx.$$

Here C_{x_0} designates a small semicircle about x_0 in the lower half-plane. Show that this reduces to

$$f(x_0) = \frac{1}{\pi i}P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}dx,$$

which is Eq. 7.69.

7.3.3 (a) For $f(z) = e^{iz}$, Eq. 7.66 does not hold at the end points, $\arg z = 0, \pi$. Show, with the help of Jordan's lemma, Section 7.2, that Eq. 7.67 still holds.
(b) For $f(z) = e^{iz}$ verify the dispersion relations, Eq. 7.71 or Eqs. 7.75 and 7.76, by direct integration.

7.3.4 With $f(x) = u(x) + iv(x)$ and $f(x) = f^*(-x)$, show that as $x_0 \to \infty$,

(a) $u(x_0) \sim -\dfrac{2}{\pi x_0^2}\displaystyle\int_0^{\infty} xv(x)\,dx,$

(b) $v(x_0) \sim \dfrac{2}{\pi x_0}\displaystyle\int_0^{\infty} u(x)\,dx.$

In quantum mechanics relations of this form are often called *sum rules*.

7.3.5 (a) Given the integral equation

$$\frac{1}{1+x_0^2} = \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{u(x)}{x-x_0}dx,$$

use Hilbert transforms to determine $u(x_0)$.
(b) Verify that the integral equation of part (a) is satisfied.
(c) From $f(z)|_{y=0} = u(x) + iv(x)$, replace x by z and determine $f(z)$. Verify that the conditions for the Hilbert transforms are satisfied.
(d) Are the crossing conditions satisfied?

ANS. (a) $u(x_0) = \dfrac{x_0}{(1+x_0^2)}$,
(c) $f(z) = (z+i)^{-1}$,

7.3.6 (a) If the real part of the complex index of refraction (squared) is constant (no optical dispersion), show that the imaginary part is zero (no absorption).
(b) Conversely, if there is absorption, show that there must be dispersion. In other words, if the imaginary part of $n^2 - 1$ is not zero, show that the real part of $n^2 - 1$ is not constant.

7.3.7 Given $u(x) = x/(x^2+1)$ and $v(x) = -1/(x^2+1)$, show by direct evaluation of each integral that

$$\int_{-\infty}^{\infty} |u(x)|^2\,dx = \int_{-\infty}^{\infty} |v(x)|^2\,dx.$$

ANS. $\displaystyle\int_{-\infty}^{\infty} |u(x)|^2\,dx = \int_{-\infty}^{\infty} |v(x)|^2\,dx = \frac{\pi}{2}.$

7.3.8 Take $u(x) = \delta(x)$, a delta function, and *assume* that the Hilbert transform equations hold.
(a) Show that

$$\delta(w) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{dy}{y(y-w)}.$$

(b) With changes of variables $w = s - t$ and $x = s - y$, transform the δ representation of part (a) into

$$\delta(s - t) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dx}{(x - s)(x - t)}.$$

Note. The δ function is discussed in Section 8.7.

7.3.9 Show that

$$\delta(x) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dt}{t(t - x)}$$

is a valid representation of the delta function in the sense that

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0).$$

Assume that $f(x)$ satisfies the condition for the existence of a Hilbert transform.
Hint. Apply Eq. 7.69 twice.

7.4 THE METHOD OF STEEPEST DESCENTS

In analyzing problems in mathematical physics, one often finds it desirable to know the behavior of a function for large values of the variable, that is, the asymptotic behavior of the function. Specific examples are furnished by the gamma function (Chapter 10) and the various Bessel functions (Chapter 11). The method of steepest descents is a method of determining such asymptotic behavior when the function can be expressed as an integral of the general form

$$I(s) = \int_C g(z)e^{sf(z)}\,dz. \tag{7.85}$$

For the present, let us take s to be real. The contour of integration C is then chosen so that the real part of $f(z)$ approaches minus infinity at both limits and that the integrand will vanish at the limits, or is chosen as a closed contour. It is further assumed that the factor $g(z)$ in the integrand is dominated by the exponential in the region of interest.

If the parameter s is large and positive, the value of the integrand will become large when the real part of $f(z)$ is large and small when the real part of $f(z)$ is small or negative. In particular, as s is permitted to increase indefinitely (leading to the asymptotic dependence), the entire contribution of the integrand to the integral will come from the region in which the real part of $f(z)$ takes on a positive maximum value. Away from this positive maximum the integrand will become negligibly small in comparison. This is seen by expressing $f(z)$ as

$$f(z) = u(x, y) + iv(x, y).$$

Then the integral may be written as

$$I(s) = \int_C g(z)e^{su(x,y)}e^{isv(x,y)}\,dz. \tag{7.86}$$

If now, in addition, we impose the condition that the imaginary part of the exponent, $iv(x, y)$, be constant in the region in which the real part takes on its maximum value, that is, $v(x, y) = v(x_0, y_0) = v_0$, we may approximate the integral by

$$I(s) \approx e^{isv_0} \int_C g(z)e^{su(x,y)}\,dz. \tag{7.87}$$

Away from the maximum of the real part, the imaginary part may be permitted to oscillate as it wishes, for the integrand is negligibly small and the varying phase factor is therefore irrelevant.

The real part of $sf(z)$ is a maximum for a given s when the real part of $f(z)$, $u(x, y)$, is a maximum. This implies that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0,$$

and therefore, by use of the Cauchy–Riemann conditions of Section 6.2

$$\frac{df(z)}{dz} = 0. \tag{7.88}$$

We proceed to search for such zeros of the derivative.

It is essential to note that the maximum value of $u(x, y)$ is the maximum only along a given contour. In the finite plane neither the real nor the imaginary part of our analytic function possesses an absolute maximum. This may be seen by recalling that both u and v satisfy Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{7.89}$$

From this, if the second derivative with respect to x is positive, the second derivative with respect to y must be negative, and therefore neither u nor v can possess an absolute maximum or minimum. Since the function $f(z)$ was taken to be analytic, singular points are clearly excluded. The vanishing of the derivative (Eq. 7.88) then implies that we have a saddle point, a stationary value, which may be a maximum of $u(x, y)$ for one contour and a minimum for another (Fig. 7.19).

Our problem, then, is to choose the contour of integration to satisfy two conditions. (1) The contour must be chosen so that $u(x, y)$ has a maximum at the saddle point. (2) The contour must pass through the saddle in such a way that the imaginary part, $v(x, y)$, is a constant. This second condition leads to the path of steepest descent and gives the method its name. From Section 6.2, especially Exercise 6.2.1, we know that the curves corresponding to $u =$ constant and $v =$ constant form an orthogonal system. This means that a curve $v = c_i$, constant, is everywhere tangential to the gradient of u, ∇u. Hence the curve $v =$ constant is the curve that gives the line of steepest descent from the saddle point.[1]

[1] The line of steepest ascent is also characterized by constant v. The saddle point must be inspected carefully to distinguish the line of steepest descent from the line of steepest ascent. This is discussed later in two examples.

But by (5.247), these commutators are $\Delta_+(x-y)$ and $\Delta_+(y-x)$. Thus the mean value in the vacuum of the time-ordered product

$$\begin{aligned}\langle 0|\mathcal{T}\{\phi(x)\phi(y)\}\,|0\rangle &= \theta(x^0-y^0)\Delta_+(x-y)+\theta(y^0-x^0)\Delta_+(y-x)\\ &= -i\Delta_F(x-y)\end{aligned} \tag{5.252}$$

is the Feynman propagator (5.241) multiplied by $-i$. □

5.19 Dispersion relations

In many physical contexts, functions occur that are analytic in the upper half-plane (UHP). Suppose for instance that $\hat{f}(t)$ is a transfer function that determines an effect $e(t)$ due to a cause $c(t)$

$$e(t) = \int_{-\infty}^{\infty} dt'\,\hat{f}(t-t')\,c(t'). \tag{5.253}$$

If the system is **causal**, then the transfer function $\hat{f}(t-t')$ is zero for $t-t'<0$, and so its Fourier transform

$$f(z) = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}}\hat{f}(t)\,e^{izt} = \int_{0}^{\infty} \frac{dt}{\sqrt{2\pi}}\hat{f}(t)\,e^{izt} \tag{5.254}$$

will be analytic in the upper half-plane and will shrink as the imaginary part of $z = x+iy$ increases.

So let us assume that the function $f(z)$ is analytic in the upper half-plane and on the real axis and further that

$$\lim_{r\to\infty} |f(re^{i\theta})| = 0 \quad \text{for} \quad 0\le\theta\le\pi. \tag{5.255}$$

By Cauchy's integral formula (5.32), if z_0 lies in the upper half-plane, then $f(z_0)$ is given by the closed counterclockwise contour integral

$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}\,dz, \tag{5.256}$$

in which the contour runs along the real axis and then loops over the semicircle

$$\lim_{r\to\infty} re^{i\theta} \quad \text{for} \quad 0\le\theta\le\pi. \tag{5.257}$$

Our assumption (5.255) about the behavior of $f(z)$ in the UHP implies that this contour (5.257) is a ghost contour because its modulus is bounded by

$$\lim_{r\to\infty} \frac{1}{2\pi}\int \frac{|f(re^{i\theta})|r}{r}\,d\theta = \lim_{r\to\infty} |f(re^{i\theta})| = 0. \tag{5.258}$$

So we may drop the ghost contour and write $f(z_0)$ as

$$f(z_0) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)}{x-z_0}\,dx. \tag{5.259}$$

Letting the imaginary part y_0 of $z_0 = x_0 + iy_0$ shrink to ϵ

$$f(x_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0 - i\epsilon}\, dx \tag{5.260}$$

and using Cauchy's trick (5.220), we get

$$f(x_0) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx + \frac{i\pi}{2\pi i} \int_{-\infty}^{\infty} f(x)\, \delta(x - x_0)\, dx \tag{5.261}$$

or

$$f(x_0) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx + \frac{1}{2} f(x_0), \tag{5.262}$$

which is the **dispersion relation**

$$f(x_0) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx. \tag{5.263}$$

If we break $f(z) = u(z) + iv(z)$ into its real $u(z)$ and imaginary $v(z)$ parts, then this dispersion relation (5.263)

$$\begin{aligned} u(x_0) + iv(x_0) &= \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{u(x) + iv(x)}{x - x_0}\, dx \\ &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x)}{x - x_0}\, dx - \frac{i}{\pi} P \int_{-\infty}^{\infty} \frac{u(x)}{x - x_0}\, dx \end{aligned} \tag{5.264}$$

breaks into its real and imaginary parts

$$u(x_0) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x)}{x - x_0}\, dx \quad \text{and} \quad v(x_0) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(x)}{x - x_0}\, dx, \tag{5.265}$$

which express u and v as **Hilbert transforms** of each other.

In applications of dispersion relations, the function $f(x)$ for $x < 0$ sometimes is either physically meaningless or experimentally inaccessible. In such cases, there may be a symmetry that relates $f(-x)$ to $f(x)$. For instance, if $f(x)$ is the Fourier transform of a real function $\hat{f}(k)$, then by equation (3.25) it obeys the symmetry relation

$$f^*(x) = u(x) - iv(x) = f(-x) = u(-x) + iv(-x), \tag{5.266}$$

which says that u is even, $u(-x) = u(x)$, and v odd, $v(-x) = -v(x)$. Using these symmetries, one may show (exercise 5.36) that the Hilbert transformations (5.265) become

$$u(x_0) = \frac{2}{\pi} P \int_0^{\infty} \frac{x\, v(x)}{x^2 - x_0^2}\, dx \quad \text{and} \quad v(x_0) = -\frac{2x_0}{\pi} P \int_0^{\infty} \frac{u(x)}{x^2 - x_0^2}\, dx, \tag{5.267}$$

which do not require input at negative values of x.

5.20 Kramers–Kronig relations

If we use $\sigma \boldsymbol{E}$ for the current density $\boldsymbol{J}$ and $\boldsymbol{E}(t) = e^{-i\omega t}\boldsymbol{E}(t)$ for the electric field, then Maxwell's equation $\nabla \times \boldsymbol{B} = \mu \boldsymbol{J} + \epsilon\mu\dot{\boldsymbol{E}}$ becomes

$$\nabla \times \boldsymbol{B} = -i\omega\epsilon\mu\left(1 + i\frac{\sigma}{\epsilon\omega}\right)\boldsymbol{E} \equiv -i\omega n^2\epsilon_0\mu_0\boldsymbol{E} \tag{5.268}$$

and reveals the squared index of refraction as

$$n^2(\omega) = \frac{\epsilon\mu}{\epsilon_0\mu_0}\left(1 + i\frac{\sigma}{\epsilon\omega}\right). \tag{5.269}$$

The imaginary part of n^2 represents the scattering of light mainly by electrons. At high frequencies in nonmagnetic materials $n^2(\omega) \to 1$, and so Kramers and Kronig applied the Hilbert-transform relations (5.267) to the function $n^2(\omega) - 1$ in order to satisfy condition (5.255). Their relations are

$$\mathrm{Re}(n^2(\omega_0)) = 1 + \frac{2}{\pi}P\int_0^\infty \frac{\omega\,\mathrm{Im}(n^2(\omega))}{\omega^2 - \omega_0^2}\,d\omega \tag{5.270}$$

and

$$\mathrm{Im}(n^2(\omega_0)) = -\frac{2\omega_0}{\pi}P\int_0^\infty \frac{\mathrm{Re}(n^2(\omega)) - 1}{\omega^2 - \omega_0^2}\,d\omega. \tag{5.271}$$

What Kramers and Kronig actually wrote was slightly different from these dispersion relations (5.270 & 5.271). H. A. Lorentz had shown that the index of refraction $n(\omega)$ is related to the forward scattering amplitude $f(\omega)$ for the scattering of light by a density N of scatterers (Sakurai, 1982)

$$n(\omega) = 1 + \frac{2\pi c^2}{\omega^2}Nf(\omega). \tag{5.272}$$

They used this formula to infer that the real part of the index of refraction approached unity in the limit of infinite frequency and applied the Hilbert transform (5.267)

$$\mathrm{Re}[n(\omega)] = 1 + \frac{2}{\pi}P\int_0^\infty \frac{\omega'\,\mathrm{Im}[n(\omega')]}{\omega'^2 - \omega^2}\,d\omega'. \tag{5.273}$$

The Lorentz relation (5.272) expresses the imaginary part $\mathrm{Im}[n(\omega)]$ of the index of refraction in terms of the imaginary part of the forward scattering amplitude $f(\omega)$

$$\mathrm{Im}[n(\omega)] = 2\pi(c/\omega)^2 N\,\mathrm{Im}[f(\omega)]. \tag{5.274}$$

And the **optical theorem**relates $\mathrm{Im}[f(\omega)]$ to the **total cross-section**

$$\sigma_{\mathrm{tot}} = \frac{4\pi}{|\boldsymbol{k}|}\mathrm{Im}[f(\omega)] = \frac{4\pi c}{\omega}\mathrm{Im}[f(\omega)]. \tag{5.275}$$

Thus we have $\mathrm{Im}[n(\omega)] = cN\sigma_{\mathrm{tot}}/(2\omega)$, and by the Lorentz relation (5.272) $\mathrm{Re}[n(\omega)] = 1 + 2\pi(c/\omega)^2 N\mathrm{Re}[f(\omega)]$. Insertion of these formulas into the Kramers–Kronig integral (5.273) gives a dispersion relation for the real part of the forward scattering amplitude $f(\omega)$ in terms of the total cross-section

$$\mathrm{Re}[f(\omega)] = \frac{\omega^2}{2\pi^2 c} P \int_0^\infty \frac{\sigma_{\mathrm{tot}}(\omega')}{\omega'^2 - \omega^2}\, d\omega'. \tag{5.276}$$

5.21 Phase and group velocities

Suppose $A(\boldsymbol{x}, t)$ is the amplitude

$$A(\boldsymbol{x}, t) = \int e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - Et)/\hbar} A(\boldsymbol{p})\, d^3p = \int e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} B(\boldsymbol{k})\, d^3k \tag{5.277}$$

where $B(\boldsymbol{k}) = \hbar^3 A(\hbar\boldsymbol{k})$ varies slowly compared to the phase $\exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)]$. The **phase velocity** $\boldsymbol{v}_p$ is the linear relation $\boldsymbol{x} = \boldsymbol{v}_p t$ between $\boldsymbol{x}$ and t that keeps the phase $\phi = \boldsymbol{p}\cdot\boldsymbol{x} - Et$ constant as a function of the time

$$0 = \boldsymbol{p}\cdot d\boldsymbol{x} - E\,dt = (\boldsymbol{p}\cdot\boldsymbol{v}_p - E)\,dt \quad\Longleftrightarrow\quad \boldsymbol{v}_p = \frac{E}{p}\hat{\boldsymbol{p}} = \frac{\omega}{k}\hat{\boldsymbol{k}}, \tag{5.278}$$

in which $p = |\boldsymbol{p}|$, and $k = |\boldsymbol{k}|$. For light in the vacuum, $\boldsymbol{v}_p = c = (\omega/k)\,\hat{\boldsymbol{k}}$.

The **group velocity** $\boldsymbol{v}_g$ is the linear relation $\boldsymbol{x} = \boldsymbol{v}_g t$ between $\boldsymbol{x}$ and t that maximizes the amplitude $A(\boldsymbol{x}, t)$ by keeping the phase $\phi = \boldsymbol{p}\cdot\boldsymbol{x} - Et$ constant as a function of the momentum $\boldsymbol{p}$

$$\nabla_p(\boldsymbol{p}\boldsymbol{x} - Et) = \boldsymbol{x} - \nabla_p E(\boldsymbol{p})\, t = 0 \tag{5.279}$$

at the maximum of $A(\boldsymbol{p})$. This **condition of stationary phase** gives the group velocity as

$$\boldsymbol{v}_g = \nabla_p E(\boldsymbol{p}) = \nabla_k \omega(\boldsymbol{k}). \tag{5.280}$$

If $E = \boldsymbol{p}^2/(2m)$, then $\boldsymbol{v}_g = \boldsymbol{p}/m$.

When light traverses a medium with a complex index of refraction $n(\boldsymbol{k})$, the wave vector $\boldsymbol{k}$ becomes complex, and its (positive) imaginary part represents the scattering of photons in the forward direction, typically by the electrons of the medium. For simplicity, we'll consider the propagation of light through a medium in one dimension, that of the forward direction of the beam. Then the (real) frequency $\omega(k)$ and the (complex) wave-number k are related by $k = n(k)\,\omega(k)/c$, and the phase velocity of the light is

$$v_{\mathrm{p}} = \frac{\omega}{\mathrm{Re}(k)} = \frac{c}{\mathrm{Re}(n(k))}. \tag{5.281}$$

If we regard the index of refraction as a function of the frequency ω, instead of the wave-number k, then by differentiating the real part of the relation $\omega n(\omega) = ck$ with respect to ω, we find

$$n_r(\omega) + \omega \frac{dn_r(\omega)}{d\omega} = c\frac{dk_r}{d\omega}, \tag{5.282}$$

in which the subscript r means real part. Thus the group velocity (5.280) of the light is

$$v_g = \frac{d\omega}{dk_r} = \frac{c}{n_r(\omega) + \omega\, dn_r/d\omega}. \tag{5.283}$$

Optical physicists call the denominator the **group index of refraction**

$$n_g(\omega) = n_r(\omega) + \omega \frac{dn_r(\omega)}{d\omega} \tag{5.284}$$

so that as in the expression (5.281) for the phase velocity $v_p = c/n_r(\omega)$, the group velocity is $v_g = c/n_g(\omega)$.

In some media, the derivative $dn_r/d\omega$ is large and positive, and the group velocity v_g of light there can be much less than c (Steinberg *et al.*, 1993; Wang and Zhang, 1995) – as slow as 17 m/s (Hau *et al.*, 1999). This effect is called **slow light**. In certain other media, the derivative $dn/d\omega$ is so negative that the group index of refraction $n_g(\omega)$ is less than unity, and in them the group velocity v_g exceeds c! This effect is called **fast light**. In some media, the derivative $dn_r/d\omega$ is so negative that $dn_r/d\omega < -n_r(\omega)/\omega$, and then $n_g(\omega)$ is not only less than unity but also less than zero. In such a medium, the group velocity v_g of light is negative! This effect is called **backwards light**.

Sommerfeld and Brillouin (Brillouin, 1960, ch. II & III) anticipated fast light and concluded that it would not violate special relativity as long as the **signal velocity** – defined as the speed of the front of a square pulse – remained less than c. Fast light does not violate special relativity (Stenner *et al.*, 2003; Brunner *et al.*, 2004) (Léon Brillouin, 1889–1969; Arnold Sommerfeld, 1868–1951).

Slow, fast, and backwards light can occur when the frequency ω of the light is near a peak or **resonance** in the total cross-section σ_{tot} for the scattering of light by the atoms of the medium. To see why, recall that the index of refraction $n(\omega)$ is related to the forward scattering amplitude $f(\omega)$ and the density N of scatterers by the formula (5.272)

$$n(\omega) = 1 + \frac{2\pi c^2}{\omega^2} N f(\omega) \tag{5.285}$$

and that the real part of the forward scattering amplitude is given by the Kramers–Kronig integral (5.276) of the total cross-section

$$\mathrm{Re}(f(\omega)) = \frac{\omega^2}{2\pi^2 c} P\int_0^\infty \frac{\sigma_{tot}(\omega')\, d\omega'}{\omega'^2 - \omega^2}. \tag{5.286}$$

So the real part of the index of refraction is

$$n_r(\omega) = 1 + \frac{cN}{\pi} P \int_0^\infty \frac{\sigma_{tot}(\omega')\, d\omega'}{\omega'^2 - \omega^2}. \tag{5.287}$$

If the amplitude for forward scattering is of the Breit–Wigner form

$$f(\omega) = f_0 \frac{\Gamma/2}{\omega_0 - \omega - i\Gamma/2} \tag{5.288}$$

then by (5.285) the real part of the index of refraction is

$$n_r(\omega) = 1 + \frac{\pi c^2 N f_0 \Gamma(\omega_0 - \omega)}{\omega^2\left[(\omega - \omega_0)^2 + \Gamma^2/4\right]} \tag{5.289}$$

and by (5.283) the group velocity is

$$v_g = c\left[1 + \frac{\pi c^2 N f_0 \Gamma\, \omega_0}{\omega^2} \frac{\left[(\omega - \omega_0)^2 - \Gamma^2/4\right]}{\left[(\omega - \omega_0)^2 + \Gamma^2/4\right]^2}\right]^{-1}. \tag{5.290}$$

This group velocity v_g is less than c whenever $(\omega - \omega_0)^2 > \Gamma^2/4$. But we get fast light $v_g > c$, if $(\omega - \omega_0)^2 < \Gamma^2/4$, and even backwards light, $v_g < 0$, if $\omega \approx \omega_0$ with $4\pi c^2 N f_0/\Gamma\omega_0 \gg 1$. Robert W. Boyd's papers explain how to make slow and fast light (Bigelow *et al.*, 2003) and backwards light (Gehring *et al.*, 2006).

We can use the principal-part identity (5.224) to subtract

$$0 = \frac{cN}{\pi} P \int_0^\infty \frac{\sigma_{tot}(\omega)}{\omega'^2 - \omega^2}\, d\omega' \tag{5.291}$$

from the Kramers–Kronig integral (5.287) so as to write the index of refraction in the regularized form

$$n_r(\omega) = 1 + \frac{cN}{\pi} \int_0^\infty \frac{\sigma_{tot}(\omega') - \sigma_{tot}(\omega)}{\omega'^2 - \omega^2}\, d\omega', \tag{5.292}$$

which we can differentiate and use in the group-velocity formula (5.283)

$$v_g(\omega) = c\left[1 + \frac{cN}{\pi} P \int_0^\infty \frac{\left[\sigma_{tot}(\omega') - \sigma_{tot}(\omega)\right](\omega'^2 + \omega^2)}{(\omega'^2 - \omega^2)^2}\, d\omega'\right]^{-1}. \tag{5.293}$$

5.22 The method of steepest descent

Suppose we want to approximate for big $x > 0$ the integral

$$I(x) = \int_a^b dz\, h(z) \exp(x f(z)), \tag{5.294}$$